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ON FINITE SUMS OF LAGUERRE POLYNOMIALS

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ABSTRACT. We evaluate a family of finite summations of Laguerre polynomials in terms of Laguerre polynomials themselves or in terms of a generalized hypergeometric function. As a byproduct, we find a Kummer-like transformation of the hypergeometric function ${}_2F_2$ in terms of a Laguerre polynomial. The results have applications in the theory of special functions as well as in analytic number theory.

0. Introduction. In recent years it has become evident that the Laguerre calculus [**3**] is important in the description of the Li criterion [**11**] for the Riemann hypothesis (RH) [**5, 6**]. The Laguerre polynomials and their properties also play an important role in many areas of mathematical physics, including random matrix theory, Fourier optics, and quantum mechanics. The Laguerre polynomials L_n^α appear prominently in the solution of the higher dimensional Kepler-Coulomb and harmonic oscillator problems, whose wavefunctions constitute "quantum shapelets" [**7**].

Since the Laguerre polynomials form a Sheffer sequence with a special generating function of exponential form, these polynomials are singled out in developing representations of special functions under fractional linear transformation [**6**]. In particular, the Laguerre polynomials have properties that are crucial in formulating recurrence and integral relations for transformations of the Riemann xi function, and therefore for describing the Li criterion. It appears that the particular Laguerre polynomials L_{n-1}^1 are very important as test functions for a Weil inner product whose nonnegativity is equivalent to the RH.

Given these several reasons to further investigate the classical orthogonal Laguerre polynomials [**4, 13**], we consider here finite sums that

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may be written in very similar form as either Laguerre polynomials themselves or as very closely related confluent or other hypergeometric functions. A very special case that has occurred very recently [**6**] provides a touchstone for these developments:

$$\sum_{j=1}^{n} \frac{L_{j-1}^{1}(x)}{j} = \frac{1}{x}[1 - L_n(x)]. \tag{0.1}$$

Our main result and its corollaries given in the next section generalize this relation. We then present a discussion that provides extension, elaboration and an alternative means to arrive at these finite sums. In fact, we find that relation (0.1) leads to a Kummer-like transformation formula for a particular generalized hypergeometric function ${}_2F_2$. A compendium of known both finite and infinite summations with Laguerre polynomials is given in [**10**, Section 48].

1. Finite sum of Laguerre polynomials. Let L_n^α be the Laguerre polynomial of degree n and parameter α (e.g., [**1**, Section 6.2]), and let ${}_pF_q$ be the generalized hypergeometric function [**2**]. *Let n and k be integers, with $1 \le k \le n$. Then we have*

Proposition 1. *We have*

$$\sum_{j=k}^{n} \frac{L_{j-k}^{k}(x)}{j} = \frac{1}{k}\binom{n}{k}\ {}_2F_2(k, k-n; k+1, k+1; x). \tag{1.1}$$

As special cases we have the following.

Corollary 1.

$$\sum_{j=1}^{n} \frac{L_{j-1}^{1}(x)}{j} = \frac{1}{x}[1 - L_n(x)]. \tag{1.2}$$

Corollary 2.

$$\sum_{j=2}^{n} \frac{L_{j-2}^{2}(x)}{j} = \frac{1}{x^2}[1 - L_n(x) - xL_{n-1}^{1}(x)]. \tag{1.3}$$

Corollary 3.

$$\sum_{j=3}^{n} \frac{L_{j-3}^{3}(x)}{j} = \frac{1}{x^3}[2 - 2L_n(x) - 3xL_{n-1}^{1}(x) + nxL_{n-1}(x)]. \tag{1.4}$$

Corollary 4.

$$\sum_{j=k}^{n} \frac{L_{j-k}^{k}(0)}{j} = \frac{1}{k}\binom{n}{k}. \tag{1.5}$$

In this section we provide two different proofs of Proposition 1, and then describe alternatives in the next.

First method of proof. We begin by using the power series form of $L_{j-k}^{k}(x)$ and then reordering sums:

(1.6)

$$\begin{aligned}
\sum_{j=k}^{n} \frac{L_{j-k}^{k}(x)}{j} &= \sum_{j=k}^{n} \frac{1}{j} \sum_{m=0}^{j-k} (-1)^m \binom{j}{m+k} \frac{x^m}{m!} \\
&= \sum_{j=k}^{n} \frac{1}{j} \sum_{m=k}^{j} (-1)^{m-k} \binom{j}{k} \frac{x^{m-k}}{(m-k)!} \\
&= \frac{1}{x^k} \sum_{m=k}^{n} \sum_{j=m}^{n} \frac{1}{j} (-1)^{m-k} \binom{j}{m} \frac{x^m}{(m-k)!} \\
&= \frac{(-1)^k}{x^k} \sum_{m=k}^{n} \frac{(-1)^m}{m} \binom{n}{m} \frac{x^m}{(m-k)!}.
\end{aligned}$$

We let $(a)_n = \Gamma(a+n)/\Gamma(a)$ denote as usual the Pochhammer symbol, where Γ is the Gamma function. In order to achieve hypergeometric form in equation (1.6) we shift the summation index and use the relations $(-n)_{m+k} = (-n)_k(k-n)_m$ and $(-n)_k = (-1)^k k!\binom{n}{k}$:

(1.7)
$$\begin{aligned}\sum_{j=k}^{n} \frac{L_{j-k}^k(x)}{j} &= \sum_{m=0}^{n-k} \frac{(-1)^m}{(m+k)} \binom{n}{m+k} \frac{x^m}{m!} \\ &= \frac{1}{k} \sum_{m=0}^{n-k} (-1)^m \binom{n}{m+k} \frac{(k)_m}{(k+1)_m} \frac{x^m}{m!} \\ &= \frac{1}{k} \sum_{m=0}^{n-k} \frac{(k)_m}{(k+1)_m} \frac{(-1)^k}{(m+k)!} (-n)_{m+k} \frac{x^m}{m!} \\ &= \frac{(-1)^k(-n)_k}{kk!} \sum_{m=0}^{n-k} \frac{(k)_m(k-n)_m}{[(k+1)_m]^2} \frac{x^m}{m!} \\ &= \frac{1}{k}\binom{n}{k}\ {}_2F_2(k, k-n; k+1, k+1; x),\end{aligned}$$

where at the end we have applied the power series definition of ${}_2F_2$.

Second method of proof. We use induction, noting that (1.1) holds for $k = 1$. We have (e.g., [**9**, page 1039] or [**1**, page 286])

$$\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x). \tag{1.8}$$

We assume that (1.1) holds at k. That it also holds at $k+1$ easily follows from the property

$$\frac{d}{dx}\ {}_2F_2(a, b; c, d; x) = \frac{ab}{cd}\ {}_2F_2(a+1, b+1; c+1, d+1; x). \tag{1.9}$$

We conclude this section by examining the $k = 0$ case corresponding to equation (1.1). We have

Proposition 2. *Let $H_n \equiv \sum_{k=1}^n 1/k$ be the nth harmonic number. Then we have*

$$\sum_{j=1}^{n} \frac{L_j(x)}{j} = -nx\ {}_3F_3(1, 1, 1-n; 2, 2, 2; x) + H_n. \tag{1.10}$$

Corollary 5. *We have*

$$\begin{aligned}\int_0^x \frac{1}{t}[L_n(t)-1]\,dt &= \sum_{j=1}^{n} \frac{L_j(x)}{j} - H_n \\ &= -nx\ {}_3F_3(1,1,1-n;2,2,2;x).\end{aligned} \tag{1.11}$$

In proving Proposition 2 we write

$$\begin{aligned}\sum_{j=1}^{n} \frac{L_j(x)}{j} &= \sum_{j=1}^{n} \frac{1}{j} \sum_{m=0}^{j} (-1)^m \binom{j}{m} \frac{x^m}{m!} \\ &= H_n + \sum_{j=1}^{n} \frac{1}{j} \sum_{m=1}^{j} (-1)^m \binom{j}{m} \frac{x^m}{m!},\end{aligned} \tag{1.12}$$

and proceed as above. Corollary 5 follows by applying the property (1.8). Equivalently, Corollary 5 may be determined by term-by-term integration. We have

$$\begin{aligned}\int_0^x \frac{1}{t}[L_n(t)-1]\,dt &= \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{m!} \int_0^x t^{m-1}\,dt \\ &= \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{m} \frac{x^m}{m!} \\ &= x \sum_{m=0}^{n-1} (-1)^{m+1} \binom{n}{m+1} \frac{1}{(m+1)^2} \frac{x^m}{m!} \\ &= x \sum_{m=0}^{n-1} (-1)^{m+1} \binom{n}{m+1} \frac{(1)_m^2}{(2)_m^2} \frac{x^m}{m!}.\end{aligned} \tag{1.13}$$

Then to reach the form in equation (1.11), we use

$$\binom{n}{m+1} = (-1)^{m+1} \frac{(-n)_{m+1}}{(m+1)!} = (-1)^{m+1} \frac{(-n)_1}{(2)_m} (1-n)_m. \tag{1.14}$$

2. Discussion: Alternative approaches and further results.

Contour integration. Contour integral representations of the Laguerre polynomials are known, including [**8**], (Vol. 2, p. 190)]
(2.1)
$$e^{-x/2}L_n^\alpha = \frac{(-1)^n}{2^\alpha}\frac{1}{2\pi i}\int_{(1+)}^{\infty} e^{-xz/2}\left(\frac{1+z}{1-z}\right)^{\nu/4}(1-z^2)^{(\alpha-1)/2}\,dz,$$

where $\nu = 4n+2\alpha+2$. The path of integration encircles $z=1$ in the positive direction, and closes at $\operatorname{Re} z = \infty$, $\operatorname{Im} z$ =constant. Equivalently, we have

$$L_n^\alpha(z) = \frac{\Gamma(n+\alpha+1)}{n!2\pi i}\int_{-\infty}^{(0+)}\left(1-\frac{z}{t}\right)^n e^t\frac{dt}{t^{\alpha+1}}, \tag{2.2}$$

where now the contour encircles the origin in the positive direction and closes at $\operatorname{Re} z = -\infty$. Additionally, contour integral representations of the confluent hypergeometric function ${}_1F_1$ may be used, as we have the relation

$$L_n^\alpha(z) = \binom{n+\alpha}{n}\,{}_1F_1(-n,\alpha+1;z). \tag{2.3}$$

We briefly illustrate the use of equation (2.2) in finding Corollary 1. We have

$$\begin{aligned}\sum_{j=1}^n \frac{L_{j-1}^1(z)}{j} &= \sum_{j=1}^n \frac{1}{2\pi i}\int_{-\infty}^{(0+)}\left(1-\frac{z}{t}\right)^{j-1}e^t\frac{dt}{t^2}\\ &= \frac{1}{z}\frac{1}{2\pi i}\int_{-\infty}^{(0+)}\left[1-\left(1-\frac{z}{t}\right)^n\right]e^t\frac{dt}{t}\\ &= \frac{1}{z}[1-L_n(z)],\end{aligned} \tag{2.4}$$

where we applied the residue theorem and the original representation (2.2). Similarly, a contour integral representation may be written for sums of the form $\sum_{j=1}^n x^j L_{j-1}^1(z)/j$ and related sums.

Use of other integral representations.

We next have

Proposition 3. *Let $\Gamma(x,y) = \Gamma(x) - \gamma(x,y)$ be the complementary incomplete Gamma function, defined by*

$$\Gamma(x,y) = \int_y^\infty e^{-t} t^{x-1}\, dt. \tag{2.5}$$

We have for $\alpha > 0$ and J_β the Bessel function of order β,

$$\sum_{j=1}^{n} \frac{L_{j-1}^{\alpha}(x)}{j} \tag{2.6}$$
$$= e^x x^{-\alpha/2} \left\{ \frac{1}{n!} \int_0^\infty t^{\alpha/2-1} \Gamma(n+1,t) J_\alpha(2\sqrt{xt})\, dt - x^{-1/2}[\Gamma(\alpha) - \Gamma(\alpha, x)] \right\}.$$

This proposition proceeds from the representation [**1**, page 286] for $\alpha > -1$,

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha/2}}{n!} \int_0^\infty t^{n+\alpha/2} J_\alpha(2\sqrt{xt}) e^{-t}\, dt. \tag{2.7}$$

We then interchange summation and integration, using [**9**, page 941]

$$\Gamma(n+1,x) = n! e^{-x} \sum_{m=0}^{n} \frac{x^m}{m!}, \tag{2.8}$$

so that

$$\sum_{j=1}^{n} \frac{L_{j-1}^{\alpha}(x)}{j} = e^x x^{-\alpha/2} \int_0^\infty t^{\alpha/2-1} \left[\frac{1}{n!} \Gamma(n+1,t) - e^{-t} \right] J_\alpha(2\sqrt{xt})\, dt. \tag{2.9}$$

With a change of variable, the second integral on the right side may be evaluated as a Mellin transform of a Gaussian function multiplying a

Bessel function [**9**, page 717]:

$$(2.10)\quad \int_0^\infty t^{\alpha/2-1}e^{-t}J_\alpha(2\sqrt{xt})\,dt = 2^{1-\alpha}x^{-\alpha/2}\int_0^\infty u^{\alpha-1}e^{-u^2/4x}J_\alpha(u)\,du = x^{-\alpha/2}\gamma(\alpha,x) = x^{-\alpha/2}[\Gamma(\alpha)-\Gamma(\alpha,x)], \quad \alpha>0.$$

Proposition 3 follows.

At $\alpha=1$, Corollary 1 may be recovered from Proposition 3 by noting that $\Gamma(1,x)=e^{-x}$ and by using Kummer's first transformation $e^{-z}\,{}_1F_1(a,b;z)={}_1F_1(b-a,b;-z)$ as applied to the Laguerre polynomial $L_n(x)={}_1F_1(-n,1;x)$.

By using the relation (2.8) and integrating termwise, or else by expressing $\gamma(\alpha,x)=(x^\alpha/\alpha)e^{-x}\,{}_1F_1(1,1+\alpha;x)$ in terms of the confluent hypergeometric function, the remaining integral in equation (2.6) may be explicitly evaluated as

Proposition 4. *For $\alpha>0$, we have*

$$(2.11)\quad \int_0^\infty t^{\alpha/2-1}\Gamma(n+1,t)J_\alpha(2\sqrt{xt})\,dt = 2^{1-\alpha}x^{-\alpha/2}\int_0^\infty u^{\alpha-1}\Gamma\left(n+1,\frac{u^2}{4x}\right)J_\alpha(u)\,du = \frac{x^{\alpha/2}\Gamma(n+\alpha+1)}{\alpha\Gamma(\alpha+1)}\,{}_2F_2(n+\alpha+1,\alpha;1+\alpha,1+\alpha;-x).$$

Proof. We write

$$(2.12)\quad \int_0^\infty u^{\alpha-1}\Gamma\left(n+1,\frac{u^2}{4x}\right)J_\alpha(u)\,du = n!\sum_{m=0}^n\frac{1}{m!}\int_0^\infty u^{\alpha-1}\left(\frac{u^2}{4x}\right)^m e^{-u^2/4x}J_\alpha(u)\,du = n!\frac{2^{\alpha-1}x^\alpha}{\Gamma(\alpha+1)}\sum_{m=0}^n\frac{\Gamma(m+\alpha)}{m!}\,{}_1F_1(\alpha+m,1+\alpha;-x),$$

where we have used [**9**, page 716]. We next interchange sums:

$$
\begin{aligned}
(2.13)\quad & \int_0^\infty u^{\alpha-1}\Gamma\left(n+1, \frac{u^2}{4x}\right) J_\alpha(u)\,du \\
&= n!\frac{2^{\alpha-1}x^\alpha}{\Gamma(\alpha+1)} \sum_{m=0}^{n} \frac{\Gamma(m+\alpha)}{m!} \sum_{\ell=0}^{\infty} \frac{(\alpha+m)_\ell}{(1+\alpha)_\ell} \frac{(-x)^\ell}{\ell!} \\
&= n!\frac{2^{\alpha-1}x^\alpha}{\Gamma(\alpha+1)} \sum_{\ell=0}^{\infty} \frac{(-x)^\ell}{(1+\alpha)_\ell \ell!} \sum_{m=0}^{n} \frac{\Gamma(m+\alpha)}{m!}(\alpha+m)_\ell \\
&= \frac{2^{\alpha-1}x^\alpha}{\Gamma(\alpha+1)} \sum_{\ell=0}^{\infty} \frac{(-x)^\ell}{(1+\alpha)_\ell \ell!} \frac{\Gamma(\alpha+\ell+n+1)}{(\alpha+\ell)} \\
&= \frac{2^{\alpha-1}x^\alpha}{\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+1)}{\alpha} \sum_{\ell=0}^{\infty} \frac{(n+\alpha+1)_\ell}{(1+\alpha)_\ell^2}(\alpha)_\ell \frac{(-x)^\ell}{\ell!} \\
&= \frac{2^{\alpha-1}x^\alpha}{\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+1)}{\alpha}\ {}_2F_2(n+\alpha+1, \alpha; 1+\alpha, 1+\alpha; -x),
\end{aligned}
$$

completing the Proposition.

Remarks. It may be verified that the use of ([**9** , page 718, 6.631.10, page 717, 6.631.5] for the $m = 0$ term) as applied to the integral of the right side of equation (2.6) nicely returns us to the finite Laguerre polynomial sum on the left side of Proposition 3.

The Laguerre polynomials may also be studied via their connection with the Whittaker function $M_{\mu,\nu}$,

$$
(2.14)\qquad L_\nu^\alpha(z) = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)} z^{-(\alpha+1)/2} e^{z/2} M_{\alpha+1/2+\nu,\alpha/2}(z).
$$

One may inquire as to the $n \to \infty$ limit in Propositions 1 and 2. We have [**5**]

$$
(2.15)\qquad \sum_{n=1}^{\infty} \frac{L_{n-1}^1(x)}{n} w^n = \frac{1}{x}\left[1 - e^{xw/(w-1)}\right], \quad |w| < 1,
$$

that is obtainable simply by integrating a standard generating function of the Laguerre polynomials. If we let $w \to 1^-$, then we obtain

$$
(2.16)\qquad \sum_{n=1}^{\infty} \frac{L_{n-1}^1(x)}{n} = \frac{1}{x}, \quad x > 0,
$$

in agreement with known results (e.g., [**9**, page 1038], [**10**, page 313]). Upon repeated differentiation, we have

$$\sum_{n=k+1}^{\infty} \frac{L_{n-k-1}^{k+1}(x)}{n} = \frac{k!}{x^{k+1}}, \quad x > 0. \tag{2.17}$$

Upon integration of equation (2.16), we have

$$\sum_{n=1}^{\infty} \frac{L_n(x)}{n} = -\ln x - \gamma, \quad x > 0, \tag{2.18}$$

where γ is the Euler constant, also in agreement with a known result [**10**, page 313], giving the obvious special case

$$\sum_{n=2}^{\infty} \frac{L_n(1)}{n} = -\gamma, \tag{2.19}$$

where we used $L_1(1) = 0$. Given the asymptotic form of H_n as $n \to \infty$, we may conclude the following from Proposition 2.

Corollary 6. *We have as $n \to \infty$,*

$$-n \ {}_3F_3(1, 1, 1-n; 2, 2, 2; 1) = -\ln n - 2\gamma + o(1). \tag{2.20}$$

This is a special case of the large n asymptotics of more general confluent functions and polynomials covered in Section 7.4.6 of [**12**].

Remarks. If we take the $n \to \infty$ limit in Proposition 3, using (2.8), we readily obtain the known extension ([**9**, page 1038] or [**10**, page 313]) of equation (2.16),

$$\sum_{j=1}^{\infty} \frac{L_{j-1}^{\alpha}(x)}{j} = e^x \frac{\Gamma(\alpha, x)}{x^{\alpha}}, \quad x > 0. \tag{2.21}$$

If we put $w = 1/2$ in equation (2.15) and integrate, we find

$$-\sum_{n=1}^{\infty} \frac{L_n(x)}{n 2^n} = \gamma + \Gamma(0, x) + \ln\left(\frac{x}{2}\right), \tag{2.22}$$

where $\Gamma(0,x) = -\mathrm{Ei}\,(-x)$ in terms of the exponential integral Ei. This equation is then equivalent to a known result [**10**, page 315]. More generally, we find by integrating equation (2.15) that

Corollary 7.

$$-\sum_{n=1}^{\infty} \frac{L_n(x)}{n} w^n = \gamma + \Gamma\left(0, \frac{wx}{1-w}\right) + \ln(wx), \quad x > 0. \tag{2.23}$$

In particular, we obtain for $w = -1$,

$$-\sum_{n=1}^{\infty} (-1)^n \frac{L_n(x)}{n} = \gamma + \Gamma\left(0, -\frac{x}{2}\right) + \ln(-x), \tag{2.24}$$

so that

$$-\sum_{n=2}^{\infty} (-1)^n \frac{L_n(1)}{n} = \gamma + \Gamma\left(0, -\frac{1}{2}\right) + \ln(-1). \tag{2.25}$$

This equation provides a companion to that of equation (2.19). The approximate numerical value of the constant on the right side is 0.12299576003835928068598 83.

We note in passing that the values $L_{n-1}^1(1)$ may be written as a certain alternating sum over the Lah numbers of combinatorics.

3. Kummer-like transformation for ${}_2F_2$. As a summary of Proposition 1, Corollary 1, and Propositions 3 and 4 at $k = \alpha = 1$, we have

Corollary 8.

$$\begin{aligned} \frac{1}{x}[1 - L_n(x)] &= n\ {}_2F_2(1, 1-n; 2, 2; x) \\ &= (n+1)e^x\ {}_2F_2(n+2, 1; 2, 2; -x) - \frac{1}{x}(e^x - 1), \end{aligned} \tag{3.1}$$

one conclusion of which is

$$L_n(x) = e^x[1 - (n+1)x\ {}_2F_2(n+2, 1; 2, 2; -x)]. \tag{3.2}$$

Corollary 8 provides a Kummer-like transformation for the hypergeometric function ${}_2F_2$. This identity, as well as the expression of equation (3.2), appears to be new. Equation (3.2) may be successively differentiated with respect to x to find ${}_2F_2$ expressions for the Laguerre polynomials L_{n-k}^k.

We may now give a direct proof of the first and last members of Corollary 8, thereby also furnishing equation (3.2).

Proof. We have

$$
\begin{aligned}
(3.3)\quad & e^x\, {}_2F_2(n+2,1;2,2;-x) \\
&= e^x \sum_{\ell=0}^{\infty} \frac{(n+2)_\ell}{[(\ell+1)!]^2}(-x)^\ell \\
&= \sum_{j,\ell=0}^{\infty} \frac{1}{j!} \frac{(n+2)_\ell}{[(\ell+1)!]^2}(-1)^\ell x^{\ell+j} \\
&= \sum_{m=\ell}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{(m-\ell)!} \frac{(n+2)_\ell}{[(\ell+1)!]^2}(-1)^\ell x^m \\
&= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{(-1)^\ell}{(m-\ell)!} \frac{(n+2)_\ell}{[(\ell+1)!]^2} x^m \\
&= \frac{1}{(n+1)} \sum_{m=0}^{\infty} \left[\frac{1}{(m+1)!} - \frac{(m-n)!}{\Gamma(-n)[(m+1)!]^2}\right] x^m \\
&= \frac{1}{(n+1)} \left[\frac{1}{x}(e^x-1) - n! \sum_{m=0}^{n-1} \frac{(-1)^{m+1}}{(n-m-1)!} \frac{x^m}{[(m+1)!]^2}\right] \\
&= \frac{1}{(n+1)} \left[\frac{1}{x}(e^x-1) - \frac{1}{x}[L_n(x)-1]\right].
\end{aligned}
$$

We may proceed similarly for more general sums with integers j_1, $j_2 \geq 1$,

$$
\begin{aligned}
(3.4)\quad & e^x\, {}_2F_2(n+2,1;1+j_1,1+j_2;-x) \\
&= e^x j_1! j_2! \sum_{\ell=0}^{\infty} \frac{(n+2)_\ell}{(\ell+j_1)!(\ell+j_2)!}(-x)^\ell
\end{aligned}
$$

$$= j_1!j_2! \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{(-1)^\ell}{(m-\ell)!} \frac{(n+2)_\ell}{(\ell+j_1)!(\ell+j_2)!} x^m$$
$$= \sum_{m=0}^{\infty} {}_3F_2(1,-m,n+2;1+j_1,1+j_2;1)\frac{x^m}{m!},$$

where the last expression follows simply from using $1/(m-\ell)! = (-1)^\ell(-m)_\ell/m!$. In this way, we find that the functions $e^x\,{}_2F_2(n+2,1;1+j_1,1+j_1;-x)$ contain a term with $L_{n-j_1+1}(x)/x^{j_1}$. We record the following special cases of equation (3.4) in the next proposition.

Proposition 5. *We have*

(3.5)
$$e^x\,{}_2F_2(n+2,1;2,3;-x) = \frac{2}{(n+1)x}\left[e^x - \frac{1}{n}L^1_{n-1}(x)\right],$$

(3.6)
$$e^x\,{}_2F_2(n+2,1;3,3;-x) = \frac{4}{n(n+1)x^2}\left[e^x(nx-1) + L_{n-1}(x)\right],$$

$$e^x\,{}_2F_2(n+2,1;3,4;-x) = \frac{6}{n(n+1)x^2} \times \left[(nx-2)e^x + \frac{2}{n-1}L^1_{n-2}(x)\right], \tag{3.7}$$

$$e^x\,{}_2F_2(n+2,1;4,4;-x) = \frac{9e^x(n^2x^2 - nx^2 - 4nx + 4x + 4)}{n(n^2-1)x^3} - \frac{36L_{n-2}(x)}{n(n^2-1)x^3}. \tag{3.8}$$

4. Final proposition. Another way to express Corollary 1, for instance, is as

$$\sum_{j=1}^{n} \frac{L^1_{j-1}(x)}{j} = \int_0^\infty f_1(t,x)\,dt = \frac{1}{x}[1 - L_n(x)], \tag{4.1a}$$

where

$$f_1(t,x) \equiv \sum_{j=1}^{n} e^{-jt} L^1_{j-1}(x). \tag{4.1b}$$

It is of interest to have an alternative form of this function. More generally, we have the following, providing another representation of the finite sum of Proposition 1.

Proposition 6. *For $n \geq k \geq 1$ integers, we have*

$$\sum_{j=k}^{n} \frac{L_{j-k}^{k}(x)}{j} = \int_0^\infty f_k(t,x)\,dt, \tag{4.2a}$$

where

$$f_k(u,x) = \binom{n}{k} e^{-ku}\,{}_1F_1(k-n,k+1;xe^{-u}) = e^{-ku} L_{n-k}^{k}(xe^{-u}). \tag{4.2b}$$

Proof. We have (cf. [**9**, page 845])

$$\begin{aligned} {}_2F_2(k,k-n;k+1,k+1;x) &= k\int_0^1 t^{k-1}\,{}_1F_1(k-n,k+1;xt)\,dt \\ &= \frac{k}{\binom{n}{k}} \int_0^1 t^{k-1} L_{n-k}^{k}(xt)\,dt. \end{aligned} \tag{4.3}$$

Therefore, with a change of variable and the use of Proposition 1 we have

$$\sum_{j=k}^{n} \frac{L_{j-k}^{k}(x)}{j} = \binom{n}{k} \int_0^\infty e^{-ku}\,{}_1F_1(k-n,k+1;xe^{-u})\,du, \tag{4.4}$$

and Proposition 6 follows.

Remark. Equation (4.3) can be proved directly by either repeated integration by parts or by the use of induction and properties (1.8) and (1.9).

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