ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 1, 2011

# NORMAL FAMILIES AND SHARED SETS OF MEROMORPHIC FUNCTIONS

JUN-FAN CHEN

ABSTRACT. Let $k$, $m$ and $n$ be three positive integers with $m \geq 2$, let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, all of whose zeros are of multiplicity at least $k+1$, and let the sets $S_1 = \{a_1, a_2, \dots, a_m\}$ and $S_2 = \{b_1, b_2, \dots, b_n\}$, where $a_1, a_2, \dots, a_m$ are $m$ distinct finite complex numbers, and $b_1, b_2, \dots, b_n$ are $n$ distinct finite complex numbers. If, for every $f \in \mathcal{F}$, $f^{(k)}(z) \in S_1 \Rightarrow f(z) \in S_2$, then $\mathcal{F}$ is normal in $D$. The condition that the zeros of functions in $\mathcal{F}$ are of multiplicity at least $k+1$ cannot be weakened, and the corresponding result is no longer true for $m = 1$.

**1. Introduction and main results.** Let $D$ be a domain on $\mathbf{C}$, and let $\mathcal{F}$ be a family of meromorphic functions defined in $D$. The family $\mathcal{F}$ is said to be normal in $D$, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ that converges, spherically locally uniformly in $D$, to a meromorphic function or $\infty$ (see Hayman [**4**], Schiff [**8**], Yang [**10**]).

Let $f$ and $g$ be two functions meromorphic in $D$ on $\mathbf{C}$, let $a \in \mathbf{C} \cup \{\infty\}$, and let $S_1$ and $S_2$ be two sets of complex numbers. If $g(z) \in S_2$ whenever $f(z) \in S_1$, then we write $f(z) \in S_1 \Rightarrow g(z) \in S_2$. If $f(z) \in S_1 \Rightarrow g(z) \in S_2$ and $g(z) \in S_2 \Rightarrow f(z) \in S_1$, then we write $f(z) \in S_1 \Leftrightarrow g(z) \in S_2$. If $f(z) \in S_1 \Leftrightarrow g(z) \in S_1$, then we say that $f$ and $g$ share the set $S_1$ in $D$. In particular, if $f(z) \in S_1 \Leftrightarrow g(z) \in S_1$ and $S_1 = \{a\}$, then we say that $f$ and $g$ share the value $a$ in $D$.

In [**9**], Schwick proved the following result.

**Theorem A.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, and let $a_1$, $a_2$ and $a_3$ be three distinct finite complex num-*

2010 AMS *Mathematics subject classification.* Primary 30D45.
*Keywords and phrases.* Meromorphic function, normal family, shared set.
Project supported by the NNSF of China (Grant No. 10771076), the NSF of Guangdong Province, China (Grant No. 07006700) and the NSF of Fujian Province, China (Grant No. 2010J05003).
Received by the editors on July 29, 2008, and in revised form on November 15, 2008.

DOI:10.1216/RMJ-2011-41-1-37 

*bers. If, for every $f \in \mathcal{F}$, $f$ and $f'$ share $a_1$, $a_2$ and $a_3$, then $\mathcal{F}$ is normal in $D$.*

Actually, Theorem A remains valid when $f$ and $f'$ share two (rather than three) distinct finite complex numbers in a domain $D$, as is shown by the following result of Pang and Zalcman [**7**].

**Theorem B.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, and let $a$ and $b$ be two distinct finite complex numbers. If, for every $f \in \mathcal{F}$, $f$ and $f'$ share $a$ and $b$, then $\mathcal{F}$ is normal in $D$.*

Afterward, Fang and Zalcman [**3**] observed that Theorem B admitted the following generalization obtained by replacing $f'$ by $f^{(k)}$ for a positive integer $k$.

**Theorem C.** *Let $k$ be a positive integer, let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, all of whose zeros are of multiplicity at least $k+1$, and let $a$ and $b$ be two nonzero finite complex numbers. If, for every $f \in \mathcal{F}$, $f^{(k)}(z) = a \Leftrightarrow f(z) = b$, then $\mathcal{F}$ is normal in $D$.*

On the other hand, Fang [**2**], Liu and Pang [**5**] extended Schwick's result in view of shared sets. Indeed, they proved the following theorem.

**Theorem D.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, and let $a_1$, $a_2$ and $a_3$ be three distinct finite complex numbers. If, for every $f \in \mathcal{F}$, $f$ and $f'$ share the set $S = \{a_1, a_2, a_3\}$, then $\mathcal{F}$ is normal in $D$.*

Now it is natural to ask whether Theorem C still holds if the assumption $f^{(k)}(z) = a \Leftrightarrow f(z) = b$ is replaced by the assumption $f^{(k)}(z) \in S_1 \Leftrightarrow f(z) \in S_2$ for two sets $S_1$ and $S_2$. In this paper, we give an affirmative answer to this question. In fact, we prove the following more general result, which is also an extension of Theorem D.

**Theorem 1.** *Let $k$, $m$ and $n$ be three positive integers with $m \geq 2$, let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, all of* *whose zeros are of multiplicity at least $k+1$, and let the sets $S_1 = \{a_1, a_2, \dots, a_m\}$ and $S_2 = \{b_1, b_2, \dots, b_n\}$, where $a_1, a_2, \dots, a_m$ are $m$ distinct finite complex numbers, and $b_1, b_2, \dots, b_n$ are $n$ distinct finite complex numbers. If, for every $f \in \mathcal{F}$, $f^{(k)}(z) \in S_1 \Rightarrow f(z) \in S_2$, then $\mathcal{F}$ is normal in $D$.*

**Example 1.** Let $k$ be a positive integer, let $D = \{z : |z| < 1\}$, let the set $S_1 = \{1\}$, and let $\mathcal{F} = \{f_n : n = 1, 2, 3, \dots\}$, where

$$f_n(z) = \frac{(z + (1/(k+1)n))^{k+1}}{k!(z + (1/n))} = \frac{z^k}{k!} + p_{k-2}(z) + \frac{(-k/n)^{k+1}}{(k+1)^{k+1} k!(z + (1/n))},$$

where $p_{k-2}(z)$ is a polynomial of degree $k-2$. Then, for every $f_n \in \mathcal{F}$, $f_n$ has a single zero of multiplicity $k+1$, and

$$f_n^{(k)}(z) = 1 - \frac{(k/n)^{k+1}}{(k+1)^{k+1}(z + (1/n))^{k+1}}.$$

Thus $f_n^{(k)}(z) \neq 1$, so that $f_n^{(k)}(z) \in S_1 \Rightarrow f_n(z) \in S_2$ for any set $S_2$; but since $f_n$ takes on the values 0 and $\infty$ in any fixed neighborhood of 0 if $n$ is sufficiently large, $\mathcal{F}$ fails to be normal in $D$. This shows that Theorem 1 is no longer true for $m = 1$.

**Example 2.** Let $k$ be a positive integer, let $D = \{z : |z| < 1\}$, let the set $S_1 = \{1/2, 1/3\}$, and let $\mathcal{F} = \{f_n(z) : n = 1, 2, 3, \dots\}$, where $f_n(z) = nz^k$. Then, for every $f_n \in \mathcal{F}$, $f_n$ has a single zero of multiplicity $k$, and $f_n^{(k)}(z) = nk!$ in $D$. It now follows that $f_n^{(k)}(z) \neq 1/2, 1/3$, so that $f_n^{(k)}(z) \in S_1 \Rightarrow f_n(z) \in S_2$ for any set $S_2$. But $\mathcal{F}$ clearly fails to be normal in $D$. This shows that the condition in Theorem 1 that the zeros of functions in $\mathcal{F}$ are of multiplicity at least $k+1$ cannot be weakened.

**2. Some lemmas.** We first require the following renormalization result, which has become a standard tool in the study of normal families.

**Lemma 1** [**6**, Lemma 2], cf. [**11**, pages 216–217]. *Let $k$ be a positive integer, and let $\mathcal{F}$ be a family of functions meromorphic in the unit disk*

*$\Delta$, all of whose zeros have multiplicity at least $k$, and suppose that there exists an $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$,*

(b) *points $z_n$, $|z_n| < r$,*

(c) *functions $f_n \in \mathcal{F}$, and*

(d) *positive numbers $\rho_n \to 0$*

*such that*

$$\frac{f_n(z_n + \rho_n \zeta)}{\rho_n^\alpha} = g_n(\zeta) \longrightarrow g(\zeta)$$

*locally uniformly with respect to the spherical metric, where $g$ is a nonconstant meromorphic function on $\mathbf{C}$, all of whose zeros have multiplicity at least $k$, such that $g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1$. In particular, $g$ has order at most two.*

**Lemma 2** [**1**, Lemma 4]. *Let $k$ be a positive integer, let $f$ be a meromorphic function of finite order, all of whose zeros are of multiplicity at least $k$, and let $a$ be a nonzero finite complex number. If $f$ and $f^{(k)}$ share $0$, and $f^{(k)}(z) \neq a$, then $f(z)$ is a constant.*

**Lemma 3** [**4**, Theorem 3.4]. *Let $k$ be a positive integer, and let $f$ be a transcendental meromorphic function. Then $f^{(k)}(z)$ assumes every finite complex number with at most one exception infinitely often.*

Finally, a simple application of Lemma 3 yields the following result.

**Lemma 4.** *Let $k$ be a positive integer, and let $f$ be a meromorphic function. If $f^{(k)}(z)$ omits two finite complex numbers, then $f^{(k)}(z)$ is a constant.*

*Proof.* It is clear that no nonconstant rational function omits two finite complex numbers. Then $f^{(k)}$ is a transcendental meromorphic function; however, Lemma 3 shows that $f^{(k)}$ assumes every finite complex number with at most one exception infinitely often, a contradiction. Thus $f^{(k)}(z)$ is a constant.

**3. Proof of Theorem 1.** Without loss of generality, we may assume that $D = \Delta$, the unit disc, $S_1 = \{a_1, a_2\}$, and $S_2 = \{b_1, b_2, \dots, b_n\}$, where $a_1$, $a_2$ are two distinct finite complex numbers, and $b_1, b_2, \dots, b_n$ are $n$ distinct finite complex numbers. Suppose that $\mathcal{F}$ is not normal on $\Delta$ and for every $f \in \mathcal{F}$, $f^{(k)}(z) \in S_1 \Rightarrow f(z) \in S_2$. Then by Lemma 1 we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, and $\rho_n \to 0^+$ such that

$$\frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} = g_n(\zeta) \longrightarrow g(\zeta)$$

locally uniformly with respect to the spherical metric, where $g$ is a nonconstant meromorphic function on $\mathbf{C}$, all of whose zeros have multiplicity at least $k+1$. Moreover, $g$ has order at most two. For convenience, for every finite complex value $a$ we define the set

$$T_a = \{f(z) : f^{(k)}(z) = a, f \in \mathcal{F}, z \in \Delta\}.$$

We will prove two statements under the assumptions above:

(i) if $0 \in S_1$ then the functions $g$ and $g^{(k)}$ share the value zero, and

(ii) if $g^{(k)}$ assumes the value $a \neq 0$, then the set $T_a$ is an infinite set.

To prove (i), note that since all zeros of $g$ are of multiplicity at least $k+1$, we have $g(\zeta) = 0 \Rightarrow g^{(k)}(\zeta) = 0$. Suppose now that $g^{(k)}(\zeta_0) = 0$. Clearly, $g^{(k)}(\zeta) \not\equiv 0$, for otherwise $g$ would be a polynomial of degree at most $k$, and so could not have zeros of multiplicity at least $k+1$. Then, by Hurwitz's theorem, there exist $\zeta_n$, $\zeta_n \to \zeta_0$, such that, for $n$ sufficiently large,

$$0 = g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n),$$

so that $f_n^{(k)}(z_n + \rho_n \zeta_n) = 0 \in S_1$. Thus $f_n(z_n + \rho_n \zeta_n) \in S_2$. If $f_n(z_n + \rho_n \zeta_n) \in S_2 \backslash \{0\}$ for all but a finite number of $n$, then $g_n(\zeta_n) \to \infty$, since $\rho_n \to 0$, and this means that $g_n(\zeta_n) \to g(\zeta_0) = \infty$, which contradicts that $g^{(k)}(\zeta_0) = 0$, as assumed. Therefore we must have that $f_n(z_n + \rho_n \zeta_n) = 0$ for $n$ sufficiently large. Hence $g_n(\zeta_n) \to g(\zeta_0) = 0$, and we have shown that $g^{(k)}(\zeta) = 0 \Rightarrow g(\zeta) = 0$, yielding that $g$ and $g^{(k)}$ share the value zero. This proves (i).

Now suppose that $g^{(k)}(\zeta_0) = a \neq 0$ and $g(\zeta_0) = b$. Note that every zero of $g$ has multiplicity at least $k+1$, so that if $b$ were 0 then $a$ would be zero also, so it follows that $b \neq 0$. For the same reason, $g^{(k)}(\zeta) \not\equiv a$. Then, by Hurwitz's theorem, in every neighborhood of $\zeta_0$ all but a finite number of functions $g_n^{(k)}(\zeta)$ assume the value $a$. Thus there exists a sequence $\zeta_n$, $\zeta_n \to \zeta_0$, such that $a = g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n)$. Also, $g_n(\zeta_n) \to b$, and so $f_n(z_n + \rho_n \zeta_n) \to 0$. But $f_n(z_n + \rho_n \zeta_n)$ cannot be zero, as this would imply that $f_n^{(k)}(z_n + \rho_n \zeta_n) = g_n^{(k)}(\zeta_n)$ is zero, which is not the case for at least all but a finite number of $n$. It then follows that $\{f_n(z_n + \rho_n \zeta_n)\}$ is an infinite set. Hence, the set $T_a$ is an infinite set. This proves (ii).

To prove the theorem, if $0 \in S_1$ then (i) applies, and thus, by Lemma 2, $g^{(k)}$ must assume all nonzero finite values. Also, $S_1$ contains a nonzero value, say $a_2$. Now by (ii), $T_{a_2}$ is an infinite set and cannot be contained in the finite set $S_2$. If $0 \notin S_1$, then, by Lemma 4, $g^{(k)}$ assumes at least one of the values in $S_1$, say $a_1$, and again by (ii) the set $T_{a_1}$ is an infinite set and cannot be contained in the finite set $S_2$. Thus, in either case, we have arrived at a contradiction, and the theorem is proved.

**Acknowledgments.** The author is very grateful to the referee for reading through the manuscript and giving a lot of valuable suggestions to greatly improve the presentation.

## REFERENCES

**1.** H.H. Chen and M.L. Fang, *Shared values and normal families of meromorphic functions*, J. Math. Anal. Appl. **260** (2001), 124–132.

**2.** M.L. Fang, *A note on sharing values and normality*, Chinese J. Math. Stud. **29** (1996), 29–32.

**3.** M.L. Fang and L. Zalcman, *Normal families and shared values of meromorphic functions* III, Comput. Methods Funct. Theory **2** (2002), 385–395.

**4.** W.K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

**5.** X.J. Liu and X.C. Pang, *Shared values and normal families*, Acta Math. Sinica Chinese Ser. **50** (2007), 409–412.

**6.** X.C. Pang and L. Zalcman, *Normal families and shared values*, Bull. London Math. Soc. **32** (2000), 325–331.

**7.** ———, *Normality and shared values*, Ark. Mat. **38** (2000), 171–182.

**8.** J. Schiff, *Normal families*, Springer-Verlag, New York, 1993.

**9.** W. Schwick, *Sharing values and normality*, Arch. Math. (Basel) **59** (1992), 50–54.

**10.** L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993.

**11.** L. Zalcman, *Normal families*: *New perspectives*, Bull. Amer. Math. Soc. **35** (1998), 215–230.

Department of Mathematics, Fujian Normal University, Fuzhou 350007, Fujian Province, P.R. China
**Email address: junfanchen@163.com**