

ASYMPTOTIC STATISTICAL CHARACTERIZATIONS OF p -HARMONIC FUNCTIONS OF TWO VARIABLES

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Dedicated to Professor Lloyd Jackson and his enduring legacy

ABSTRACT. Generalizing the well known mean-value property of harmonic functions, we prove that a p -harmonic function of two variables satisfies, in a viscosity sense, two asymptotic formulas involving its local statistics. Moreover, we show that these asymptotic formulas characterize p -harmonic functions when $1 < p < \infty$. An example demonstrates that, in general, these formulas do not hold in a non-asymptotic sense.

1. Introduction. A fundamental and fascinating fact about harmonic functions is their characterization by the mean value property [4]: the continuous function u is harmonic in the domain $\Omega \subset \mathbf{R}^N$ if and only if

$$(1) \quad u(x) = \int_{\partial B_r(x)} u(s) ds = \int_{B_r(x)} u(y) dy \quad \text{for each } x \in \Omega,$$

where $B_r(x) \Subset \Omega$ is a ball with center x and radius $r > 0$, $\partial B_r(x)$ is its boundary, and $\int_E f$ denotes the average of f over the set E . Ostensibly, identity (1) says nothing about derivatives and could be studied entirely within the category of continuous functions. It is the prototypical *statistical* characterization of solutions of a PDE, and it is natural to wonder if this is peculiar to Laplace's equation. In other words, can one characterize solutions of other PDEs in a statistical way that avoids any explicit mention of derivatives?

Recent work shows that such statistical characterizations exist, in a certain sense, for p -harmonic functions, i.e., solutions of the quasilinear PDE

$$(2) \quad -\Delta_p u := -\operatorname{div} (|Du|^{p-2} Du) = 0, \quad \text{for } 1 < p < \infty.$$

2010 AMS *Mathematics subject classification.* Primary 35J92, 35J70, 35D40.
Keywords and phrases. p -Laplacian, mean-value property, p -harmonic functions, viscosity solutions.

Received by the editors on February 2, 2010.

DOI:10.1216/RMJ-2011-41-2-493 Copyright ©2011 Rocky Mountain Mathematics Consortium

More precisely, p -harmonic functions are usually defined to be weak solutions of (2); thanks to work by Juutinen et al. [8], however, weak solutions of (2) are the same as viscosity solutions of (2). Viscosity techniques are particularly relevant to the present work, as Manfredi et al. [10] used such methods to prove that the continuous function u is p -harmonic in the domain $\Omega \subset \mathbf{R}^N$ if and only if the functional equation

$$(3) \quad u(x) = \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u \right\} + \beta \int_{B_\varepsilon(x)} u(y) dy + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0$$

holds in the viscosity sense for all $x \in \Omega$. The constants α and β are determined by the exponent p and the dimension N :

$$\alpha := \frac{p-2}{p+N} \quad \text{and} \quad \beta := \frac{2+N}{p+N}.$$

This characterization also holds for ∞ -harmonic functions, where the ∞ -Laplacian Δ_∞ has the formal definition

$$(4) \quad \Delta_\infty u := \frac{1}{|Du|^2} \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

for smooth u .

To establish their results, the authors of [10] combine several interesting facts. First, calculating formally yields

$$(5) \quad \Delta_p u = |Du|^{p-2} (\Delta u + (p-2)\Delta_\infty u),$$

an identity that plays a central role in both [8] and [10]. Using it, Juutinen et al. proved that u is a viscosity solution of (2) if and only if

$$-\Delta u - (p-2)\Delta_\infty u = 0$$

in the viscosity sense, about which more will be said below. Manfredi et al. then invoke the identities

$$(6) \quad u(x) - \int_{B_\varepsilon(x)} u(y) dy = -\frac{\varepsilon^2}{2(N+2)} \Delta u(x) + o(\varepsilon^2)$$

and

$$(7) \quad u(x) - \frac{1}{2} \left\{ \max_{y \in \overline{B_\varepsilon(x)}} u(y) + \min_{y \in \overline{B_\varepsilon(x)}} u(y) \right\} = -\frac{\varepsilon^2}{2} \Delta_\infty u(x) + o(\varepsilon^2),$$

valid for smooth functions as $\varepsilon \rightarrow 0$, to obtain their asymptotic characterization (3). Here and in what follows, a function is called smooth if it is C^2 .

The decomposition (5) can be written in various ways, a fact that we exploit to obtain new statistical characterizations of p -harmonic functions of two variables. Specifically, if we define the 1-Laplacian Δ_1 on smooth functions by

$$(8) \quad \Delta_1 u := |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right),$$

then the formal relationship

$$\Delta_1 = \Delta - \Delta_\infty$$

holds and immediately yields two alternatives to (5):

$$(9) \quad \Delta_p u = |Du|^{p-2} ((p-1)\Delta u + (2-p)\Delta_1 u),$$

and

$$(10) \quad \Delta_p u = |Du|^{p-2} (\Delta_1 u + (p-1)\Delta_\infty u).$$

Using these identities and the Taylor approximation

$$(11) \quad u(x) - \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{u(s)\} = -\frac{\varepsilon^2}{2} \Delta_1 u(x) + o(\varepsilon^2),$$

valid for smooth functions u of two variables as $\varepsilon \rightarrow 0$ when $|Du(x)| \neq 0$, we prove the following:

Theorem 1. *Suppose that $1 < p < \infty$ and $\Omega \subset \mathbf{R}^2$ is open, and let u be a continuous function on Ω . The following are equivalent:*

- (1) u is p -harmonic in Ω .

(2) At each $x \in \Omega$, the equation

$$(12) \quad u(x) = \left(\frac{2}{p} - 1\right) \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{u(s)\} \\ + \left(2 - \frac{2}{p}\right) \int_{\partial B_\varepsilon(x)} u(s) ds + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0$$

holds in the viscosity sense.

(3) At each $x \in \Omega$, the equation

$$(13) \quad u(x) = \frac{1}{p} \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{u(s)\} \\ + \left(\frac{p-1}{2p}\right) \left(\max_{y \in B_\varepsilon(x)} \{u(y)\} + \min_{y \in B_\varepsilon(x)} \{u(y)\} \right) \\ + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0$$

holds in the viscosity sense.

The median operator occurring here is defined as expected: if u is continuous on Ω , $x \in \Omega$ and $\overline{B_\varepsilon(x)} \subset \Omega$,

$$m = \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{u(s)\}$$

if and only if

$$|\{s \in \partial B_\varepsilon(x) : u(s) \geq m\}| = |\{s \in \partial B_\varepsilon(x) : u(s) \leq m\}|,$$

where $|E|$ is the 1-dimensional Hausdorff measure of the set E . We remark that if u is smooth and $|Du(x)| \neq 0$, then (12) and (13) hold in the usual non-viscosity sense if and only if $\Delta_p u(x) = 0$. This follows from Lemmas 1 and 2 below.

Considering (1), it is natural to ask if the formulas (12) and (13) hold in a non-asymptotic sense. More precisely, if u is p -harmonic in Ω , do the equations

$$(14) \quad u(x) = \left(\frac{2}{p} - 1\right) \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{u(s)\} + \left(2 - \frac{2}{p}\right) \int_{\partial B_\varepsilon(x)} u(s) ds$$

(15)

$$u(x) = \frac{1}{p} \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{u(s)\} + \left(\frac{p-1}{2p}\right) \left(\max_{y \in B_\varepsilon(x)} \{u(y)\} + \min_{y \in B_\varepsilon(x)} \{u(y)\} \right)$$

necessarily hold at all $x \in \Omega$ for all $\varepsilon > 0$ sufficiently small? The answer to this question is no, and in subsection 2.2 we provide an example demonstrating that these equations do not hold in general even for smooth p -harmonic functions.

On the way to proving Theorem 1 in subsection 2.2, we provide a simple analytic proof of identity (11). We should point out, however, that the relationship between median values and the 1-Laplacian has appeared before, either explicitly or implicitly. In [11], for example, Oberman uses a discrete median scheme of forward Euler type to approximate solutions of the parabolic mean curvature equation,

$$(16) \quad \frac{\partial u}{\partial t} - \Delta_1 u = 0 \quad \text{for } t > 0, \quad u(\cdot, 0) = u_0,$$

in two space dimensions. Unlike many other proposed algorithms for this equation, Oberman's median scheme is provably convergent, an easy consequence of the main theorem in [1].

Kohn and Serfaty [9] discuss a different convergent approximation scheme for the initial-value problem (16) that can be described geometrically as follows. Let $\Gamma(0)$ be a simple closed curve in the plane, let $\Gamma(t)$ be the curve obtained from $\Gamma(0)$ by letting it evolve by mean curvature for time t , and fix a small $\varepsilon > 0$. The curve $\Gamma(t + (\varepsilon^2/2))$ is approximately the locus of all centers of circles of radius ε with antipodal points on $\Gamma(t)$; one can approximate $\Gamma(t + (\varepsilon^2/2))$ by tracking the center of a segment of length 2ε as its endpoints traverse the curve $\Gamma(t)$. This is the basic idea behind our proof of (11), even though Kohn and Serfaty never mention medians in [9]. Related papers that use similar ideas without explicitly connecting the 1-Laplacian and median values include, but are certainly not limited to, [3, 13].

The present work is actually closely related to the work of Jackson and it is our pleasure to briefly discuss this connection. Over the past thirty or so years, viscosity solutions have become a standard tool in the study of nonlinear PDEs. However the contemporary viscosity approach is similar in some ways to the earlier abstract Perron method of Jackson and Jackson and Beckenbach as in [2, 6, 7]. In fact, for a class of second-order elliptic PDEs, viscosity subsolutions and the subfunctions of Beckenbach and Jackson are equivalent (see [5]). Furthermore, Jackson applied this abstract Perron method to obtain existence and uniqueness results for the minimal surface equation in two

independent variables [7]; this work is closely related to ongoing work on 1-harmonic functions [12], as the level sets of 1-harmonic functions are minimal surfaces (cf. [14]).

2. New results.

2.1. Definitions. Before proving Theorem 1, we review the necessary definitions and related results.

Definition 1. Suppose that $1 < p < \infty$, and let Ω be a domain in \mathbf{R}^2 .

(1) The lower semicontinuous function u is *p-superharmonic* in Ω in the viscosity sense if and only if the equivalent inequalities

$$(17) \quad (1-p)\Delta\varphi + (p-2)\Delta_1\varphi \geq 0 \quad \text{and} \quad -\Delta_1\varphi + (1-p)\Delta_\infty\varphi \geq 0$$

hold at $x \in \Omega$ for any smooth function φ such that $|D\varphi(x)| \neq 0$ and $u - \varphi$ has a strict minimum at x .

(2) The upper semicontinuous function u is *p-subharmonic* in Ω in the viscosity sense if and only if the equivalent inequalities

$$(18) \quad (1-p)\Delta\varphi + (p-2)\Delta_1\varphi \leq 0 \quad \text{and} \quad -\Delta_1\varphi + (1-p)\Delta_\infty\varphi \leq 0$$

hold at $x \in \Omega$ for any smooth function φ such that $|D\varphi(x)| \neq 0$ and $u - \varphi$ has a strict maximum at x .

(3) u is *p-harmonic* in Ω if it is both *p-superharmonic* and *p-subharmonic* in Ω .

The legitimacy of this definition follows from [8] and the formal identities (5), (9) and (10) above, as checking *p-harmonic* in the viscosity sense reduces to evaluating $-\Delta_p\varphi$ for smooth functions φ away from critical points. We refer to [8, 10] for more details.

Definition 2. Let $1 < p < \infty$, let Ω be a domain in \mathbf{R}^2 , and consider the equation

$$(19) \quad u(x) = \left(\frac{2}{p} - 1\right) \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{u(s)\} + \left(2 - \frac{2}{p}\right) \int_{\partial B_\varepsilon(x)} u(s) ds + o(\varepsilon^2)$$

as $\varepsilon \rightarrow 0$.

(1) u is a *supersolution* of (19) in the viscosity sense if and only if the inequality

$$(20) \quad \varphi(x) \geq \left(\frac{2}{p} - 1\right) \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{\varphi(s)\} + \left(2 - \frac{2}{p}\right) \int_{\partial B_\varepsilon(x)} \varphi(s) ds + o(\varepsilon^2)$$

as $\varepsilon \rightarrow 0$

holds at $x \in \Omega$ for any smooth function φ such that $|D\varphi(x)| \neq 0$ and $u - \varphi$ has a strict minimum at x .

(2) u is a *subsolution* of (19) in the viscosity sense if and only if the inequality

$$(21) \quad \varphi(x) \leq \left(\frac{2}{p} - 1\right) \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{\varphi(s)\} + \left(2 - \frac{2}{p}\right) \int_{\partial B_\varepsilon(x)} \varphi(s) ds$$

+ $o(\varepsilon^2)$ as $\varepsilon \rightarrow 0$

holds at $x \in \Omega$ for any smooth function φ such that $|D\varphi(x)| \neq 0$ and $u - \varphi$ has a strict maximum at x .

(3) u is a *solution* of (19) in the viscosity sense if and only if it is both a subsolution and a supersolution.

2.2. Proof of Theorem 1. We begin with asymptotic formulas valid for smooth functions that will be used to establish our main result. The following lemma can be established using Taylor expansion; we omit the routine proof.

Lemma 1. *Let Ω be a domain in \mathbf{R}^2 , let $x \in \Omega$, and let φ be a smooth function on Ω . Then*

$$(22) \quad \varphi(x) - \int_{\partial B_\varepsilon(x)} \varphi(s) ds = -\frac{\varepsilon^2}{4} \Delta \varphi(x) + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0.$$

Lemma 2. *Let Ω be a domain in \mathbf{R}^2 , let $x = (x_1, x_2) \in \Omega$, and let φ be a smooth function on Ω with $|D\varphi(x)| \neq 0$. Then*

$$(23) \quad \varphi(x) - \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{\varphi(s)\} = -\frac{\varepsilon^2}{2} \Delta_1 \varphi(x) + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. The Implicit Function Theorem guarantees that, for $\varepsilon > 0$ sufficiently small, the level sets of φ form a one-parameter family

of smooth, non-intersecting curves that foliate the closed ball $\overline{B_\varepsilon(x)}$. Consequently, the median of φ over $\partial B_\varepsilon(x)$,

$$M_\varepsilon := \operatorname{median}_{s \in \partial B_\varepsilon(x)} \{\varphi(s)\},$$

is the value corresponding to the level set that intersects $\partial B_\varepsilon(x)$ in antipodal points; for each $\varepsilon > 0$, there is a unique angle $\theta_\varepsilon \in [0, 2\pi)$ such that

$$(24) \quad M_\varepsilon = \varphi(x_1 + \varepsilon \cos \theta_\varepsilon, x_2 + \varepsilon \sin \theta_\varepsilon) = \varphi(x_1 - \varepsilon \cos \theta_\varepsilon, x_2 - \varepsilon \sin \theta_\varepsilon).$$

Let \mathbf{v}_ε denote the unit vector $(\cos \theta_\varepsilon, \sin \theta_\varepsilon)$, and define

$$D\varphi^\perp(x) := (-\varphi_2(x), \varphi_1(x)).$$

The derivatives of φ below are evaluated at x , which we omit for simplicity. Taylor expanding about x yields

$$(25) \quad M_\varepsilon = \varphi(x + \varepsilon \mathbf{v}_\varepsilon) = \varphi(x) + \varepsilon D\varphi \cdot \mathbf{v}_\varepsilon + \frac{\varepsilon^2}{2} \mathbf{v}_\varepsilon^\top D^2 \varphi \mathbf{v}_\varepsilon + o(\varepsilon^2)$$

and

$$(26) \quad M_\varepsilon = \varphi(x - \varepsilon \mathbf{v}_\varepsilon) = \varphi(x) - \varepsilon D\varphi \cdot \mathbf{v}_\varepsilon + \frac{\varepsilon^2}{2} \mathbf{v}_\varepsilon^\top D^2 \varphi \mathbf{v}_\varepsilon + o(\varepsilon^2).$$

Since these expressions both equal M_ε ,

$$\varepsilon D\varphi \cdot \mathbf{v}_\varepsilon = o(\varepsilon^2),$$

we therefore have

$$(27) \quad \mathbf{v}_\varepsilon = \frac{D\varphi^\perp}{|D\varphi|} + \mathbf{w}_\varepsilon,$$

where

$$\varepsilon D\varphi \cdot \mathbf{w}_\varepsilon = o(\varepsilon^2),$$

and we see (among other things) that the sequence $\{\mathbf{v}_\varepsilon\}$ of unit vectors converges:

$$\mathbf{v}_\varepsilon \longrightarrow \frac{D\varphi^\perp}{|D\varphi|} \quad \text{as } \varepsilon \downarrow 0.$$

Using the decomposition (27) in the right-hand side of either (25) or (26) yields (cf. [9])

$$(28) \quad \varphi(x) - M_\varepsilon = -\frac{\varepsilon^2}{2} \frac{(D\varphi^\perp)^\top}{|D\varphi|} D^2\varphi \frac{D\varphi^\perp}{|D\varphi|} + o(\varepsilon^2) = -\frac{\varepsilon^2}{2} \Delta_1\varphi + o(\varepsilon^2),$$

proving the lemma. \square

With these lemmas, Theorem 1 is easily established using the same approach as in [10]: apply the asymptotic formulas for smooth functions to the viscosity formulation.

Proof. Suppose that u is continuous in Ω and that φ is a smooth function for which $|D\varphi(x)| \neq 0$ and $u - \varphi$ has a strict minimum at $x \in \Omega$. Using Lemmas 1 and 2 and observing that $(2/p - 1) + (2 - 2/p) = 1$, it follows that the first inequality in (17) holds if and only if (20) holds. Thus u is p -superharmonic in the viscosity sense if and only if it is a viscosity supersolution of (12). The analogous argument establishes the equivalence of p -subharmonicity and being a subsolution of (12).

The equivalence of the first and third statements of the theorem is proved similarly, using identity (7) instead of Lemma 1. \square

2.3. Necessity of asymptotic nature of Theorem 1. In this section, we present an example to show that (14) and (15) do not hold for p -harmonic functions in general. In fact, these equations do not even necessarily hold for all $\varepsilon > 0$ sufficiently small, so that the asymptotic results appearing in Theorem 1 are, in general, the best available.

For any $1 < p < 2$, the function $u_p(x) = |x|^{(p-2)/(p-1)}$ is smooth and p -harmonic in $\mathbf{R}^2 \setminus \{0\}$ and is known as the fundamental solution of the p -Laplacian (see for example [8]). Let $x = (x_1, 0)$ where $x_1 > 0$ and let $0 < \varepsilon < x_1$. Because u_p is radial and radially decreasing, it is not hard to see that

$$(29) \quad \text{median}_{\partial B_\varepsilon(x)} u_p = (x_1^2 + \varepsilon^2)^{(p-2)/2(p-1)}.$$

The mean of u_p on $\partial B_\varepsilon(x)$ is

$$(30) \quad \frac{1}{2\pi} \int_0^{2\pi} (x_1^2 + 2x_1\varepsilon \cos \theta + \varepsilon^2)^{(p-2)/2(p-1)} d\theta.$$

Using (29) and (30), (14) at x with $u = u_p$ becomes

$$(31) \quad |x_1|^{(p-2)/(p-1)} \\ = \left(\frac{2}{p} - 1\right) (x_1^2 + \varepsilon^2)^{(p-2)/2(p-1)} \\ + \left(2 - \frac{2}{p}\right) \frac{1}{2\pi} \int_0^{2\pi} (x_1^2 + 2x_1\varepsilon \cos \theta + \varepsilon^2)^{(p-2)/2(p-1)} d\theta.$$

If (31) holds for all ε sufficiently small we can differentiate it with respect to ε to obtain

$$(32) \quad (2-p)(x_1^2 + \varepsilon^2)^{(p-2)/2(p-1)-1} \varepsilon \\ = \frac{2-2p}{2\pi} \int_0^{2\pi} (x_1^2 + 2x_1\varepsilon \cos \theta + \varepsilon^2)^{(p-2)/2(p-1)-1} (x_1 \cos \theta + \varepsilon) d\theta.$$

Now let $x_1 = 1$ and $p = 3/2$. The last equation is then

$$(33) \quad (1/2)(1 + \varepsilon^2)^{-3/2} \varepsilon = \frac{-1}{2\pi} \int_0^{2\pi} (1 + 2\varepsilon \cos \theta + \varepsilon^2)^{-3/2} (\cos \theta + \varepsilon) d\theta,$$

which holds if and only if

$$(34) \quad -\varepsilon = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{1 + 2\varepsilon \cos \theta + \varepsilon^2}{1 + \varepsilon^2}\right)^{-3/2} (\cos \theta + \varepsilon) d\theta \\ = \frac{1}{\pi} \int_0^{2\pi} \left(1 + \frac{2\varepsilon \cos \theta}{1 + \varepsilon^2}\right)^{-3/2} (\cos \theta + \varepsilon) d\theta.$$

Using the binomial formula:

$$(35) \quad \left(1 + \frac{2\varepsilon \cos \theta}{1 + \varepsilon^2}\right)^{-3/2} = 1 - \frac{3}{2} \left(\frac{2\varepsilon \cos \theta}{1 + \varepsilon^2}\right) + \frac{15}{8} \left(\frac{2\varepsilon \cos \theta}{1 + \varepsilon^2}\right)^2 - \frac{35}{16} \left(\frac{2\varepsilon \cos \theta}{1 + \varepsilon^2}\right)^3$$

plus higher order terms. Therefore the integrand in (34) is equal to

$$(36) \quad \cos \theta - \frac{3\varepsilon}{1 + \varepsilon^2} \cos^2 \theta + \frac{15}{2} \frac{\varepsilon^2 \cos^3 \theta}{(1 + \varepsilon^2)^2} - \frac{35}{2} \frac{\varepsilon^3 \cos^4 \theta}{(1 + \varepsilon^2)^3} \\ + \varepsilon - \frac{3\varepsilon^2 \cos \theta}{(1 + \varepsilon^2)} + \frac{15}{2} \frac{\varepsilon^3 \cos^2 \theta}{(1 + \varepsilon^2)^2}$$

plus terms of order 4 and higher. Using (36) in the integral in (34), noting that odd powers of $\cos \theta$ integrate to zero and recalling that $\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$ and $\int_0^{2\pi} \cos^4 \theta \, d\theta = (3/4)\pi$, we obtain

$$(37) \quad \frac{1}{\pi} \int_0^{2\pi} \left(\frac{1 + 2\varepsilon \cos \theta + \varepsilon^2}{1 + \varepsilon^2} \right)^{-3/2} (\cos \theta + \varepsilon) \, d\theta \approx -\varepsilon - (21/8)\varepsilon^3,$$

which is strictly less than $-\varepsilon$ if ε is sufficiently small so that (34) does not hold. As a result, (31) cannot hold for all ε sufficiently small.

The same example can be used to show that (15) also fails in general, even if ε is small. Again let $p = 3/2$ and $x = (1, 0)$, and let $0 < \varepsilon < 1$. The maximum value of u_p on $\overline{B_\varepsilon(x)}$ is $1/(1 - \varepsilon)$, and the minimum on the same ball is $1/(1 + \varepsilon)$. Using (29), in this case (15) becomes

$$(38) \quad 1 = \frac{2}{3} (1 + \varepsilon^2)^{-1/2} + \frac{1}{6} \left(\frac{1}{1 - \varepsilon} + \frac{1}{1 + \varepsilon} \right)$$

which one can easily see does not hold, even if $\varepsilon > 0$ is restricted to being smaller than some ε_0 .

3. Concluding remarks. The asymptotic characterizations of p -harmonic functions in [10] are valid in N dimensions. It would be interesting to extend the results presented here to higher dimensions. The only part of the proof of Theorem 1 that requires two dimensions is Lemma 2. If an N -dimensional version of Lemma 2, perhaps involving the median on an $(N - 1)$ -dimensional sphere, were established, new asymptotic statistical characterizations of p -harmonic functions would follow.

We presented an example showing that, in general, only asymptotic characterizations of this type are possible. However, this is not the case for $p = 2$. A natural question is: do the equations (14) and (15) hold either globally or locally for any other values of p ? Concrete examples in [12] show that the limiting cases of (14) and (15) can hold when $p = 1$, but more work on this question needs to be done.

Finally, we did not consider the extreme cases $p = 1$ and $p = \infty$, although we remark that if p is formally allowed to be ∞ in (13) the resulting characterization is the same as that in [10].

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