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SHARP OSCILLATION CRITERIA FOR A CLASS OF FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with the oscillatory and the nonoscillatory behavior of the fourth-order nonlinear differential equation

$$(\Delta) \qquad \big(p(t)|x''|^{\alpha-1}\,x''\big)'' + q(t)|x|^{\beta-1}x = 0,$$

where $\alpha > 0$, $\beta > 0$ are constants and $p, q : [a,\infty) \to (0,\infty)$ are continuous functions. We will establish necessary and sufficient condition for oscillation of all solutions of sub-half-linear equation (Δ) (for $\beta < \alpha$) as well as of super-half-linear equation (Δ) (for $\beta > \alpha$).

1. Introduction. We consider the fourth-order quasilinear differential equation

$$(\Delta) \qquad \big(p(t)|x''|^{\alpha-1}\,x''\big)'' + q(t)|x|^{\beta-1}x = 0,$$

where $\alpha > 0$, $\beta > 0$ are constants and $p, q : [a,\infty) \to (0,\infty)$ are continuous functions. If we use the notation

$$\varphi_\gamma(\xi) = |\xi|^{\gamma-1}\,\xi, \quad \xi \in \mathbf{R},\ \gamma > 0,$$

the equation (Δ) can be expressed in the form

$$\big(p(t)\varphi_\alpha(x'')\big)'' + q(t)\varphi_\beta(x) = 0.$$

The equation (Δ) is called *super-half-linear* if $\beta > \alpha$ and *sub-half-linear* if $\beta < \alpha$.

Keywords and phrases. Fourth order nonlinear differential equation, positive solution, oscillation, sub-half-linear, super-half-linear.

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By a solution of (Δ) we mean a function $x : [T_x, \infty) \to \mathbf{R}$, which has the property that $p(t)\varphi_\alpha(x''(t))$ is twice continuously differentiable and satisfies the equation (Δ) for every $t \in [T_x, \infty)$. Those solutions of (Δ) which vanish identically in some neighborhood of infinity will be excluded from our consideration. A solution of (Δ) is called *oscillatory* if it has an infinite sequence of zeros tending to infinity; otherwise it is called *nonoscillatory*.

The study of oscillation theory of fourth-order nonlinear differential equations was initiated by Kusano and Naito [**4, 5**]. They completely characterized the oscillatory behavior of solutions of the equation (Δ) with $\alpha = 1$. The general case of (Δ) with $\alpha > 0$ has been considered by Kamo and Usami [**3**], Naito and Wu [**7, 8**] and Wu [**9**]. These authors have made efforts at obtaining necessary and sufficient conditions for oscillation of all solutions of (Δ) under suitable combinations of the convergence or divergence conditions of the integrals

$$\int_a^\infty \frac{t}{(p(t))^{1/\alpha}}\,dt, \qquad \int_a^\infty \left(\frac{t}{p(t)}\right)^{1/\alpha} dt.$$

The object of this paper is to consider the equation (Δ) under the following condition on $p(t)$

$$\text{(P)} \qquad \int_a^\infty t\left(\frac{t}{p(t)}\right)^{1/\alpha} dt < \infty.$$

The equation (Δ) with $p(t)$ satisfying (P) has recently been investigated from a viewpoint of nonoscillation by Kusano and Tanigawa in [**6**]. They have made a detailed analysis of the structure of nonoscillatory (or equivalently, positive) solutions of (Δ). However, the problem of providing necessary and sufficient conditions for the oscillation of all solutions of (Δ), under condition (P), still remains open. Our task is to proceed further exactly in this direction and to establish sharp criteria for oscillation of all solutions of (Δ).

Since the oscillation of all solutions of (Δ) is equivalent to the absence of nonoscillatory solutions, in order to establish sharp oscillation criteria for (Δ) it is sufficient to derive sharp conditions under which all possible nonoscillatory solutions of (Δ) are eliminated. This is exactly the procedure that we are going to follow in the present paper.

Naturally, extensive use is made of the results obtained in [**6**] about the asymptotic behavior of nonoscillatory solutions of (Δ). Therefore, the present paper, together with the paper [**6**], provides the complete characterization of the oscillatory solutions as well as of the nonoscillatory solutions of (Δ) under assumption (P).

We emphasize that the structure of the set of nonoscillatory solution of the fourth-order equation (Δ) is quite similar to the set of nonoscillatory solutions of the second order quasi-linear differential equation

$$\big(p(t)|x'|^{\alpha-1}\,x'\big)' + q(t)|x|^{\beta-1}x = 0, \quad \alpha,\ \beta > 0,$$

which includes the Emden-Fowler differential equation

$$\text{(EF)} \qquad x''(t) + q(t)|x(t)|^{\beta-1}x(t) = 0, \quad \beta > 0,$$

as a special case. The analysis of (EF) started in connection with astrophysical studies around the turn of the 20th century, and since then (EF) has been the object of intensive investigations. The equation (EF) is said to be *sublinear* or *superlinear* according as $0 < \beta < 1$ or $\beta > 1$. We emphasize two fundamental oscillation theorems for equation (EF) due to Atkinson [**1**] and Belohorec [**2**], which characterize the oscillatory solutions of the superlinear and the sublinear cases of (EF).

Theorem A [**1**]. *Let $q : [a,\infty) \to (0,\infty)$ be continuous. Then, all solutions of equation* (EF) *with $\beta > 1$ are oscillatory if and only if*

$$\int^{\infty} t\,q(t)\,dt = \infty.$$

Theorem B [**2**]. *Let $q : [a,\infty) \to (0,\infty)$ be continuous. Then, all solutions of equation* (EF) *with $0 < \beta < 1$ are oscillatory if and only if*

$$\int^{\infty} t^{\beta}\,q(t)\,dt = \infty.$$

Incidentally, it turns out that these theorems will play an important role in some parts of the proof of our main results: Theorems 3.1 and 3.2.

The paper is organized as follows. In Section 2 we begin with the classification of all nonoscillatory (positive) solutions of (Δ) according to their asymptotic behavior as $t \to \infty$. As a result, the set of nonoscillatory solutions of (Δ) is divided into six subclasses and for each subclass an integral criterion is formulated for the existence of solution belonging to that particular subclass. These facts have already been proved in paper [**6**], but their explicit statements are needed for later purposes. The main results are stated in Section 3 and proved in Section 5. The preparatory Section 4 is devoted to a collection of some lemmas which are crucial in proofs of our main results. The example illustrating the oscillatory and nonoscillatory behavior of solutions of (Δ) will be presented in Section 6.

2. Classification of positive solutions. We begin by classifying the nonoscillatory solutions of (Δ) according to their asymptotic behavior as $t \to \infty$. If $x(t)$ satisfies (Δ), then so does $-x(t)$, and so there is no loss of generality in restricting our attention to the set of positive solutions. In this section we state and list some of the basic results regarding the classification of positive solutions of (Δ). These results have already been obtained in [**6**] and they will be essential for the proof of our main theorems.

Let $x(t)$ be a positive solution of the equation (Δ). Since, from (Δ), $(p(t)|x''(t)|^{\alpha-1}x''(t))'$ is eventually monotone, it follows that all the functions $(p(t)|x''(t)|^{\alpha-1}x''(t))'$, $x''(t)$ and $x'(t)$ are eventually monotone and one-signed. Hence, the next eight cases can be considered:

	$(p(t)\varphi_\alpha(x''))'$	x''	x'		$(p(t)\varphi_\alpha(x''))'$	x''	x'
(a)	$+$	$+$	$+$	(e)	$-$	$+$	$+$
(b)	$+$	$+$	$-$	(f)	$-$	$+$	$-$
(c)	$+$	$-$	$+$	(g)	$-$	$-$	$+$
(d)	$+$	$-$	$-$	(h)	$-$	$-$	$-$

If $x'(t) < 0$ and $x''(t) < 0$ eventually, then $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the positivity of solution $x(t)$. Therefore, cases (d) and (h) never hold. Similarly, since $(p(t)\varphi_\alpha(x''(t)))'' < 0$, if $(p(t)\varphi_\alpha(x''(t)))' < 0$, then $\lim_{t\to\infty} p(t)\varphi_\alpha(x''(t)) = -\infty$, that is, $x''(t) < 0$ for large t. This observation rules out cases (e) and (f).

Accordingly, the following four types of combination of the signs of $x'(t)$, $x''(t)$ and $(p(t)|x''|^{\alpha-1}x'')'$ are possible for an eventually positive solution $x(t)$ of the equation (Δ):

$$\begin{array}{llll}
\text{(I)} & x'(t) > 0, & x''(t) > 0, & \big(p(t)|x''|^{\alpha-1}x''\big)' > 0,\\
\text{(II)} & x'(t) < 0, & x''(t) > 0, & \big(p(t)|x''|^{\alpha-1}x''\big)' > 0,\\
\text{(III)} & x'(t) > 0, & x''(t) < 0, & \big(p(t)|x''|^{\alpha-1}x''\big)' > 0,\\
\text{(IV)} & x'(t) > 0, & x''(t) < 0, & \big(p(t)|x''|^{\alpha-1}x''\big)' < 0.
\end{array}$$

In determining the asymptotic behavior of positive solutions of (Δ), a crucial role is played by the functions:

$$\begin{array}{ll}
\psi_1(t) = \displaystyle\int_t^\infty \frac{s-t}{p^{1/\alpha}(s)}\,ds, & \psi_2(t) = \displaystyle\int_t^\infty \frac{(s-t)s^{1/\alpha}}{p^{1/\alpha}(s)}\,ds,\\
\psi_3(t) = 1, & \psi_4(t) = t,
\end{array}$$

which are in fact the particular solutions of the unperturbed differential equation $(p(t)|x''|^{\alpha-1}x'')'' = 0$. As a result of further analysis of the four types (I)–(IV) of solutions mentioned above, Kusano and Tanigawa in [**6**] have shown that the following six types are possible for the asymptotic behavior of positive solutions of (Δ):

$$\begin{array}{llll}
\text{(A)} & x(t) \sim c_1\psi_1(t),\ t\to\infty, & \text{(B)} & x(t) \sim c_2\psi_2(t),\ t\to\infty,\\
\text{(C)} & x(t) \sim c_3\psi_3(t),\ t\to\infty, & \text{(D)} & x(t) \sim c_4\psi_4(t),\ t\to\infty,\\
\text{(E)} & \psi_1(t) \prec x(t) \prec \psi_2(t),\ t\to\infty, & \text{(F)} & \psi_3(t) \prec x(t) \prec \psi_4(t),\ t\to\infty,
\end{array}$$

where $c_i > 0$, $i = 1,2,3,4$, are constants and the symbol $f(t) \sim g(t)$, $t \to \infty$ is used to mean the asymptotic equivalence:

$$f(t) \sim g(t), \quad t\to\infty \iff \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1,$$

while the symbol $f(t) \prec g(t)$, $t \to \infty$ is used to express the relation

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty.$$

The interrelation between the types (I)–(IV) of the derivatives of solutions and the types (A)–(F) of the asymptotic behavior of solutions is as follows:

▹ All solutions of type (I) have the asymptotic behavior of type (A);

▹ A solution of type (II) has the asymptotic behavior of one of the types (A), (B), (C) and (E);

▹ A solution of type (III) has the asymptotic behavior of one of the types (C) and (D);

▹ A solution of type (IV) has the asymptotic behavior of one of the types (C), (D) and (F).

The following four theorems proven in [**6**] characterize the existence of a solution of (Δ) having the asymptotic behavior of types (A), (B), (C) or (D).

Theorem 2.1. *The equation* (Δ) *has a positive solution* $x(t)$ *of type* (A) *if and only if*

$$(C_1) \qquad \int_a^\infty t\,\psi_1^\beta(t)\,q(t)\,dt < \infty.$$

Theorem 2.2. *The equation* (Δ) *has a positive solution* $x(t)$ *of type* (B) *if and only if*

$$(C_2) \qquad \int_a^\infty \psi_2^\beta(t)\,q(t)\,dt < \infty.$$

Theorem 2.3. *The equation* (Δ) *has a positive solution* $x(t)$ *of type* (C) *if and only if*

$$(C_3) \qquad \int_a^\infty \frac{t}{(p(t))^{1/\alpha}}\left(\int_a^t (t-s)q(s)\,ds\right)^{1/\alpha} dt < \infty.$$

Theorem 2.4. *The equation* (Δ) *has a positive solution* $x(t)$ *of type* (D) *if and only if*

$$(C_4) \qquad \int_a^\infty \frac{1}{(p(t))^{1/\alpha}}\left(\int_a^t (t-s)\,s^\beta\,q(s)\,ds\right)^{1/\alpha} dt < \infty.$$

Unlike the solutions of types (A)–(D), a solution of type (E) (or type (F)) does not have the definite asymptotic behavior as $t \to \infty$. These solutions decay faster than $\psi_2(t)$ but slower than $\psi_1(t)$ (or it grows faster than $\psi_3(t)$ but slower than $\psi_4(t)$). Thus, a solution of type (E) (or type (F)) may well be called an "intermediate" solution between $\psi_1(t)$ and $\psi_2(t)$ (or between $\psi_3(t)$ and $\psi_4(t)$). This "intermediate" nature prevents us from characterizing the existence of solutions of types (E) and (F). Namely, in [**6**] necessary or sufficient conditions have been established for the existence of such solutions. This fact will raise difficulties that should be overcome in the process of proving our main results.

3. Main results. We now state the main results of this paper which provide sharp oscillation criteria for the sub-half-linear and the super-half-linear equation (Δ).

Theorem 3.1 (Super-half-linear case). *Let $\alpha \le 1 < \beta$. All solutions of the equation (Δ) are oscillatory if and only if*

$$(C_5) \qquad \int_a^\infty t\,\psi_1^\beta(t)\,q(t)\,dt = \infty.$$

Theorem 3.2 (Sub-half-linear case). *Let $\beta < 1 \le \alpha$. All solutions of equation (Δ) are oscillatory if and only if*

$$(C_6) \qquad \int_a^\infty \frac{1}{(p(t))^{1/\alpha}} \left(\int_a^t (t-s)s^\beta\,q(s)\,ds \right)^{1/\alpha} dt = \infty.$$

4. Auxiliary lemmas. The proofs of the main theorems mentioned above are presented in Section 5. In order to make the arguments thereof clear-cut we collect here four necessary calculus lemmas, the last two of which are concerned with the "intermediate" solution of types (E) and (F).

Lemma 4.1. *Let $\alpha \le 1 < \beta$. Then*
(a) *the condition (C_5) implies that*

$$\int_a^\infty \psi_2^\beta(t)\, q(t)\, dt = \infty; \tag{C_7}$$

(b) *the condition (C_3) implies condition (C_1);*
(c) *the condition (C_4) implies condition (C_1).*

Proof. $((C_5) \Rightarrow (C_7))$. This is obvious, since for $\beta > \alpha$, we have

$$\psi_2(t) \ge t^{1/\alpha}\, \psi_1(t),$$
$$\text{i.e. } \psi_2^\beta(t) \ge t^{\beta/\alpha}\, \psi_1^\beta(t) \ge t\, \psi_1^\beta(t), \quad t \ge t_0 \ge \max\{1, a\}.$$

$((C_3) \Rightarrow (C_1))$. Let (C_3) hold and define

$$I(t) = \int_a^t (t-s) q(s)\, ds, \quad \chi(t) = \int_t^\infty \frac{(s-t)\, s}{(p(s))^{1/\alpha}}\, ds, \quad t \ge a.$$

Since, $1/\alpha \ge 1$, we have $t^{1/\alpha} \ge t$ and

$$\psi_2(t) = \int_t^\infty \frac{(s-t) s^{1/\alpha}}{(p(s))^{1/\alpha}}\, ds \ge \int_t^\infty \frac{(s-t)\, s}{(p(s))^{1/\alpha}}\, ds = \chi(t) \ge t\, \psi_1(t), \tag{4.1}$$

for $t \ge t_0 \ge 1$, so that $\chi(t)$ is well-defined.

Without loss of generality, we may suppose that $\lim_{t\to\infty} I(t) = \infty$, so that we can choose some $t_0 > \max\{1, a\}$, such that

$$I(t) \ge 1 \text{ and } \psi_1(t) \le 1 \text{ for } t \ge t_0. \tag{4.2}$$

Now, using (4.1), (4.2) and the fact that $\beta > 1 \geq \alpha$, we have

$$\begin{aligned}\infty &> \int_{t_0}^{\infty} \frac{t}{(p(t))^{1/\alpha}} \left(\int_a^t (t-s) q(s)\, ds \right)^{1/\alpha} dt \\ &\geq \int_{t_0}^{\infty} \frac{t}{(p(t))^{1/\alpha}} \left(\int_a^t (t-s) q(s)\, ds \right) dt \\ &\geq \int_{t_0}^{\infty} \left(\int_s^{\infty} \frac{(t-s)\, t}{(p(t))^{1/\alpha}}\, dt \right) q(s)\, ds \\ &= \int_{t_0}^{\infty} \chi(s)\, q(s)\, ds \geq \int_{t_0}^{\infty} s\, \psi_1(s)\, q(s)\, ds \\ &\geq \int_{t_0}^{\infty} s\, \psi_1^{\beta}(s)\, q(s)\, ds, \quad t \geq t_0,\end{aligned}$$

which shows that (C_3) implies (C_1).

$((C_4) \Rightarrow (C_1))$. Suppose that (C_4) is satisfied and define

$$J(t) = \int_a^t (t-s) s^{\beta}\, q(s)\, ds, \quad t \geq a. \tag{4.3}$$

We may suppose that $\lim_{t\to\infty} J(t) = \infty$. Let $t_0 > \max\{1, a\}$ be such that

$$J(t) \geq 1 \text{ and } \psi_1(t) \leq 1, \quad t \geq t_0.$$

Then, for all $t \geq t_0$, we have

$$\begin{aligned}\infty &> \int_{t_0}^{\infty} \frac{1}{(p(t))^{1/\alpha}} \left(\int_a^t (t-s) s^{\beta}\, q(s)\, ds \right)^{1/\alpha} dt \\ &\geq \int_{t_0}^{\infty} \frac{1}{(p(t))^{1/\alpha}} \left(\int_a^t (t-s) s^{\beta}\, q(s)\, ds \right) dt \\ &\geq \int_{t_0}^{\infty} \left(\int_s^{\infty} \frac{t-s}{(p(t))^{1/\alpha}}\, dt \right) s^{\beta}\, q(s)\, ds \\ &= \int_{t_0}^{\infty} s^{\beta}\, \psi_1(s)\, q(s)\, ds \geq \int_{t_0}^{\infty} s\, \psi_1^{\beta}(s)\, q(s)\, ds,\end{aligned}$$

which shows that (C_1) is satisfied. □

Remark 1. Note that in the proof of part (a) of Lemma 4.1, we just need that $\alpha < \beta$.

Remark 2. We have used in the proof of Lemma 4.1 (parts (b) and (c)), the statement that without loss of generality we may suppose that $\lim_{t\to\infty} I(t) = \infty$ or $\lim_{t\to\infty} J(t) = \infty$. We would like to give here a brief explanation for this statement, since it will be further used in the proofs, from time to time. Observe, that the divergence of $I(t)$ as $t \to \infty$ was assumed to have the inequalities $(I(t))^{1/\alpha} \geq I(t)$ and $(J(t))^{1/\alpha} \geq J(t)$ holding for $\alpha \leq 1$ and for all large t. It may happen that $I(t)$ tends to a positive constant as $t \to \infty$. In this case, there exist positive constants k_1, k_2, such that $k_1 \leq I(t) \leq k_2$ for large t. Accordingly, we have

$$\big(I(t)\big)^{(1/\alpha)-1} \geq \frac{k_1^{1/\alpha}}{k_2} = K, \text{ i.e. } \big(I(t)\big)^{1/\alpha} \geq K\, I(t),$$

for large t, which is a sufficient argument to verify that statement (b) holds. The same remark could be applied to $J(t)$ as well.

Lemma 4.2. *Let $\beta < 1 \leq \alpha$. Then,*

(a) *the condition (C_6) implies condition (C_5);*

(b) *the condition (C_6) implies condition (C_7);*

(c) *the condition (C_6) implies that*

$$(C_8) \qquad \int_a^\infty \frac{t}{(p(t))^{1/\alpha}} \left(\int_a^t (t-s) q(s)\, ds \right)^{1/\alpha} dt = \infty;$$

(d) *the condition (C_6) implies that*

$$(C_9) \qquad \int_a^\infty t^{\beta/\alpha} \psi_1^\beta(t)\, q(t)\, dt = \infty.$$

Proof. $((C_6) \Rightarrow (C_5))$. Let (C_6) hold and suppose that $J(t) \to \infty$ as $t \to \infty$, where $J(t)$ is defined by (4.3). Choose $t_0 > \max\{a, 1\}$ such that

$$J(t) = \int_a^t (t-s) s^\beta q(s)\, ds \geq 1 \text{ and } \psi_1(t) \leq 1 \text{ for } t \geq t_0.$$

Then, using integration by parts and the fact that $1/\alpha \le 1$, $\beta < 1$, we see that

$$\begin{aligned}
\int_{t_0}^{t} (p(s))^{-1/\alpha} \bigg(\int_a^s (s-r) r^\beta \, q(r)\, dr \bigg)^{1/\alpha} ds
&\le \int_{t_0}^{t} (p(s))^{-1/\alpha} \bigg(\int_a^s (s-r) r^\beta \, q(r)\, dr \bigg) ds \\
&= \int_{t_0}^{t} \psi_1''(s) \bigg(\int_a^s (s-r) r^\beta \, q(r)\, dr \bigg) ds \\
&= \bigg[\psi_1'(s) \int_a^s (s-r) r^\beta \, q(r)\, dr - \psi_1(s) \int_a^s \; r^\beta q(r)\, dr \bigg] \bigg|_{s=t_0}^{s=t} \\
&\quad + \int_{t_0}^{t} \psi_1(s)\, s^\beta q(s)\, ds \\
&\le K + \int_{t_0}^{t} \psi_1(s)\, s^\beta q(s)\, ds \\
&\le K + \int_{t_0}^{t} s\, \psi_1^\beta(s)\, q(s)\, ds, \quad t \ge t_0,
\end{aligned}$$

where

$$K = -\psi_1'(t_0) \int_a^{t_0} (t_0 - r)\, r^\beta \, q(r)\, dr + \psi_1(t_0) \int_a^{t_0} \; r^\beta \, q(r)\, dr.$$

Therefore, (C_6) implies (C_5).

$((C_6) \Rightarrow (C_7))$. For $\beta < 1$, we have

$$\begin{aligned}
\bigg(\int_{t_0}^{t} (t-s) s^\beta \, q(s)\, ds \bigg)^{1/\alpha} &\le t^{\beta/\alpha} \bigg(\int_{t_0}^{t} (t-s)\, q(s)\, ds \bigg)^{1/\alpha} \\
&\le t^{1/\alpha} \bigg(\int_{t_0}^{t} (t-s)\, q(s)\, ds \bigg)^{1/\alpha}
\end{aligned} \tag{4.4}$$

for all $t \ge t_0 \ge \max\{1, a\}$. Accordingly, (C_6) clearly implies that

$$\int_{t_0}^{\infty} t^{1/\alpha} (p(t))^{-1/\alpha} \bigg(\int_{t_0}^{t} (t-s) q(s)\, ds \bigg)^{1/\alpha} dt = \infty. \tag{4.5}$$

Choose $t_1 > t_0$, such that

$$\int_{t_0}^{t} (t-s)\,q(s)\,ds \geq 1 \text{ and } \psi_2(t) \leq 1 \text{ for } t \geq t_1.$$

Then, by integration by parts, we obtain

$$\begin{aligned}
&\int_{t_1}^{t} s^{1/\alpha}(p(s))^{-1/\alpha}\bigg(\int_{t_0}^{s}(s-r)q(r)\,dr\bigg)^{1/\alpha} ds \\
&\qquad\qquad = \int_{t_1}^{t} \psi_2''(s)\bigg(\int_{t_0}^{s}(s-r)q(r)\,dr\bigg)^{1/\alpha} ds \\
&\qquad\qquad \leq \int_{t_1}^{t} \psi_2''(s)\bigg(\int_{t_0}^{s}(s-r)q(r)\,dr\bigg) ds \\
&\qquad\qquad = \bigg[\psi_2'(s)\int_{t_0}^{s}(s-r)q(r)\,dr - \psi_2(s)\int_{t_0}^{s} q(r)\,dr\bigg]\bigg|_{s=t_1}^{s=t} \\
&\qquad\qquad\quad + \int_{t_1}^{t} \psi_2(s)q(s)\,ds \\
&\qquad\qquad \leq K + \int_{t_1}^{t} \psi_2^{\beta}(s)q(s)\,ds, \quad t \geq t_1,
\end{aligned}$$

where

$$K = -\psi_2'(t_1)\int_{t_0}^{t_1}(t_1-r)q(r)\,dr + \psi_2(t_1)\int_{t_0}^{t_1} q(r)\,dr.$$

From the last inequality and (4.5), we have that (C_7) is satisfied.

$((C_6) \Rightarrow (C_8))$. Since $\beta < \alpha$, we have for all large t

$$\begin{aligned}
\bigg(\int_{a}^{t}(t-s)s^{\beta}\,q(s)\,ds\bigg)^{1/\alpha} &\leq t^{\beta/\alpha}\bigg(\int_{a}^{t}(t-s)\,q(s)\,ds\bigg)^{1/\alpha} \\
&\leq t\bigg(\int_{a}^{t}(t-s)\,q(s)\,ds\bigg)^{1/\alpha},
\end{aligned}$$

so that, clearly (C_6) implies (C_8).

$((C_6) \Rightarrow (C_9))$. If (C_6) holds, it implies that for any $t_0 > a$

$$\int_{t_0}^{\infty}(p(t))^{-1/\alpha}\bigg(\int_{t_0}^{t}(t-s)s^{\beta}\,q(s)\,ds\bigg)^{1/\alpha} dt = \infty. \tag{4.6}$$

Choose $t_0 > 1$, such that $\psi_1(t) \le 1$ for $t \ge t_0$. In view of the basic integral condition (P), (4.6) implies that

$$\lim_{t\to\infty} \int_{t_0}^{t} (t-s)s^{\beta}\, q(s)\, ds = \infty,$$

so that, by L'Hopital's rule, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s)s^{\beta}\, q(s)\, ds = \lim_{t\to\infty} \int_{t_0}^{t} s^{\beta}\, q(s)\, ds \in (0,\infty],$$

which shows that there exists some constant $k > 0$ and some $t_1 > t_0$ such that

$$\int_{t_0}^{t} (t-s)s^{\beta}\, q(s)\, ds \ge k\, t \text{ for every } t \ge t_1. \tag{4.7}$$

For all $t > t_1$, using integration by parts, we obtain

$$\begin{aligned}
&\int_{t_1}^{t} (p(s))^{-1/\alpha} \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{1/\alpha} ds \\
&\qquad = \int_{t_1}^{t} \psi_1''(s) \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{1/\alpha} ds \\
&\qquad = \psi_1'(s) \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{1/\alpha} \Bigg|_{s=t_1}^{s=t} \\
&\qquad\quad - \frac{1}{\alpha} \int_{t_1}^{t} \psi_1'(s) \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{(1/\alpha)-1} \\
&\qquad\qquad \times \left(\int_{t_0}^{s} r^{\beta} q(r)\, dr \right) ds.
\end{aligned}$$

From the last equality, using (4.7) and noting that $1/\alpha \le 1$ and that $\psi_1'(t)$ is negative, we get

$$\begin{aligned}
&\int_{t_1}^{t} (p(s))^{-1/\alpha} \left(\int_{t_0}^{s} (s-r)\, r^{\beta} q(r)\, dr \right)^{1/\alpha} ds \\
&\qquad \le k_1 - k_2 \int_{t_1}^{t} \psi_1'(s)\, s^{(1/\alpha)-1} \int_{t_0}^{s} r^{\beta}\, q(r)\, dr\, ds \\
&\qquad \le k_1 - k_2 \int_{t_1}^{t} \psi_1'(s) \int_{t_0}^{s} r^{(1/\alpha)-1+\beta}\, q(r)\, dr\, ds, \quad t \ge t_1,
\end{aligned} \tag{4.8}$$

where

$$k_1 = -\psi_1'(t_1)\bigg(\int_{t_0}^{t_1} (t_1 - r)\, r^\beta q(r)\, dr\bigg)^{1/\alpha} > 0, \quad k_2 = \frac{1}{\alpha}\, k^{(1/\alpha)-1} > 0.$$

Now, from (4.8), using the fact that

$$0 \le \frac{(1-\alpha)\,(\beta-1)}{\alpha} = \frac{\beta}{\alpha} - \bigg(\frac{1}{\alpha} - 1 + \beta\bigg),$$

we have

$$\begin{aligned}
\int_{t_1}^{t} (p(s))^{-1/\alpha}\bigg(\int_{t_0}^{s} (s-r)\, r^\beta q(r)\, dr\bigg)^{1/\alpha} ds & \\
&\le k_1 - k_2 \int_{t_1}^{t} \psi_1'(s) \int_{t_0}^{s} r^{\beta/\alpha}\, q(r)\, dr\, ds \\
&= k_1 + k_2 \bigg[\bigg(-\psi_1(s) \int_{t_0}^{s} r^{\beta/\alpha}\, q(r)\, dr\bigg)\bigg|_{s=t_1}^{s=t} \\
&\qquad + \int_{t_1}^{t} \psi_1(s)\, s^{\beta/\alpha}\, q(s)\, ds\bigg] \\
&\le k_3 + k_2 \int_{t_1}^{t} \psi_1(s)\, s^{\beta/\alpha}\, q(s)\, ds \\
&\le k_3 + k_2 \int_{t_1}^{t} s^{\beta/\alpha}\, \psi_1^\beta(s)\, q(s)\, ds, \quad t \ge t_1,
\end{aligned}$$

where $k_3 = k_1 + k_2\, \psi_1(t_1) \int_{t_0}^{t_1} r^{\beta/\alpha}\, q(r)\, dr$. Letting $t \to \infty$, we conclude that (C_6) implies (C_9). $\square$

Lemma 4.3. *If $x(t)$ is a positive "intermediate" solution of type* (E)*, then it satisfies*

(4.9)
$$x(t) \ge \int_t^\infty \frac{s-t}{(p(s))^{1/\alpha}} \bigg(\int_{t_0}^{s} \int_r^\infty q(\mu) x^\beta(\mu)\, d\mu\, dr\bigg)^{1/\alpha} ds, \quad t \ge t_0,$$

and

$$\int^\infty t\, q(t)\, x^\beta(t)\, dt = \infty. \tag{4.10}$$

Proof. Let $x(t)$ be a solution of (Δ) of type (E). It has derivatives of type (II), i.e.,

$$x'(t) < 0, \quad x''(t) > 0, \quad \Big(p(t)\varphi_\alpha(x''(t))\Big)' > 0$$

for all large t, say for $t \geq t_0$. Since the function $(p(t)\varphi_\alpha(x''(t))'$ decreases to a finite limit $w_3 \geq 0$ as $t \to \infty$, integrating the equation (Δ) twice, first from t to ∞ and afterwards on $[t_0, t]$ yields

(4.11)
$$x''(t) = (p(t))^{-1/\alpha}\left[\xi_2 + \int_{t_0}^{t}\left(w_3 + \int_s^\infty q(r)x^\beta(r)\,dr\right)ds\right]^{1/\alpha}, \quad t \geq t_0,$$

where $\xi_2 = p(t_0)(x''(t_0))^\alpha > 0$. Since $x'(t)$ is a negative increasing function, there exists a finite limit $w_1 = \lim_{t\to\infty} x'(t) \leq 0$. If $w_1 < 0$, then the inequality $x'(t) \leq w_1$, $t \geq t_0$, yields $\lim_{t\to\infty} x(t) = -\infty$, an impossibility, so that w_1 must be zero. Since

$$\int_{t_0}^{t}\left(w_3 + \int_s^\infty q(r)x^\beta(r)\,dr\right)ds = O(t), \quad t \to \infty,$$

condition (P) enables us to integrate (4.11) over $[t, \infty)$ and we get

(4.12)
$$-x'(t) = \int_t^\infty (p(s))^{-1/\alpha}\left[\xi_2 + \int_{t_0}^{s}\left(w_3 + \int_r^\infty q(\mu)x^\beta(\mu)\,d\mu\right)dr\right]^{1/\alpha} ds,$$
$$t \geq t_0.$$

Noting that the limit $w_0 = \lim_{t\to\infty} x(t) \geq 0$ is finite, integrating (4.12) from t to ∞, we obtain

$$(4.13)\quad x(t) = w_0 + \int_t^\infty \frac{s-t}{(p(s))^{1/\alpha}}\Bigg[\xi_2 + \int_{t_0}^{s}\left(w_3 + \int_r^\infty q(\mu)x^\beta(\mu)\,d\mu\right)dr\Bigg]^{1/\alpha} ds, \quad t \geq t_0.$$

If $w_0 > 0$, then $x(t) \sim w_0 = w_0\psi_3(t)$ as $t \to \infty$, so that the solution $x(t)$ would be of type (C). Therefore, we must have that $w_0 = 0$. Then, we have $\lim_{t\to\infty} x(t)/\psi_2(t) = w_3^{1/\alpha}$, which for $w_3 = 0$ implies that

$x(t) \prec \psi_2(t)$, $t \to \infty$. Accordingly, from (4.13), for $w_3 = w_0 = 0$, we conclude that (4.9) is satisfied for any solution $x(t)$ of type (E).

Furthermore, from (4.13), we find that

$$\lim_{t\to\infty} \frac{x(t)}{\psi_1(t)} = \left[\xi_2 + \int_{t_0}^{\infty} \int_s^{\infty} q(\mu) x^{\beta}(\mu)\, d\mu\, ds\right]^{1/\alpha}. \tag{4.14}$$

If we suppose that (4.10) does not hold, the right hand side of (4.14) becomes a positive constant c, which implies that $x(t) \sim c\,\psi_1(t)$ as $t \to \infty$. This, however, contradicts the assumption that $x(t)$ is of type (E). Therefore, we conclude that $x(t)$ must satisfy (4.10), which completes the proof. □

Lemma 4.4. *If $x(t)$ is a positive "intermediate" solution of type* (F)*, then it satisfies*

$$x(t) \geq \int_{t_0}^{t} \int_s^{\infty} (p(r))^{-1/\alpha} \left(\int_{t_0}^{r} (r-\mu) q(\mu) x^{\beta}(\mu)\, d\mu \right)^{1/\alpha} dr\, ds, \ \ t \geq t_0, \tag{4.15}$$

and

$$\int_{t_0}^{\infty} t\,(p(t))^{-1/\alpha} \left(\int_{t_0}^{t} (t-s) q(s)\, x^{\beta}(s)\, ds \right)^{1/\alpha} dt = \infty. \tag{4.16}$$

Proof. Let $x(t)$ be a positive solution of type (F). Then it has derivatives of type (IV), i.e.,

$$x'(t) > 0, \quad x''(t) < 0, \quad \big(p(t)\varphi_{\alpha}(x''(t))\big)' < 0$$

for all large t, say for $t \geq t_0$. We now integrate equation (Δ) twice over $[t_0, t]$, obtaining

$$-x''(t) = (p(t))^{-1/\alpha} \left(\xi_2 + \xi_3 (t-t_0) + \int_{t_0}^{t} (t-s) q(s)\, x^{\beta}(s)\, ds \right)^{1/\alpha}, \ t \geq t_0, \tag{4.17}$$

where $\xi_2 = p(t_0)(-x''(t_0))^{\alpha} > 0$ and $\xi_3 = -(p(t)(-x''(t))^{\alpha})'|_{t=t_0} > 0$. The existence of the finite limit $w_1 = \lim_{t\to\infty} x'(t) \geq 0$ implies the

integrability of the right hand side of (4.17) on $[t_0, \infty)$. Integrating (4.17) twice more, first on $[t, \infty)$ and afterwards on $[t_0, t]$, we obtain for all $t \geq t_0$:
(4.18)
$$x(t) = \xi_0 + w_1(t - t_0) + \int_{t_0}^{t} \int_{s}^{\infty} (p(r))^{-1/\alpha} \bigg(\xi_2 + \xi_3(r - t_0) + \int_{t_0}^{r} (r - \mu) q(\mu)\, x^{\beta}(\mu)\, d\mu \bigg)^{1/\alpha} dr\, ds,$$

where $\xi_0 = x(t_0) > 0$. From the above representation for $x(t)$, we see that if $w_1 > 0$ the solution $x(t)$ would be of type (D), which is impossible. Therefore, $w_1 = 0$ in (4.18). Furthermore, if we suppose that (4.16) fails to hold, i.e.,

$$\int_{t_0}^{\infty} t\, (p(t))^{-1/\alpha} \bigg(\int_{t_0}^{t} (t - s) q(s)\, x^{\beta}(s)\, ds \bigg)^{1/\alpha} dt < \infty,$$

then from (4.18) it follows that $\lim_{t\to\infty} x(t) = w_0 > 0$. This shows that $x(t)$ is a solution of type (C), contradicting the basic assumption. Therefore, (4.16) necessarily holds, and the proof of the lemma is completed. □

5. Proofs of the main results. We are now in a position to prove the two theorems stated in Section 3. In each of those two theorems the "only if" part is easier to handle, so we will direct our efforts at the verification of the "if" part, that is, to the verification that the condition (C_5) or (C_6) guarantees the oscillation of all solutions of (Δ), or equivalently, the nonexistence of any types of nonoscillatory solutions for (Δ).

Proof of Theorem 3.1. (The "only if" part): Suppose that (C_5) is not satisfied. Then, by the "if" part of Theorem 2.1 equation (Δ) has a positive solution of type (A). Therefore, (C_5) must hold if all solutions of (Δ) are oscillatory.

(The "if" part): We will prove that (C_5) ensures the oscillation of all solutions of (Δ), or equivalently, the nonexistence of any positive solution of (Δ). Since any positive solution of (Δ) falls into one of the six types (A)–(F) mentioned in Section 2, it is sufficient to verify that (C_5) eliminates positive solutions of (Δ) of all these six types.

Elimination of solution $x(t)$ of type (A): This is nothing but what the "only if" part of Theorem 2.1 asserts.

Elimination of solution $x(t)$ of type (B): Since by (a) of Lemma 4.1, (C_5) implies (C_7), from the "only if" part of Theorem 2.2 we see that (Δ) admits no solution of type (B).

Elimination of solution $x(t)$ of type (C): Suppose that (Δ) has a solution of type (C). Then, by the "only if" part of Theorem 2.3 condition (C_3) holds, and hence condition (C_1) holds by (b) of Lemma 4.1. But this is the contradiction with assumption (C_5).

Elimination of solution $x(t)$ of type (D): Suppose that the equation (Δ) has a solution of type (D). Then, by the "only if" part of Theorem 2.4 condition (C_4) holds. By (c) of Lemma 4.1 condition (C_4) implies condition (C_1), contradicting assumption (C_5).

Elimination of solution $x(t)$ of type (E): Suppose that there exists a solution $x(t)$ of type (E). Then, by Lemma 4.3, $x(t)$ satisfies (4.9), which implies that

(5.1)
$$x(t) \geq \left(\int_t^\infty \frac{s-t}{(p(s))^{1/\alpha}}\,ds \right) \cdot \left(\int_{t_0}^t \int_s^\infty q(r)x^\beta(r)\,dr\,ds \right)^{1/\alpha}, \quad t \geq t_0.$$

Put

$$X(t) = \int_{t_0}^t \int_s^\infty q(r)x^\beta(r)\,dr\,ds, \quad t \geq t_0.$$

We then have

$$X''(t) = -q(t)x^\beta(t) \text{ or } x(t) = \left(-\frac{X''(t)}{q(t)} \right)^{1/\beta},$$

which, combined with (5.1), gives the following differential inequality for $X(t)$:

$$\left(-\frac{X''(t)}{q(t)} \right)^{1/\beta} \geq X^{1/\alpha}(t)\,\psi_1(t),$$

or

$$X''(t) + \psi_1^\beta(t)\,q(t)\,(X(t))^{\beta/\alpha} \leq 0, \quad t \geq t_0.$$

It is not difficult to verify that there exists a positive solution $\widetilde{X}(t)$ of the Emden-Fowler differential equation

$$\widetilde{X}''(t) + \psi_1^{\beta}(t)q(t)(\widetilde{X}(t))^{\beta/\alpha} = 0, \tag{5.2}$$

which is superlinear because of $\beta > \alpha$. Atkinson's theorem (Theorem A) then implies that

$$\int_{t_0}^{\infty} t\, \psi_1^{\beta}(t)q(t)\, dt < \infty,$$

which contradicts (C_5).

Elimination of solution $x(t)$ of type (F): Let us suppose that equation (Δ) possesses a solution of type (F). Then, by Lemma 4.4, $x(t)$ satisfies (4.15) and (4.16). Since (4.16) implies that

$$\int_{t_0}^{r} (r-\mu)q(\mu)x^{\beta}(\mu)\, d\mu \longrightarrow \infty, \quad r \to \infty,$$

there exists some $t_1 > t_0$ such that

$$\int_{t_0}^{r} (r-\mu)q(\mu)x^{\beta}(\mu)\, d\mu \geq 1 \text{ and } \psi_1(r) \leq 1 \text{ for } r \geq t_1.$$

Using the above inequalities and the fact that $1/\alpha \geq 1$, we see that

$$\begin{aligned}
\int_{s}^{\infty} (p(r))^{-1/\alpha} \bigg(\int_{t_0}^{r} (r-\mu)q(\mu)x^{\beta}(\mu)\, d\mu \bigg)^{1/\alpha} dr& \\
&\geq \int_{s}^{\infty} (p(r))^{-1/\alpha} \bigg(\int_{t_0}^{r} (r-\mu)q(\mu)x^{\beta}(\mu)\, d\mu \bigg) dr \\
&\geq \int_{s}^{\infty} \bigg(\int_{r}^{\infty} (r-\mu)(p(r))^{-1/\alpha}\, dr \bigg) q(\mu)x^{\beta}(\mu)\, d\mu \\
&= \int_{s}^{\infty} \psi_1(\mu)\, q(\mu)x^{\beta}(\mu)\, d\mu, \quad s \geq t_1.
\end{aligned}$$

Consequently, we find from (4.15) that

$$x(t) \geq \int_{t_1}^{t} \int_{s}^{\infty} \psi_1(r)\, q(r)x^{\beta}(r)\, dr\, ds, \quad t \geq t_1. \tag{5.3}$$

We put

$$Y(t) = \int_{t_1}^{t} \int_{s}^{\infty} \psi_1(r)\, q(r) x^{\beta}(r)\, dr\, ds, \quad t \geq t_1.$$

Then, we have $Y''(t) + \psi_1(t) q(t) x^{\beta}(t) = 0$ for $t \geq t_1$ and, since $x(t) \geq Y(t)$, $t \geq t_1$ from (5.3), we have that

$$Y''(t) + \psi_1(t) q(t) Y^{\beta}(t) \leq 0, \quad t \geq t_1.$$

Therefore, there exists a positive solution of the Emden-Fowler equation

$$\widetilde{Y}''(t) + \psi_1(t) q(t) \widetilde{Y}^{\beta}(t) = 0, \quad t \geq t_1, \tag{5.4}$$

which is superlinear because of $\beta > 1$. Applying Theorem A to (5.4), we have that

$$\int_{t_1}^{\infty} t\, \psi_1(t) q(t)\, dt < \infty,$$

which implies that

$$\int_{t_1}^{\infty} t\, \psi_1^{\beta}(t) q(t)\, dt < \infty.$$

But, this clearly contradicts condition (C_5), and so equation (Δ) cannot possess a positive solution of type (F). This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. (The "only if" part): Suppose that condition (C_6) fails to hold. Then, by the "if" part of Theorem 2.4 equation (Δ) has a positive solution of type (D), and so (C_6) must hold if all solutions of (Δ) are oscillatory.

(The "if" part): Assume that (C_6) is satisfied. We will show that (C_6) is sufficient to eliminate all six types (A)–(F) of positive solutions of the sub-half-linear equation (Δ).

Elimination of solution $x(t)$ of type (A): Since by (a) of Lemma 4.2 (C_6) implies (C_5), the application of the "only if" part of Theorem 2.1 ensures the nonexistence of a positive type (A)-solution of (Δ).

Elimination of solution $x(t)$ of type (B): By (b) of Lemma 4.2 condition (C_6) implies (C_7). In view of the fact that (C_7) is the negation of (C_2), from the "only if" part of Theorem 2.2 it follows that equation (Δ) cannot have a positive type (B)-solution.

Elimination of solution $x(t)$ of type (C): By (c) of Lemma 4.2, (C_6) implies (C_8), so that the application of the "only if" part of Theorem 2.3 eliminates solutions of type (C).

Elimination of solution $x(t)$ of type (D): This is equivalent to the "only if" part of Theorem 2.4.

Elimination of solution $x(t)$ of type (E): We first note that condition (C_9) holds by (d) of Lemma 4.2. Suppose that equation (Δ) has a solution $x(t)$ of type (E). Then, by Lemma 4.3, $x(t)$ satisfies (4.9). Following exactly the same steps of elimination of type (E)-solution as in the proof of Theorem 3.1, we are led to the conclusion that the Emden-Fowler differential equation (5.2) possesses a positive solution. Since $\beta < \alpha$, in this case equation (5.2) is sublinear and, accordingly, Belohorec's theorem (Theorem B) implies that

$$\int_{t_0}^{\infty} t^{\beta/\alpha} \psi_1^{\beta}(t)\, q(t)\, dt < \infty, \tag{5.5}$$

which contradicts (C_9).

Elimination of solution $x(t)$ of type (F): Let us suppose that equation (Δ) has a solution $x(t)$ of type (F). Note that $x(t)$ has derivatives of type (IV), so that $x'(t) > 0$, $x''(t) < 0$ for all large t, say for $t \geq t_0$. By Lemma 4.4, $x(t)$ satisfies integral inequality (4.15), which implies that

$$x(t) \geq (t - t_0) \int_{t}^{\infty} (p(s))^{-1/\alpha} \left(\int_{t_0}^{s} (s-r) q(r) x^{\beta}(r)\, dr \right)^{1/\alpha} ds, \quad t \geq t_0. \tag{5.6}$$

Put

$$\Psi(t) = \frac{x(t)}{t - t_0}, \quad t > t_0, \tag{5.7}$$

$$X(t) = \int_{t}^{\infty} (p(s))^{-1/\alpha} \Psi^{\beta/\alpha}(s) \left(\int_{t_0}^{s} (s-r)(r-t_0)^{\beta} q(r)\, dr \right)^{1/\alpha} ds, \; t > t_0. \tag{5.8}$$

Since

$$x(t)-x(t_0)=\int_{t_0}^{t} x'(s)\,ds \geq (t-t_0)\,x'(t), \quad t\geq t_0,$$

we have

$$\Psi'(t)=\frac{(t-t_0)x'(t)-x(t)}{(t-t_0)^2}<0, \quad t>t_0,$$

which implies that $\Psi(t)$ is decreasing on (t_0,∞), so that the function $X(t)$ is well-defined by (5.8). Now, from (5.6), we have

(5.9)

$$\begin{aligned}\Psi(t) &\geq \int_t^\infty (p(s))^{-1/\alpha}\left(\int_{t_0}^{s}(s-r)q(r)(r-t_0)^\beta\Psi^\beta(r)\,dr\right)^{1/\alpha} ds\\ &\geq \int_t^\infty (p(s))^{-1/\alpha}\Psi^{\beta/\alpha}(s)\left(\int_{t_0}^{s}(s-r)(r-t_0)^\beta q(r)\,dr\right)^{1/\alpha} ds\\ &= X(t), \quad t>t_0.\end{aligned}$$

Let $t_1>t_0$ be fixed arbitrarily. Combining (5.9) with the relation

$$X'(t)=-(p(t))^{-1/\alpha}\Psi^{\beta/\alpha}(t)\left(\int_{t_0}^{t}(t-s)(s-t_0)^\beta\,q(s)\,ds\right)^{1/\alpha},$$

we obtain the differential inequality

(5.10)

$$-X'(t)\geq (p(t))^{-1/\alpha}\left(\int_{t_0}^{t}(t-s)(s-t_0)^\beta\,q(s)\,ds\right)^{1/\alpha}\Big(X(t)\Big)^{\beta/\alpha},\ t\geq t_1.$$

Dividing (5.10) by $(X(t))^{\beta/\alpha}$ and integrating from t_1 to ∞, we conclude that

$$\begin{aligned}&\int_{t_1}^\infty (p(s))^{-1/\alpha}\left(\int_{t_0}^{s}(s-r)(r-t_0)^\beta q(r)\,dr\right)^{1/\alpha} ds\\ &\leq -\int_{t_1}^\infty \Big(X(s)\Big)^{-\beta/\alpha} X'(s)\,ds\\ &= \frac{\alpha}{\alpha-\beta}\Big(X(t_1)\Big)^{1-(\beta/\alpha)}<\infty,\end{aligned}$$

which contradicts assumption (C_6). Thus, (Δ) admits no solution of type (F), if (C_6) is satisfied. This completes the proof of Theorem 3.2. □

Remark. We conjecture that the number 1 could be removed from the super-half-linearity condition $\alpha \le 1 < \beta$ and the sub-half-linearity condition $\alpha \ge 1 > \beta$, i.e., that Theorems 3.1 and 3.2 are still in effect if $\alpha < \beta$ and $\beta < \alpha$, respectively.

6. Example. We present here an example which illustrates our main oscillation theorems and supplements nonoscillation results of [**6**].

Example. Consider the equation

$$(\Delta_1) \qquad \left(t^{\lambda}|x''|^{\alpha-1}\, x''\right)'' + t^{-\mu}\,|x|^{\beta-1}x = 0,$$

where α, β and $\lambda > 2\alpha + 1$ are fixed positive constants and $\mu \in \mathbf{R}$, $\mu \ne 2$, $\mu \ne 2 + \beta$ is a varying parameter. The assumption $\lambda > 2\alpha + 1$ ensures that condition (P) is satisfied. Moreover, functions $\psi_1(t)$, $\psi_2(t)$ given by (2.1) have the asymptotic behavior

$$\begin{aligned}\psi_1(t) &\sim c_1(\alpha,\lambda)\, t^{2-(\lambda/\alpha)}, \quad t \to \infty;\\ \psi_2(t) &\sim c_2(\alpha,\lambda)\, t^{2-(1-\lambda)/\alpha}, \quad t \to \infty.\end{aligned}$$

It is easy to verify that

$$\begin{gathered}\int_a^t (t-s)q(s)\,ds \sim k_1(\mu)\, t^{2-\mu}, \quad t \to \infty\\ \int_a^t (t-s)s^{\beta}\, q(s)\,ds \sim k_2(\mu)\, t^{2+\beta-\mu}, \quad t \to \infty,\end{gathered}$$

and that for (Δ_1):

(C_1) is equivalent to $\mu > 2 + (2 - (\lambda/\alpha))\,\beta = \mu_A$;

(C_2) is equivalent to $\mu > 1 + (2 - (\lambda - 1)/\alpha)\,\beta = \mu_B$;

(C_3) is equivalent to $\mu > 2 + 2\alpha - \lambda = \mu_C$;

(C_4) is equivalent to $\mu > 2 + \alpha + \beta - \lambda = \mu_D$.

Using that $\lambda > 2\alpha + 1 > 2\alpha$ and $\lambda > \alpha + 1$, it follows by an easy calculation that

$$\begin{gathered}\mu_A < \mu_B < \mu_C < \mu_D, \text{ if } \alpha < \beta,\\ \mu_D < \mu_C < \mu_B < \mu_A, \text{ if } \alpha > \beta.\end{gathered}$$

Accordingly, the sub-half-linear equation (Δ_1) has solutions of all types (A)–(D) if $\mu > \mu_D$ and the super-half-linear equation (Δ_1) has solutions of all types (A)–(D) if $\mu > \mu_A$. Also, the sub-half-linear equation (Δ_1) has no solution of types (A)–(D) if $\mu \le \mu_A$ and the super-half-linear equation (Δ_1) has no solution of types (A)–(D) if $\mu \le \mu_D$.

Conditions (C_2), (C_5) guarantee the existence of an "intermediate" solution of type (E), by Theorem 2.9 in [**6**] and by Theorem 2.10 in [**6**], the conditions (C_4), (C_8) ensure the existence of an "intermediate" solution of type (F). Therefore, we may conclude that:

(a) the super-half-linear equation (Δ_1) has a solution of type (E) if $\mu_B < \mu \le \mu_A$;

(b) the super-half-linear equation (Δ_1) has a solution of type (F) if $\mu_D < \mu \le \mu_C$.

Theorems 2.9 and 2.10 in [**6**] are not applicable to the sub-half-linear equation (Δ_1), since (C_2) and (C_5) as well as (C_4) and (C_8) are inconsistent if $\alpha < \beta$.

Suppose that $\alpha \le 1 < \beta$. Then, Theorem 3.1 leads to the conclusion that all solutions of (Δ_1) are oscillatory if and only if

$$\mu \le 2 + \left(2 - \frac{\lambda}{\alpha}\right)\beta = \mu_A = \min\{\mu_A, \mu_B, \mu_C, \mu_D\}. \tag{6.1}$$

Suppose that $\beta \le 1 < \alpha$. Then, from Theorem 3.2 we conclude that all solutions of (Δ_1) are oscillatory if and only if

$$\mu \le 2 + \alpha + \beta - \lambda = \mu_D = \min\{\mu_A, \mu_B, \mu_C, \mu_D\}. \tag{6.2}$$

Note that as is shown in [**6**], Theorem 2.11, all solutions of (Δ) are oscillatory if

$$\int_a^\infty \psi_1^\beta(t)\, q(t)\, dt = \infty \tag{6.3}$$

and

$$\int_a^\infty \frac{1}{(p(t))^{1/\alpha}} \left(\int_a^t (t-s) q(s)\, ds\right)^{1/\alpha} dt = \infty \tag{6.4}$$

hold. For equation (Δ_1), (6.3) is equivalent to $\mu \leq 1+(2-(\lambda/\alpha))\beta = \mu_1$ and (6.4) is equivalent to $\mu \leq 2+\alpha-\lambda = \mu_2$. Notice that, if $\alpha < \beta$ then $\mu_1 < \mu_2$ and $\mu_1 < \mu_A$, and if $\alpha > \beta$ then $\mu_2 < \mu_1$ and $\mu_2 < \mu_D$. Therefore, the above-mentioned oscillation criterion provide us that the sub-half-linear (super-half-linear) equation (Δ_1) is oscillatory if $\mu < \mu_1 < \mu_A$ ($\mu < \mu_2 < \mu_D$). Thus, our main results Theorems 3.1 and 3.2 give sharper oscillation conditions.

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REFERENCES

1. F.V. Atkinson, *On second order nonlinear oscillation*, Pacific J. Math. **5** (1955), 643–647.

2. S. Belohorec, *Oscillatory solution of certain nonlinear differential equations of second order*, Mat. Fyz. Casopis Sloven Akad. Vied. **11** (1961), 250–255.

3. K.-I. Kamo and H. Usami, *Oscillation theorems for fourth-order quasilinear ordinary differential equations*, Stud. Sci. Math. Hung. **39** (2002), 385–406.

4. T. Kusano and M. Naito, *Nonlinear oscillation of fourth order differential equations*, Canada. J. Math. **28** (1976), 840–852.

5. ———, *On fourth-order non-linear oscillation*, J. London Math. Soc. **14** 1976, 91–105.

6. T. Kusano and T. Tanigawa, *On the structure of positive solutions of a class of fourth order nonlinear differential equations*, Ann. Mat. Pura Appl. **85** (2006), 521–536.

7. M. Naito and F. Wu, *A note on the existence and asymptotic behavior of nonoscillatory solutions of fourth order quasilinear differential equations*, Acta Math. Hung. **102** (2004), 177–202.

8. ———, *On the existence of eventually positive solutions of fourth order quasilinear differential equations*, Nonlinear Anal. Theory Meth. Appl. **57** (2004), 253–263.

9. F. Wu, *Nonoscillatory solutions of fourth order quasilinear differential equations, Funkcial. Ekvac.* **45** (2002), 71–88.

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