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COMMUTATIVE GROUP ALGEBRAS WHOSE QUOTIENT RINGS BY NILRADICALS ARE GENERATED BY IDEMPOTENTS

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ABSTRACT. We give a condition for the reduced ring $(RG)_{\mathrm{red}}$ to be generated by idempotents over R, where R is a commutative ring, G is an abelian group and RG is the group algebra of G over R.

1. Introduction. This paper is a continuation of [**5**], in which we gave conditions for the group algebra RG of an abelian group G over a commutative ring R with identity to be generated by idempotents over R. In studying algebras generated by idempotents, we frequently encounter R-algebras A such that A_{red} is generated by idempotents over R, but A itself is not, where $A_{\mathrm{red}} = A/\mathrm{nil}(A)$, the quotient ring of A by its nilradical. In view of this, in the paper we take up a problem asking conditions for $(RG)_{\mathrm{red}}$ to be generated by idempotents. The main result is as follows.

Theorem. *Let R be a commutative ring with identity and G an abelian group. Then the following conditions are equivalent.*

(1) $(RG)_{\mathrm{red}}$ *is generated by idempotents over R.*

(2) *G is a torsion group, $n \in n^2 R_{\mathrm{red}}$ and the nth cyclotomic polynomial $\phi_n(X)$ has a root in R_{red} for every positive integer n such that $n = \mathrm{ord}(g)$, the order of g, for some $g \in G$.*

Rings are assumed to be commutative R-algebras. For a ring S, we denote by $U(S)$ the group of units in S, and by $S[X]$ the polynomial ring in one indeterminate X over S. We write $\mathbf{Z}$ for the ring of integers. When considering an integer n as an element of S, we assume that n stands for $n \cdot 1_S$ as usual. For basic results and undefined terminology, our general references are [**1, 2, 6**].

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2. Proof of the theorem. Let A be an R-algebra. Then we set $\Delta(A) = R[E]$, where E is the set of all idempotents of A. Note that A_{red} is canonically an R_{red}-algebra, so that $\Delta(A_{\mathrm{red}}) = A_{\mathrm{red}}$ if and only if A_{red} is generated by idempotents over R_{red}.

Lemma 2.1. *Let A and B be R-algebras, and let $\rho: A \to B$ be an R-algebra homomorphism. Then ρ induces an R-algebra homomorphism $\overline{\rho}: A_{\mathrm{red}} \to B_{\mathrm{red}}$ such that $\pi_B \circ \rho = \overline{\rho} \circ \pi_A$, where $\pi_A: A \to A_{\mathrm{red}}$ and $\pi_B: B \to B_{\mathrm{red}}$ are canonical R-algebra homomorphisms, respectively. If ρ is surjective, then $\overline{\rho}$ is surjective, and if $\ker \rho \subseteq \mathrm{nil}(A)$, then $\overline{\rho}$ is injective.*

Proof. The assertion is easily verified, and we omit the proof. □

Let G be an abelian group. Then, for a prime number p, we set

$$G_p = \{g \in G \mid \mathrm{ord}\,(g) = p^n \text{ for some } n \geq 0\}$$

and

$$e_p(G) = \sup\{\mathrm{ord}\,(g) \mid g \in G_p\}.$$

Thus G_p is a subgroup of G. It is known that if G is a torsion group, then G_p is a direct summand of G, and in particular there exists a surjective homomorphism $G \to G_p$. Furthermore, we set

$$\mathrm{supp}\, G = \{p \mid e_p(G) \neq 1\}.$$

Note that we have

$$\mathrm{supp}\,(G/G_p) = \mathrm{supp}\, G \setminus \{p\}$$

and

$$e_q(G/G_p) = e_q(G)$$

for $q \in \mathrm{supp}\,(G/G_p)$.

Lemma 2.2. *Suppose that R is reduced. Then RG is reduced if and only if p is a regular element of R for every $p \in \mathrm{supp}\, G$.*

For the proof of the above lemma, see [**2**, Chapter 3, Theorem 4.2 and Corollary 4.3].

Lemma 2.3. *Suppose that* $\operatorname{char} R = p^n$*, where* p *is a prime number and* n *is a positive integer. Then* $(RG)_{\mathrm{red}} \cong R_{\mathrm{red}}(G/G_p)$*.*

Proof. By Lemma 2.1, we know that the natural R-algebra homomorphism $RG \to R_{\mathrm{red}}G$ induces an R-algebra isomorphism $(RG)_{\mathrm{red}} \cong (R_{\mathrm{red}}G)_{\mathrm{red}}$. Thus, replacing R by R_{red} if necessary, we may assume that R is reduced. Note that $\operatorname{char} R = p$ in this case. By [**2**, Chapter 2, Proposition 2.10], we have

$$RG/RG \cdot I(G_p) \cong R(G/G_p), \tag{2.1}$$

where $I(G_p)$ is the augmentation ideal of the group algebra RG_p; namely, $I(G_p)$ is the kernel of the R-algebra homomorphism $RG_p \to R$ induced from the group homomorphism $G_p \to \{e\}$, where $\{e\}$ denotes the trivial group. Hence, for the assertion it suffices to show $RG \cdot I(G_p) = \operatorname{nil}(RG)$. Since $R(G/G_p)$ is reduced by Lemma 2.2, it follows from (2.1) that $RG \cdot I(G_p) \supseteq \operatorname{nil}(RG)$. Conversely, note that $g - 1 \in \operatorname{nil}(RG)$ for every $g \in G_p$, because

$$(g-1)^{p^m} = g^{p^m} - 1 = 0,$$

where $p^m = \operatorname{ord}(g)$. Since $I(G_p)$ is a free R-module generated by elements $g-1$, $g \in G_p$, from this we have $I(G_p) \subseteq \operatorname{nil}(RG)$. Therefore $RG \cdot I(G_p) = \operatorname{nil}(RG)$, which completes the proof. □

Lemma 2.4. *If* $(RG)_{\mathrm{red}}$ *is generated by idempotents over* R_{red}*, then* G *is a torsion group.*

Proof. Note that $(RG)_{\mathrm{red}}$ is integral over R_{red}, because $(RG)_{\mathrm{red}}$ is generated by idempotents over R_{red} and every idempotent is integral over R_{red}. Hence if there were $g \in G$ with $\operatorname{ord}(g) = \infty$, then $(R\langle g\rangle)_{\mathrm{red}}$ would also be integral over R_{red}. However since $R\langle g\rangle = R[X, X^{-1}]$, the Laurent polynomial ring in one variable X, we have $(R\langle g\rangle)_{\mathrm{red}} = R_{\mathrm{red}}[X, X^{-1}]$, which is not integral over R_{red}. This is a contradiction. □

For a positive integer n, we set $\mathfrak{a}_n = 0 :_R n$. Note that $\mathfrak{a}_n = R$ if and only if $\operatorname{char} R \mid n$.

Lemma 2.5. *The following assertions hold for a positive integer n.*

(1) *If R is reduced, then $R/\mathfrak{a}_n$ is also reduced and n is a regular element of $R/\mathfrak{a}_n$.*

(2) *The following conditions are equivalent.*

(i) $n \in U(R/\mathfrak{a}_n)$.

(ii) $R \cong R/\mathfrak{a}_n \times R/nR$.

(iii) $n \in n^2R$.

Proof. For simplicity we set $\mathfrak{a} = \mathfrak{a}_n$. Since the assertions trivially hold when $\mathfrak{a} = R$, we may assume that $\mathfrak{a} \neq R$.

(1) Let a be an element of R such that $na^m \in \mathfrak{a}$ for some $m > 0$. Then $n^2a^m = 0$, and hence $(na)^{2m} = 0$, which implies $na = 0$ because R is reduced. Thus $a \in \mathfrak{a}$ and $\overline{a} = 0$, where $\overline{a}$ is the residue class of a in $R/\mathfrak{a}$. In particular, if $b \in \sqrt{\mathfrak{a}}$, then $b^m \in \mathfrak{a}$ for some $m > 0$, so that $nb^m \in \mathfrak{a}$. Thus $b \in \mathfrak{a}$ by what we have proved, and hence $\sqrt{\mathfrak{a}} = \mathfrak{a}$. Therefore $R/\mathfrak{a}$ is reduced. Similarly, if $n\overline{c} = 0$ for $c \in R$, then $nc \in \mathfrak{a}$, and hence $\overline{c} = 0$. Thus n is a regular element of $R/\mathfrak{a}$.

(2) We have $n \in U(R/\mathfrak{a})$ if and only if $\mathfrak{a} + nR = R$, which implies (i) $\Leftrightarrow$ (ii), because $n\mathfrak{a} = 0$. We will show the equivalence (i) $\Leftrightarrow$ (iii). If $n \in U(R/\mathfrak{a})$, then $R = \mathfrak{a} + nR$, and hence $nR = n\mathfrak{a} + n^2R = n^2R$. Thus $n \in n^2R$. Conversely, if $n \in n^2R$, then $n(1 - nx) = 0$ for some $x \in R$, which means $1 - nx \in \mathfrak{a}$. Thus $n \in U(R/\mathfrak{a})$, as desired. □

Lemma 2.6. *Suppose that $R = R_{\mathrm{red}}$ and $(RG)_{\mathrm{red}} = \Delta((RG)_{\mathrm{red}})$. Then, for $p \in \operatorname{supp} G$, we have $(R/\mathfrak{a}_p)G_p = \Delta((R/\mathfrak{a}_p)G_p)$, where $\mathfrak{a}_p = 0 :_R p$.*

Proof. We set $\mathfrak{a} = \mathfrak{a}_p$ for simplicity. Since G_p is a direct summand of G, there exists a surjective group homomorphism $G \to G_p$, which induces a surjective R-algebra homomorphism $\sigma\colon (R/\mathfrak{a})G \to (R/\mathfrak{a})G_p$. Then, setting $\tau\colon RG \to (R/\mathfrak{a})G$ to be the surjective R-algebra homomorphism induced from $R \to R/\mathfrak{a}$, we have a surjective R-algebra homomorphism $\sigma \circ \tau : RG \to (R/\mathfrak{a})G_p$. Note that $(R/\mathfrak{a})G_p$ is reduced by Lemma 2.2, because $R/\mathfrak{a}$ is reduced and p is a regular element of $R/\mathfrak{a}$ by Lemma 2.5. Thus, by Lemma 2.1, we know that $(R/\mathfrak{a})G_p$ is a surjective image of $(RG)_{\mathrm{red}}$. Since $(RG)_{\mathrm{red}}$ is generated by idempotents, $(R/\mathfrak{a})G_p$ is also generated by idempotents, as claimed. □

Lemma 2.7. *Let p be a prime number such that $p \in p^2R$, and let $q = p^m$, where m is a positive integer. If $\phi_q(X)$ has a root in $R/\mathfrak{a}_p$, then $\phi_q(X)$ has a root in R.*

Proof. Note that if α is a root of $\phi_q(X)$ in $R/\mathfrak{a}_p$, then $(\alpha, 1)$ is a root of $\phi_q(X)$ in $R/\mathfrak{a}_p \times R/pR$. Since $R \cong R/\mathfrak{a}_p \times R/pR$ by Lemma 2.5, we know that $\phi_q(X)$ has a root in R. □

Lemma 2.8. *Suppose that $(R/\mathfrak{a}_p)G_p = \Delta((R/\mathfrak{a}_p)G_p)$, where $p \in \operatorname{supp} G$. Then $p \in p^2R$ and $\phi_{p^m}(X)$ has a root in R for every positive integer m with $p^m \leq e_p(G)$.*

Proof. Since $(R/\mathfrak{a}_p)G_p = \Delta((R/\mathfrak{a}_p)G_p)$, it follows from [**5**, Theorem 4.2] that $p \in U(R/\mathfrak{a}_p)$ and $\phi_{p^m}(X)$ has a root in $R/\mathfrak{a}_p$. Hence $p \in p^2R$ by Lemma 2.5, and $\phi_{p^m}(X)$ has a root in R by Lemma 2.7. □

Lemma 2.9. *Suppose that $R = R_{\mathrm{red}}$, and let H be a cyclic group with $|H| = q$, where $q = p^m$, a power of a prime number p. Then $(RH)_{\mathrm{red}} = \Delta((RH)_{\mathrm{red}})$ if and only if $p \in p^2R$ and $\phi_q(X)$ has a root in R.*

Proof. We set $A = RH$ for simplicity. First suppose that $A_{\mathrm{red}} = \Delta(A_{\mathrm{red}})$, and let $C = (R/\mathfrak{a}_p)H$. Since $R/\mathfrak{a}_p$ is reduced and p is a regular element of $R/\mathfrak{a}_p$ by Lemma 2.5, it then follows from Lemma 2.2 that $C = C_{\mathrm{red}}$. Hence, by Lemma 2.1, there exists a surjective R-algebra homomorphism $A_{\mathrm{red}} \to C$, so that we have $C = \Delta(C)$. Thus $p \in U(R/\mathfrak{a}_p)$ and $\phi_q(X)$ has a root in $R/\mathfrak{a}_p$ by [**5**, Lemma 3.3]. Therefore $p \in p^2R$ by Lemma 2.5, and $\phi_q(X)$ has a root in R by Lemma 2.7.

Conversely, suppose that $p \in p^2R$ and $\phi_q(X)$ has a root in R. Then it follows from Lemma 2.5 that $R \cong R/\mathfrak{a}_p \times R/pR$, and hence setting

$D = (R/pR)H$ we have $A \cong_R C \times D$, which implies

$$A_{\text{red}} \cong_R C_{\text{red}} \times D_{\text{red}}. \tag{2.2}$$

Note that $D_{\text{red}} \cong (R/pR)_{\text{red}}$ by Lemma 2.3, because $\operatorname{char}(R/pR) = p$ and $H = H_p$. Hence D_{red} is a surjective image of R, and therefore $D_{\text{red}} = \Delta(D_{\text{red}})$. Moreover, since $\phi_q(X)$ has a root in $R/\mathfrak{a}_p$, we have $C = \Delta(C)$ by [**5**, Lemma 3.3], so that $C_{\text{red}} = \Delta(C_{\text{red}})$. It thus follows from (2.2) that $A_{\text{red}} = \Delta(A_{\text{red}})$, which completes the proof. $\square$

Lemma 2.10. *Let S be an R-algebra, and let H be a cyclic group with $|H| = p^m$, where p is a prime number and m is a positive integer. If $(RH)_{\text{red}}$ is generated by idempotents over R, then $(SH)_{\text{red}}$ is generated by idempotents over S.*

Proof. Note that S_{red} is canonically an R_{red}-algebra. Since $(RH)_{\text{red}} \cong (R_{\text{red}}H)_{\text{red}}$ and $(SH)_{\text{red}} \cong (S_{\text{red}}H)_{\text{red}}$, replacing R and S by R_{red} and S_{red}, respectively, we may assume that R and S are reduced rings. Now suppose that $(RH)_{\text{red}}$ is generated by idempotents over R. Then it follows from Lemma 2.9 that $p \in p^2R$ and $\phi_q(X)$ has a root in R. Thus, a fortiori, $p \in p^2S$ and $\phi_q(X)$ has a root in S, so that $(SH)_{\text{red}}$ is generated by idempotents over S again by Lemma 2.9. This completes the proof. $\square$

Lemma 2.11. *Let $H_1, \dots, H_n$ be cyclic groups with $|H_i| = p_i^{m_i}$ for each i, where p_i is a prime number and m_i is a positive integer. Set $H = H_1 \times \cdots \times H_n$. If $\Delta((RH_i)_{\text{red}}) = (RH_i)_{\text{red}}$ for each i, then $\Delta((RH)_{\text{red}}) = (RH)_{\text{red}}$.*

Proof. We use induction on n. Let $H' = H_1 \times \cdots \times H_{n-1}$ and set $S = RH'$. Then, by the induction hypothesis, S_{red} is generated by idempotents over R. On the other hand, Lemma 2.10 implies that $(SH_n)_{\text{red}}$ is generated by idempotents over S. Note that $RH \cong_R SH_n$. Thus $\Delta((RH)_{\text{red}}) = (RH)_{\text{red}}$. $\square$

We are now ready to prove the following

Theorem 2.12. *Let R be a commutative ring with identity and G an abelian group. Then the following conditions are equivalent.*

(1) $(RG)_{\mathrm{red}}$ *is generated by idempotents over* R.

(2) G *is a torsion group,* $p \in p^2 R_{\mathrm{red}}$ *and* $\phi_{p^m}(X)$ *has a root in* R_{red} *for every* $p \in \operatorname{supp} G$ *and positive integer* m *with* $p^m \leq e_p(G)$.

(3) G *is a torsion group,* $n \in n^2 R_{\mathrm{red}}$ *and* $\phi_n(X)$ *has a root in* R_{red} *for every positive integer* n *such that* $n = \operatorname{ord}(g)$ *for some* $g \in G$.

Proof. We may assume that $R = R_{\mathrm{red}}$. Note that if $p \in p^2 R$, then $p^m \in p^{2m} R$ for every $m > 0$. Hence $p \in p^2 R$ for every $p \in \operatorname{supp} G$ if and only if $n \in n^2 R$ for every n such that $n = \operatorname{ord}(g)$ for some $g \in G$. The equivalence (2) $\Leftrightarrow$ (3) thus follows from [**5**, Corollary 3.5].

(1) $\Rightarrow$ (2) This follows from Lemmas 2.4, 2.6 and 2.8.

(2) $\Rightarrow$ (1) Let f be an element of RG. Then there exists a finite subgroup H of G such that $f \in RH$. Write $H = H_1 \times \cdots \times H_n$, where each H_i is a cyclic group whose order is a power of a prime number. It then follows from Lemma 2.9 that $\Delta((RH_i)_{\mathrm{red}}) = (RH_i)_{\mathrm{red}}$ for every i. Therefore, by Lemma 2.11, we know that

$$\bar{f} \in (RH)_{\mathrm{red}} = \Delta((RH)_{\mathrm{red}}) \subseteq \Delta((RG)_{\mathrm{red}}),$$

where $\bar{f}$ is the residue class of f in $(RG)_{\mathrm{red}}$. This implies $\Delta((RG)_{\mathrm{red}}) = (RG)_{\mathrm{red}}$, which completes the proof. □

Remark 2.13. For an integer n, we have $n \in n^2 R_{\mathrm{red}}$ if and only if $n^k \in n^{k+1} R$ for some positive integer k. In fact, suppose that $n \in n^2 R_{\mathrm{red}}$. Then $n(1 - nx) \in \operatorname{nil}(R)$ for some $x \in R$, which implies $n^k (1 - nx)^k = 0$ for a sufficiently large positive integer k. Thus $n^k \in n^{k+1} R$. Conversely suppose that $n^k \in n^{k+1} R$ for some $k > 0$. Then $n^k(1 - ny) = 0$ for some $y \in R$, so that $(n(1 - ny))^k = 0$. Thus $n(1 - ny) \in \operatorname{nil}(R)$, and hence $n \in n^2 R_{\mathrm{red}}$.

Remark 2.14. If $\operatorname{char} R > 0$, then $p \in p^2 R_{\mathrm{red}}$ for every prime number p, so that $n \in n^2 R_{\mathrm{red}}$ for every integer n. In fact, let $r = \operatorname{char} R_{\mathrm{red}}$, which is a square-free positive integer. If p is not a divisor of r, then $p \in U(R_{\mathrm{red}})$, and hence the assertion is obvious. If $p \mid r$, then set $s = r/p$, and let a and b be integers satisfying $ap + bs = 1$. Then $p = ap^2 + br$, so that $p \in p^2 R_{\mathrm{red}}$. Therefore, for the case of positive characteristic, we have the following

Theorem 2.15. *Let R be a commutative ring with identity and G an abelian group. If $\operatorname{char} R > 0$, then the following conditions are equivalent.*

(1) *$(RG)_{\mathrm{red}}$ is generated by idempotents over R.*

(2) *G is a torsion group, and $\phi_{p^m}(X)$ has a root in R_{red} for every $p \in \operatorname{supp} G$ and positive integer m with $p^m \leq e_p(G)$.*

(3) *G is a torsion group, and $\phi_n(X)$ has a root in R_{red} for every positive integer n such that $n = \operatorname{ord}(g)$ for some $g \in G$.*

We conclude this paper by showing that $\Delta((RG)_{\mathrm{red}}) = (RG)_{\mathrm{red}}$ is equivalent to $\Delta(RG) = RG$ in the case where $\operatorname{char} R = 0$ and R is indecomposable. For this purpose we need some results.

Lemma 2.16. *The following assertions hold.*

(1) *For $a \in R$, if $\overline{a} \in U(R_{\mathrm{red}})$, then $a \in U(R)$, where $\overline{a}$ denotes the image of a in R_{red}.*

(2) *Let n be a positive integer such that $n \in U(R_{\mathrm{red}})$. If $\phi_n(X)$ has a root in R_{red}, then $\phi_n(X)$ has a root in R.*

Proof. Since the assertion (1) is easily verified, we give a proof only for (2). Let c be an element of R whose image $\overline{c}$ in R_{red} is a root of $\phi_n(X)$. Then $(\phi_n(c))^m = 0$ for some $m > 0$. Since $n \in U(R)$ by (1), we can define a $\mathbf{Z}$-algebra homomorphism

$$\rho\colon S = \mathbf{Z}[n^{-1}][X]/(\phi_n(X)^m) \longrightarrow R$$

by $\rho(\overline{X}) = c$, where $\overline{X}$ is the image of X in S. On the other hand it follows from Lemma 2.17 below that S contains a root u of $\phi_n(X)$. Then $\rho(u)$ is a root of $\phi_n(X)$ in R, which completes the proof. □

Lemma 2.17. *Let n and m be positive integers. Then there exists a polynomial $u(X)$ in $\mathbf{Z}[n^{-1}][X]$ such that $\phi_n(u(X))$ is divisible by $\phi_n(X)^m$ in $\mathbf{Z}[n^{-1}][X]$.*

Proof. It suffices to prove the assertion for the case $m = 2$. Indeed, if $f(X)$ is a polynomial in $\mathbf{Z}[n^{-1}][X]$ such that $\phi_n(f(X))$ is divisible by $\phi_n(X)^2$ in $\mathbf{Z}[n^{-1}][X]$, then $\phi_n(f(f(X)))$ is divisible by $\phi_n(X)^4$, and $\phi_n(f(f(f(X))))$ is divisible by $\phi_n(X)^8$, and so on. Thus, for any $r > 0$, we can find $u(X)$ in $\mathbf{Z}[n^{-1}][X]$ such that $\phi_n(u(X))$ is divisible by $\phi_n(X)^{2^r}$ in $\mathbf{Z}[n^{-1}][X]$.

Now suppose that $m = 2$, and set $f(X) = X(n+1-X^n)/n$. Then $f(X) \in \mathbf{Z}[n^{-1}][X]$ and $f(\xi) = \xi$, where ξ denotes a primitive nth root of unity. Hence, setting $F(X) = \phi_n(f(X))$, we have $F(\xi) = \phi_n(\xi) = 0$. Moreover, since $f'(X) = (n+1)(1-X^n)/n$, we have $f'(\xi) = 0$, which implies $F'(\xi) = 0$, because $F'(X) = f'(X){\phi_n}'(f(X))$. From this we can easily see that $F(X)$ is divisible by $\phi_n(X)^2$ in $\mathbf{Z}[n^{-1}][X]$. This completes the proof. □

Theorem 2.18. *Let R be a commutative ring with identity and G an abelian group. Then the following conditions are equivalent.*

(1) *RG is generated by idempotents over R.*

(2) *$R_{\mathrm{red}}G$ is generated by idempotents over R_{red}.*

Proof. Since there exists a natural surjection $RG \to R_{\mathrm{red}}G$, the implication (1) $\Rightarrow$ (2) is obvious, while (2) $\Rightarrow$ (1) follows from Lemma 2.16 and [**5**, Theorem 4.2]. □

Now we have the following

Theorem 2.19. *Let R be a commutative ring with identity and G an abelian group. If R is indecomposable and $\operatorname{char} R = 0$, then the following conditions are equivalent.*

(1) *RG is generated by idempotents over R.*

(2) *$(RG)_{\mathrm{red}}$ is generated by idempotents over R.*

Proof. We have only to show (2) $\Rightarrow$ (1). Note that R_{red} is also indecomposable. Indeed, let e be an element of R whose image $\overline{e}$ in R_{red} is an idempotent. Then $\overline{e}(1-\overline{e}) = 0$, so that $e^n(1-e)^n = 0$ for some $n > 0$. Since $(e^n, (1-e)^n)R = R$ and R is indecomposable, it then follows that $e^n = 0$ or $(1-e)^n = 0$. Thus $\overline{e} = 0$ or $1 - \overline{e} = 0$, as desired. Moreover we have $\operatorname{char} R_{\mathrm{red}} = 0$. Hence, replacing R by R_{red}, we may assume that $R = R_{\mathrm{red}}$ by Theorem 2.18.

Let p be an element of $\operatorname{supp} G$. Then, by Theorem 2.12, we have $p \in p^2R$, so that $R \cong R/\mathfrak{a}_p \times R/pR$ by Lemma 2.5. Since R is indecomposable, from this it follows that $\mathfrak{a}_p = R$ or $pR = R$. However, if $\mathfrak{a}_p = R$, then $p = 0$ in R, which contradicts $\operatorname{char} R = 0$. Thus $pR = R$, and hence $\operatorname{supp} G \subseteq U(R)$. The assertion then follows from Theorem 2.12 and [**5**, Theorem 4.2]. $\square$

Remark 2.20. Theorem 2.19 does not hold if $\operatorname{char} R > 0$. In fact, let $R = \mathbf{F}_p$, the prime field of characteristic $p > 0$, and let G be a cyclic group of order p. Then $RG = R[X]/(X-1)^p$, which is indecomposable, and hence $\Delta(RG) \neq RG$. However, we have $(RG)_{\text{red}} = R$, which is trivially generated by idempotents over R.

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