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RANDOMLY ORTHOGONAL FACTORIZATIONS OF BIPARTITE GRAPHS

SIZHONG ZHOU AND JIANCHENG WU

ABSTRACT. Let $G = (X, Y, E(G))$ be a bipartite graph with vertex set $V(G) = X \cup Y$ and edge set $E(G)$, and let g, f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. A (g, f)-factor of G is a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$; a (g, f)-factorization of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors. Let $F = \{F_1, F_2, \dots, F_m\}$ be a factorization of G, and let H be a subgraph of G with mr edges. If F_i, $1 \leq i \leq m$, has exactly r edges in common with H, we say that F is r-orthogonal to H. In this paper it is proved that every bipartite $(0, mf - m + 1)$-graph has $(0, f)$-factorizations randomly r-orthogonal to any given subgraph with mr edges if $f(x) \geq 3r - 2$ for any $x \in V(G)$.

1. Introduction. Orthogonal factorizations in graphs are very useful in combinatorial design, network design, circuit layout and so on [**2**]. Graphs considered in this paper will be finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x is denoted by $d_G(x)$. Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Then a (g, f)-factor of G is a spanning subgraph F of G with $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A subgraph H of G is called an m-subgraph if H has m edges in total. A (g, f)-factorization $F = \{F_1, F_2, \dots, F_m\}$ of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \dots, F_m$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, where a and b are two nonnegative integers, then a (g, f)-factorization of G is called an $[a, b]$-factorization

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of G. Let H be an mr-subgraph of a graph G. A (g, f)-factorization $F = \{F_1, F_2, \dots, F_m\}$ is r-orthogonal to H if $|E(H) \cap E(F_i)| = r$ for $1 \le i \le m$. If for any partition $\{A_1, A_2, \dots, A_m\}$ of $E(H)$ with $|A_i| = r$ there is a (g, f)-factorization $F = \{F_1, F_2, \dots, F_m\}$ of G such that $A_i \subseteq E(F_i)$, $1 \le i \le m$, then we say that G has (g, f)-factorizations randomly r-orthogonal to H. Other definitions and terminology can be found in [**4**].

A graph denoted by $G = (X, Y, E(G))$ is a bipartite graph with vertex bipartition (X, Y) and edge set $E(G)$. Alspach et al. [**2**] posed the following problem: given a subgraph H, does there exist a factorization F of G with some fixed type orthogonal to H? Li and Liu [**8**] gave a sufficient condition for a graph to have a (g, f)-factorization orthogonal to any given m-subgraph. Lam et al. [**6**] studied orthogonal factorizations of graphs. Anstee and Caccetta [**3**] discussed orthogonal matchings. Feng [**5**] proved that every $(0, mf - m + 1)$-graph has a $(0, f)$-factorization orthogonal to any given m-subgraph. Liu and Zhu [**9**] proved that every bipartite $(mg + m - 1, mf - m + 1)$-graph has randomly k-orthogonal (g, f)-factorizations. Now we consider the r-orthogonal factorizations of graphs. The purpose of this paper is to solve some problems on orthogonal factorizations for bipartite $(0, mf - m + 1)$-graphs. It is shown that a bipartite $(0, mf - m + 1)$-graph G has $(0, f)$-factorizations randomly r-orthogonal to any given mr-subgraph if $f(x) \ge 3r - 2$ for any $x \in V(G)$.

2. Preliminary results. Let G be a graph, and let S and T be two disjoint subsets of $V(G)$. We denote by $E_G(S, T)$ the set of edges with one end in S and the other in T, and by $e_G(S, T)$ the cardinality of $E_G(S, T)$. For $S \subset V(G)$ and $A \subset E(G)$, $G - S$ is the subgraph obtained from G by deleting the vertices in S together with the edges to which the vertices in S are incident, and $G - A$ is the subgraph obtained from G by deleting the edges in A, and $G[S]$ (respectively $G[A]$) is the subgraph of G induced by S (respectively A). For a subset X of $V(G)$, we write $f(X) = \sum_{x \in X} f(x)$ for any function f defined on $V(G)$, and define $f(\varnothing) = 0$. Specially, $d_G(X) = \sum_{x \in X} d_G(x)$.

Folkman and Fulkerson obtained the following necessary and sufficient condition for the existence of a (g, f)-factor in a bipartite graph, see [**1**, Theorem 6.8].

Lemma 2.1. *Let $G = (X, Y, E(G))$ be a bipartite graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if for all $S \subseteq X$ and $T \subseteq Y$,*

$$\gamma_{1G}(S, T, g, f) = f(S) - g(T) + d_{G-S}(T) \ge 0$$

and

$$\gamma_{2G}(S, T, g, f) = f(T) - g(S) + d_{G-T}(S) \ge 0.$$

Note that $d_{G-S}(T) = e_G(T, X \setminus S)$ and $d_{G-T}(S) = e_G(S, Y \setminus T)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$ and let $S \subseteq X$, $T \subseteq Y$. Set

$$E_{iS} = E_i \cap E_G(S, Y \setminus T), \qquad E_{iT} = E_i \cap E_G(T, X \setminus S) \text{ for } i = 1, 2,$$

and set

$$\alpha_S = |E_{1S}|, \quad \alpha_T = |E_{1T}|, \quad \beta_S = |E_{2S}|, \quad \beta_T = |E_{2T}|.$$

It is easily seen that $\alpha_S \le d_{G-T}(S)$, $\alpha_T \le d_{G-S}(T)$, $\beta_S \le d_{G-T}(S)$ and $\beta_T \le d_{G-S}(T)$.

Liu and Zhu [**9**] gave the following necessary and sufficient condition for a bipartite graph to admit a (g, f)-factor containing E_1 and excluding E_2.

Lemma 2.2. *Let $G = (X, Y, E(G))$ be a bipartite graph, let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$, and let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \varnothing$ if and only if for all $S \subseteq X$, $T \subseteq Y$,*

$$\gamma_{1G}(S, T, g, f) \ge \alpha_S + \beta_T$$

and

$$\gamma_{2G}(S, T, g, f) \ge \alpha_T + \beta_S.$$

In the following, we always assume that G is a bipartite $(0, mf-m+1)$-graph, where $m \geq 1$ is an integer. Define

$$\begin{aligned} g(x) &= \max\{0, d_G(x) - ((m-1)f(x) - (m-1) + 1)\}, \\ \triangle_1\ (x) &= \frac{1}{m} d_G(x) - g(x) \end{aligned}$$

and

$$\triangle_2\ (x) = f(x) - \frac{1}{m} d_G(x).$$

By the definitions of $g(x)$, $\triangle_1\ (x)$ and $\triangle_2\ (x)$, we have the following lemma.

Lemma 2.3. *For all $x \in V(G)$, the following inequalities hold*:

(1) *If $m \geq 2$, then $0 \leq g(x) < f(x)$.*

(2) *If $g(x) = d_G(x) - ((m-1)f(x) - (m-1) + 1)$, then $\triangle_1\ (x) \geq 1/m$.*

(3) *$\triangle_2\ (x) \geq (m-1)/m$.*

Proof. (1) Note that G is a bipartite $(0, mf-m+1)$-graph, where $m \geq 2$ is an integer. Then $0 \leq mf(x) - m + 1$ implies that $f(x) \geq (m-1)/m$. Note that $f(x)$ is nonnegative integer-valued function. Then $f(x) \geq 1$.

If $g(x) = 0$, then $0 \leq g(x) < f(x)$.

If $g(x) = d_G(x) - ((m-1)f(x) - (m-1) + 1)$, then

$$\begin{aligned} f(x) - g(x) &= f(x) - d_G(x) + (m-1)f(x) - (m-1) + 1 \\ &= mf(x) - m + 2 - d_G(x) \\ &\geq mf(x) - m + 2 - (mf(x) - m + 1) = 1. \end{aligned}$$

Hence, we get that

$$0 \leq g(x) < f(x).$$

(2) If $g(x) = d_G(x) - ((m-1)f(x) - (m-1) + 1)$, then

$$\begin{aligned}
\triangle_1 (x) &= \frac{1}{m} d_G(x) - g(x) \\
&= \frac{1}{m} d_G(x) - [d_G(x) - ((m-1)f(x) - (m-1) + 1)] \\
&= \frac{1-m}{m} d_G(x) + (m-1)f(x) - (m-1) + 1 \\
&\geq \frac{1-m}{m}(mf(x) - m + 1) \\
&\quad + (m-1)f(x) - (m-1) + 1 \\
&= (1-m)f(x) + (m-1) \\
&\quad - \frac{m-1}{m} + (m-1)f(x) - (m-1) + 1 \\
&= \frac{1}{m}.
\end{aligned}$$

(3) We have

$$\begin{aligned}
\triangle_2 (x) &= f(x) - \frac{1}{m} d_G(x) \\
&\geq f(x) - \frac{1}{m}(mf(x) - m + 1) \\
&= f(x) - f(x) + \frac{m-1}{m} = \frac{m-1}{m}.
\end{aligned}$$

This completes the proof. □

Lemma 2.4. *For any $S \subseteq X$ and $T \subseteq Y$, the following equalities hold*:

$$\gamma_{1G}(S, T, g, f) = \triangle_1 (T) + \triangle_2 (S) + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S)$$

and

$$\gamma_{2G}(S, T, g, f) = \triangle_1 (S) + \triangle_2 (T) + \frac{m-1}{m} d_{G-T}(S) + \frac{1}{m} d_{G-S}(T).$$

Proof. We prove only the first equality. The second can be verified similarly. According to the definition of γ_{1G}, we have

$$\begin{aligned}\gamma_{1G}(S,T,g,f) &= f(S) - g(T) + d_{G-S}(T)\\ &= d_G(T) - e_G(S,T) - g(T) + f(S)\\ &= \left(\frac{1}{m}d_G(T) - g(T)\right) + \left(f(S) - \frac{1}{m}d_G(S)\right)\\ &\quad + \frac{m-1}{m}d_{G-S}(T) + \frac{1}{m}d_{G-T}(S)\\ &= \triangle_1(T) + \triangle_2(S) + \frac{m-1}{m}d_{G-S}(T) + \frac{1}{m}d_{G-T}(S).\end{aligned}$$

This completes the proof. □

Let $S \subseteq X$ and $T \subseteq Y$, and

$$S_0 = \{x | x \in S, f(x) = 1\}, \qquad S_1 = S \setminus S_0,$$
$$T_0 = \{x | x \in T, g(x) = 0\}, \qquad T_1 = T \setminus T_0.$$

Hence, we get that

$$\begin{aligned} S &= S_0 \cup S_1, & S_0 \cap S_1 &= \varnothing,\\ T &= T_0 \cup T_1, & T_0 \cap T_1 &= \varnothing,\\ \alpha_S &= \alpha_{S_0} + \alpha_{S_1}, & \alpha_T &= \alpha_{T_0} + \alpha_{T_1},\\ \beta_T &= \beta_{T_0} + \beta_{T_1}, & \beta_S &= \beta_{S_0} + \beta_{S_1}.\end{aligned}$$

Lemma 2.5. *Let E_1 and E_2 be two disjoint subsets of $E(G)$.*

(1) *If*

$$\gamma_{1G}(S_1,T_1,g,f) = f(S_1) - g(T_1) + d_{G-S_1}(T_1) \geq \alpha_{S_1} + \beta_{T_1},$$

then

$$\gamma_{1G}(S,T,g,f) = f(S) - g(T) + d_{G-S}(T) \geq \alpha_S + \beta_T.$$

(2) *If*

$$\gamma_{2G}(S_1,T_1,g,f) = f(T_1) - g(S_1) + d_{G-T_1}(S_1) \geq \alpha_{T_1} + \beta_{S_1},$$

then

$$\gamma_{2G}(S,T,g,f) = f(T) - g(S) + d_{G-T}(S) \geq \alpha_T + \beta_S.$$

Proof. We prove only the first result. The second can be verified similarly.

Note that $d_{G-S}(T_0) - g(T_0) = d_{G-S}(T_0) \geq \alpha_{T_0}$, and $0 \leq d_G(x) \leq mf(x) - m + 1$, so that for all $x \in S_0$, $d_G(x) = 0$ or $d_G(x) = 1$. This implies

$$|S_0| \geq d_G(S_0) = d_{G-T}(S_0) + e_G(S_0, T) \geq \alpha_{S_0} + e_G(S_0, T_1).$$

If $\gamma_{1G}(S_1, T_1, g, f) \geq \alpha_{S_1} + \beta_{T_1}$, then

$$\begin{aligned}
\gamma_{1G}(S,T,g,f) &= f(S) + d_{G-S}(T) - g(T) \\
&= f(S_1) + |S_0| + d_{G-S}(T_1) + d_{G-S}(T_0) - g(T_1) \\
&\geq f(S_1) + \alpha_{S_0} + e_G(S_0, T_1) + d_{G-S}(T_1) + \beta_{T_0} - g(T_1) \\
&\geq f(S_1) + \alpha_{S_0} + d_{G-S_1}(T_1) + \beta_{T_0} - g(T_1) \\
&= \gamma_{1G}(S_1, T_1, g, f) + \alpha_{S_0} + \beta_{T_0} \\
&\geq \alpha_{S_1} + \beta_{T_1} + \alpha_{S_0} + \beta_{T_0} \\
&= \alpha_S + \beta_T.
\end{aligned}$$

This completes the proof. □

Lemma 2.6 [**5**]. *Let G be a $(0, mf - m + 1)$-graph. Let f be an integer-valued function defined on $V(G)$ such that $f(x) \geq 0$, and let H be an m-subgraph of G. Then G has a $(0, f)$-factorization orthogonal to H.*

3. Main result and proof. In this section, we are going to state our main theorem and present a proof of it.

Let G be a bipartite graph, let H be an mr-subgraph of G, and let E_1 be an arbitrary subset of $E(H)$ with $|E_1| = r$. Put $E_2 = E(H) \setminus E_1$. Then $|E_2| = (m-1)r$. For any two subsets $S \subseteq X$ and $T \subseteq Y$, let E_{iS}, E_{iT} for $i = 1, 2$, α_S, α_T, β_S and β_T be defined as in Section 2. It follows instantly from the definitions that

$$\alpha_S \leq r, \quad \alpha_T \leq r, \quad \beta_S \leq (m-1)r \text{ and } \beta_T \leq (m-1)r.$$

Define $g(x)$ as before. The proof of theorem relies heavily on the following lemma.

Lemma 3.1. *Let $G = (X, Y, E(G))$ be a bipartite $(0, mf - m + 1)$-graph with $m \geq 2$ and $f(x) \geq 3r - 2$ with $r \geq 2$. Then G admits a (g, f)-factor F_1 such that $E_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \varnothing$.*

Proof. By Lemma 2.2, it suffices to show that, for any two subsets $S \subseteq X$ and $T \subseteq Y$, we have

$$\gamma_{1G}(S, T, g, f) \geq \alpha_S + \beta_T$$

and

$$\gamma_{2G}(S, T, g, f) \geq \alpha_T + \beta_S.$$

Alternately, by Lemma 2.5, it suffices to show that for S_1 and T_1 (define S_1 and T_1 as before), we have

$$\gamma_{1G}(S_1, T_1, g, f) \geq \alpha_{S_1} + \beta_{T_1}$$

and

$$\gamma_{2G}(S_1, T_1, g, f) \geq \alpha_{T_1} + \beta_{S_1}.$$

We prove only the first inequality. The second can be justified similarly. Now let us distinguish among four cases.

Case 1. If $S_1 = \varnothing$, $T_1 = \varnothing$, then $\alpha_{S_1} = 0$ and $\beta_{T_1} = 0$.

According to Lemma 2.4, we obtain

$$\begin{aligned}\gamma_{1G}(S_1, T_1, g, f) = &\triangle_1 (T_1) + \triangle_2 (S_1) \\ &+ \frac{m-1}{m} d_{G-S_1}(T_1) + \frac{1}{m} d_{G-T_1}(S_1) = 0 = \alpha_{S_1} + \beta_{T_1}.\end{aligned}$$

Case 2. If $S_1 = \varnothing$, $T_1 \neq \varnothing$, then $\alpha_{S_1} = 0$.

In view of the definition of T_1, it is easy to see that $g(x) \geq 1$ for all $x \in T_1$. Note that $g(x) = \max\{0, d_G(x) - ((m-1)f(x) - (m-1) + 1)\}$. For all $x \in T_1$, we have

$$g(x) = d_G(x) - ((m-1)f(x) - (m-1) + 1) \geq 1.$$

Thus, we get

$$\begin{aligned} d_G(x) &\geq (m-1)f(x) - (m-1) + 2 \\ &\geq (m-1)(3r-2) - m + 3 = 3mr - 3r - 3m + 5 \end{aligned} \tag{1}$$

for all $x \in T_1$.

From Lemma 2.3, Lemma 2.4 and (1), we get that

$$\begin{aligned} \gamma_{1G}(S_1, T_1, g, f) &\geq \frac{m-1}{m} d_G(T_1) \geq \frac{m-1}{m} d_G(x), \quad x \in T_1 \\ &\geq \frac{m-1}{m}(3mr - 3m - 3r + 5) \\ &= (m-1)r + \frac{m-1}{m}(2mr - 3m - 3r + 5) \\ &= (m-1)r + \frac{m-1}{m}((2m-3)r - 3m + 5) \\ &\geq (m-1)r + \frac{m-1}{m}(4m - 6 - 3m + 5) \\ &= (m-1)r + \frac{(m-1)^2}{m} \\ &\geq (m-1)r \geq \beta_{T_1} = \alpha_{S_1} + \beta_{T_1}. \end{aligned}$$

Case 3. If $S_1 \neq \varnothing$, $T_1 = \varnothing$, then $\beta_{T_1} = 0$. Thus, we have

$$\begin{aligned} \gamma_{1G}(S_1, T_1, g, f) &= d_{G-S_1}(T_1) - g(T_1) + f(S_1) \\ &= f(S_1) \geq (3r-2)|S_1| \\ &\geq 3r - 2 \geq r \geq \alpha_{S_1} = \alpha_{S_1} + \beta_{T_1}. \end{aligned}$$

Case 4. $S_1 \neq \varnothing$, $T_1 \neq \varnothing$. Note that $d_{G-T_1}(S_1) \geq \alpha_{S_1}$. By Lemma 2.3, Lemma 2.4 and (1), we get that

$$\begin{aligned}
\gamma_{1G}(S_1,T_1,g,f) &\geq \triangle_1\ (T_1)+\triangle_2\ (S_1)+\frac{m-1}{m}d_{G-S_1}(T_1)+\frac{1}{m}d_{G-T_1}(S_1)\\
&\geq \frac{1}{m}|T_1|+\frac{m-1}{m}|S_1|+\frac{m-1}{m}d_{G-S_1}(T_1)\\
&\quad+\frac{1}{m}d_{G-T_1}(S_1)\\
&=\frac{1}{m}|T_1|+\frac{m-1}{m}(d_{G-S_1}(T_1)+|S_1|)+\frac{1}{m}d_{G-T_1}(S_1)\\
&\geq \frac{1}{m}|T_1|+\frac{1}{m}d_{G-T_1}(S_1)+\frac{m-1}{m}d_G(x),\quad x\in T_1\\
&\geq \frac{m-1}{m}(3mr-3r-3m+5)+\frac{1}{m}d_{G-T_1}(S_1)+\frac{1}{m}\\
&=(m-1)r+\frac{(m-1)r}{m}+\frac{1}{m}d_{G-T_1}(S_1)\\
&\quad+\frac{(m-1)(2mr-3m-4r+5)+1}{m}\\
&\geq \beta_{T_1}+\frac{m-1}{m}\alpha_{S_1}+\frac{1}{m}\alpha_{S_1}\\
&\quad+\frac{(m-1)(2(2m-4)-3m+5)+1}{m}\\
&=\alpha_{S_1}+\beta_{T_1}+\frac{(m-1)(m-3)+1}{m}\\
&\geq \alpha_{S_1}+\beta_{T_1}\ (\text{since } m\geq 2 \text{ is an integer}).
\end{aligned}$$

For S_1 and T_1, we always have

$$\gamma_{1G}(S_1,T_1,g,f)\geq \alpha_{S_1}+\beta_{T_1},$$

and this completes the proof. □

Now we are ready to prove the theorem.

Theorem 1. *Let G be a bipartite $(0, mf-m+1)$-graph, let f be an integer-valued function defined on $V(G)$ such that $f(x)\geq 3r-2$ for all $x\in V(G)$, and let H be an mr-subgraph of G. Then G has $(0,f)$-factorizations randomly r-orthogonal to H.*

Proof. According to Lemma 2.6, the theorem is trivial for $r = 1$. In the following, we consider $r \geq 2$. Let $\{A_1, A_2, \ldots, A_m\}$ be any partition of $E(H)$ with $|A_i| = r$, $1 \leq i \leq m$. We prove that there is a $(0, f)$-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of G such that $A_i \subseteq E(F_i)$ for all $1 \leq i \leq m$. We apply induction on m. The assertion is trivial for $m = 1$. Supposing the statement holds for $m - 1$, let us proceed to the induction step.

Let $E_2 = E(H) \setminus A_1$. By Lemma 3.1, G has a (g, f)-factor F_1 such that $A_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \varnothing$. Clearly, F_1 is also a $(0, f)$-factor of G. Set $G' = G - E(F_1)$. It follows from the definition of $g(x)$ that

$$\begin{aligned} 0 &\leq d_{G'}(x) = d_G(x) - d_{F_1}(x) \leq d_G(x) - g(x) \\ &\leq d_G(x) - [d_G(x) - ((m-1)f(x) - (m-1) + 1)] \\ &= (m-1)f(x) - (m-1) + 1. \end{aligned}$$

Hence, G' is a bipartite $(0, (m-1)f - (m-1) + 1)$-graph. Let $H' = G[E_2]$. Then the induction hypothesis guarantees the existence of a $(0, f)$-factorization $F' = \{F_2, \ldots, F_m\}$ in G' which satisfies $A_i \subseteq E(F_i)$, $2 \leq i \leq m$. Hence, G has $(0, f)$-factorizations which are randomly r-orthogonal to H. This completes the proof. □

Remark 3.1. Obviously, the lower bound 0 in Theorem 1 is sharp in any sense. The upper bound $mf - m + 1$ is necessary in the proof of Lemma 3.1. In this sense, the result of Theorem 1 is best possible. In the proof of Theorem 1, it is required that $f(x) \geq 3r - 2$ for all $x \in V(G)$. We do not know whether the condition $f(x) \geq 3r - 2$ can be improved.

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School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, P.R. China
Email address: zsz_cumt@163.com

School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, P.R. China
Email address: wjch78@sina.com