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ON POLAR LEGENDRE POLYNOMIALS

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ABSTRACT. In this paper we introduce a new class of polynomials $\{P_n\}$, called polar Legendre polynomials. They appear as solutions of an inverse Gauss problem for the equilibrium position of a field of forces with $n+1$ unit masses. We study algebraic, differential, and asymptotic properties of these polynomials, which are simultaneously orthogonal with respect to a differential operator and a discrete-continuous Sobolev type inner product.

1. Introduction. Let $\{L_n\}_{n\in\mathbf{N}}$ be the monic Legendre polynomials. It is well known that L_n satisfies the following orthogonality relation

$$\int_{-1}^{1} L_n(x)x^k\,dx = 0, \quad k = 0, 1, \dots, n-1, \tag{1}$$

the second order linear differential equation

$$-n(n+1)L_n(z) = \big((1-z^2)L_n'(z)\big)', \tag{2}$$

and the so-called *structure relation* [**7**, (4.5.5)]

$$(z^2-1)L_n'(z) = n\,L_{n+1}(z) - \frac{n^2(n+1)}{4n^2-1}L_{n-1}(z). \tag{3}$$

For a fixed complex number ζ, that in the sequel is called the pole, let us define $P_n = P_{\zeta,n}$ as a monic polynomial such that

$$(n+1)\,L_n(z) = ((z-\zeta)\,P_n(z))' = P_n(z) + (z-\zeta)P_n'(z). \tag{4}$$

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P_n is called the nth *polar Legendre polynomial*. Obviously, P_n is a monic polynomial of degree n. From (1) and (4) the following relation reads

$$\int_{-1}^{1} [P_n(x) + (x-\zeta)P_n'(x)]x^k\,dx = 0, \quad k = 0, 1, \dots, n-1. \tag{5}$$

This type of "orthogonality relation" generated by differential operators was introduced in [**2**], where the existence and uniqueness conditions for more general differential expressions were studied in detail.

Notice that the polynomial

$$\Pi_{\zeta,n+1}(z) = (z-\zeta)\,P_n(z) \tag{6}$$

is a primitive of $(n+1)\,L_n(z)$, such that $\Pi_{\zeta,n+1}(\zeta) = 0$; that is,

$$\Pi_{\zeta,n+1}(z) = (n+1)\int_{\zeta}^{z} L_n(t)\,dt. \tag{7}$$

$\Pi_{\zeta,n+1}$ will be called the *primitive Legendre polynomial*. The properties of $P_n = P_{\zeta,n}$ and $\Pi_{\zeta,n+1}$ are clearly closely related.

It is important to observe that, since the functions that we are considering are entire functions, we can assume that the definite integrals appearing in this paper are line integrals defined on a straight line segment with initial point in the lower limit of integration and end point in the upper limit of integration.

Now, combining (2) and (4) and integrating from ζ to z, we get the fundamental formula

$$n\,(z-\zeta)\,P_n(z) = (1-\zeta^2)L_n'(\zeta) - (1-z^2)L_n'(z). \tag{8}$$

Furthermore, from (4) it is easy to see that $\Pi_{\zeta,n+1}(z)$ is the $(n+1)$th monic orthogonal polynomial with respect to the Sobolev-type inner product (called "discrete-continuous type," see [**1**])

$$\langle p, q\rangle = p(\zeta)q(\zeta) + \int_{-1}^{1} p'(x)\,q'(x)\,dx.$$

This inner product was firstly introduced in [**4**], where necessary and sufficient conditions under which such Sobolev-type orthogonal polynomials satisfy a linear differential equation of spectral type with polynomial coefficients are studied.

The location of critical points of a given class of polynomials has many physical and geometrical interpretations. Let us consider, for instance, a field of forces given by a system of n masses m_j, $1 \leq j \leq n$, at the fixed points z_j, $1 \leq j \leq n$, that repels a movable unit mass at z according to the inverse distance law. Let $Q_m(z) = (z - z_1)^{m_1} \cdot (z - z_2)^{m_2} \cdots (z - z_n)^{m_n}$ where $m = m_1 + m_2 + \cdots + m_n$. The logarithmic derivative of $Q_m(z)$ is

$$(9) \quad \frac{d(\log(Q_m(z)))}{dz} = \frac{Q_m'(z)}{Q_m(z)} = \frac{m_1}{(z - z_1)} + \frac{m_2}{(z - z_2)} + \cdots + \frac{m_n}{(z - z_n)}.$$

The conjugate of $m_j/(z - z_j)$ is a vector directed from z_j to z, so this vector represents the force at the movable unit mass z due to a single fixed particle at z_j. By (9) the positions of equilibrium in the field of force coincide with those zeros of Q_m' that are not zeros of Q_m. In particular, all multiple zeros of Q_m yield equilibrium positions. This result is known as Gauss's theorem ([**6**, Chapter 3, Theorem 1.2.1]).

Now, let us consider the following inverse problem: if $z_1', z_2', \ldots, z_n'$ are the zeros of the orthogonal polynomial L_n which we assume to be the equilibrium positions of a field of forces with $n+1$ unit masses, one of which is given at point ζ, what is the location of the remaining masses?

Let P_n be the monic polynomial whose zeros are the remaining equilibrium positions. From (4) and (6)

$$(10) \qquad \frac{(n+1)\, L_n(z)}{(z - \zeta) P_n(z)} = \frac{1}{z - \zeta} + \frac{P_n'(z)}{P_n(z)} = \frac{\Pi_{\zeta,n+1}'(z)}{\Pi_{\zeta,n+1}(z)}.$$

Then, according to (9), (10) and the above interpretation of the logarithmic derivative, the location of the remaining unit masses is the zeros of polynomial P_n or, equivalently, the poles of (10). For this reason P_n is said to be a polar polynomial.

The main purpose of this paper is to study some algebraic, differential and analytic properties of the polar Legendre polynomials or, equivalently, of the primitive Legendre polynomials. The paper is organized as follows. In Section 2 we study the orthogonality relations and recurrence relations of polar Legendre polynomials and Section 3 is devoted to the study of the location of zeros and the asymptotic behavior of zeros and polynomials for the family $\{P_n\}$.

2. Orthogonality and recurrence relations. Besides the well-known results on orthogonality mentioned in the previous section, the following additional orthogonality relations between $\{L_n\}$ and $\{P_n\}$ hold.

Theorem 1. *The polar Legendre polynomial P_n with pole $\zeta \in \mathbf{C}$ verifies*

$$\int_{-1}^{1} [P_n(x) + (x-\zeta)P_n'(x)]\, L_m(x)\, dx = \begin{cases} 0 & m \neq n, \\ (n+1)\|L_n\|^2 & m = n, \end{cases} \tag{11}$$

where $\|L_n\|^2 = \int_{-1}^{1} L_n^2(x)\, dx$.

Furthermore, if $n > 0$ then

$$\int_{-1}^{1} (x-\zeta)P_n(x)L_m(x)\, dx = \begin{cases} (2/n)(1-\zeta^2)L_n'(\zeta) & m = 0, \\ 0 & 0 < m < (n-1), \\ -[n(n+1)/(4n^2-1)]\|L_{n-1}\|^2 & m = n-1, \\ 0 & m = n, \\ \|L_{n+1}\|^2 & m = n+1, \\ 0 & (n+1) < m. \end{cases} \tag{12}$$

Proof. From (1) and (4), the proof of (11) is straightforward. To prove (12), by the fundamental formula (8) and the structure relation (3),

$$\begin{aligned}
\int_{-1}^{1} & (x-\zeta)P_n(x)L_m(x)\,dx \\
&= \int_{-1}^{1}\left(\frac{1-\zeta^2}{n}L_n'(\zeta)+\frac{x^2-1}{n}L_n'(x)\right)L_m(x)\,dx \\
&= \frac{1-\zeta^2}{n}L_n'(\zeta)\int_{-1}^{1}L_m(x)\,dx \\
&\quad + \frac{1}{n}\int_{-1}^{1}(x^2-1)L_n'(x)\,L_m(x)\,dx \\
&= \frac{1-\zeta^2}{n}L_n'(\zeta)\int_{-1}^{1}L_m(x)\,dx \\
&\quad + \int_{-1}^{1}L_{n+1}(x)\,L_m(x)\,dx \\
&\quad - \frac{n(n+1)}{4n^2-1}\int_{-1}^{1}L_{n-1}(x)\,L_m(x)\,dx,
\end{aligned}$$

and by (1), formula (12) is also straightforward. □

As a consequence of these orthogonality relations, let us prove now a recurrence relation for the polar Legendre polynomials.

Theorem 2. *The polar Legendre polynomials $\{P_n\}$ with pole $\zeta \in \mathbf{C}$, satisfy the following recurrence relation*

$$P_{n+1}(z) = z\,P_n(z) + a_n P_{n-1}(z) + b_n, \tag{13}$$

for $n > 1$, where $P_0(z) \equiv 1$ and $P_1(z) = z + \zeta$,

$$a_n = \frac{1-n^2}{4n^2-1} \text{ and } b_n = \frac{\zeta^2-1}{n}L_n'(\zeta). \tag{14}$$

Proof. Let $\{\alpha_{n,0},\, \alpha_{n,1}, \dots, \alpha_{n,n}\}$ be such that

$$(z-\zeta)P_n(z) = \sum_{k=0}^{n+1}\alpha_{n,k}P_k(z), \text{ where } \alpha_{n,n+1} \equiv 1. \tag{15}$$

Let us consider

$$(16)\quad \begin{aligned}(z-\zeta)\,[P_n(z)+(n+1)L_n(z)] \\ &= (z-\zeta)\left[P_n(z)+((z-\zeta)P_n(z))'\right] \\ &= \sum_{k=0}^{n+1}\alpha_{n,k}\,[P_k(z)+(z-\zeta)P_k'(z)]\,.\end{aligned}$$

By the orthogonality relation (11), we have

$$(17)\quad \begin{aligned}\sum_{k=0}^{n+1}\alpha_{n,k}\int_{-1}^{1}L_m(x)[P_k(x)+(x-\zeta)P_k'(x)]\,dx \\ &= \alpha_{n,m}\,(m+1)\,\|L_m\|^2,\end{aligned}$$

for $m = 0,\ 1,\dots,n$.

On the other hand, let us denote

$$I_{n,m} = \int_{-1}^{1} L_m(x)(x-\zeta)\,[P_n(x)+(n+1)L_n(x)]\ dx.$$

Thus, from (1) and (12) we get

$$(18)\qquad I_{n,m} = \begin{cases}(2/n)(1-\zeta^2)L_n'(\zeta) & m=0,\\ 0 & 0<m<(n-1),\\ [n(n^2-1)/(4n^2-1)]\,\|L_{n-1}\|^2 & m=n-1,\\ -\zeta(n+1)\,\|L_n\|^2 & m=n.\end{cases}$$

In the case $m = n-1$, we have used the identity $\|L_n\|^2 = (n^2/4n^2-1)\times \|L_{n-1}\|^2$, which can be obtained from (2) and (13).

Thus, multiplying (16) by L_m, integrating over $[-1,1]$, and using (17)–(18), we get

$$\alpha_{n,m} = \begin{cases}(1-\zeta^2/n)L_n'(\zeta) & m=0\\ 0 & 0<m<(n-1)\\ (n^2-1)/(4n^2-1) & m=n-1\\ -\zeta & m=n.\end{cases}$$

Replacing these values in (15) we get (13). $\square$

3. Zeros and asymptotics. Let us now study the zero distribution for the polar Legendre polynomials. The next lemma summarizes several direct consequences of the formulas contained in the introductory section, in special the fundamental formula (8).

Lemma 1. *The polar Legendre polynomials* $\{P_n\}$ *with pole* $\zeta \in \mathbf{C}$ *satisfy*

1. *If* n *is odd and* $\zeta \in \mathbf{R}^* = \mathbf{R} \setminus \{0\}$, *then* $x = -\zeta$ *is a zero of* P_n.

2. *The zeros of the polar polynomial* P_n *have multiplicity at most* 2 *and their multiple zeros are located on* $[-1, 1]$.

3. *If* $\zeta = 1$, $\zeta = -1$ *or* $L'_n(\zeta) = 0$, *then the zeros of* $P_n(x)$ *are* -1 *or* 1 *and the* $(n-1)$ *critical points of the* n*th Legendre polynomial* L_n.

4. *All the zeros of* P_n *are located on the lemniscate*

$$\Lambda_n(\zeta) := \left\{ z \in \mathbf{C} : \prod_{k=0}^{n} |z - x_{n,k}| = \rho_n(\zeta) \right\}, \tag{19}$$

where $\rho_n(\zeta) = \prod_{k=0}^{n} |\zeta - x_{n,k}|$, $x_{n,0} = -1$, $x_{n,n} = 1$, *and* $x_{n,1}, x_{n,2}, \dots,$ $x_{n,n-1}$ *are the* $(n-1)$ *critical points of the Legendre polynomial* L_n.

Proof. In order to prove 1, using (6), (7) and the fact that if n is odd then all the powers of L_n are odd, one gets $P_n(-\zeta) = 0$.

Now, assume that w is a zero of P_n of multiplicity greater than or equal to 3. Notice that by (4) a zero of P_n with multiplicity greater than 2 is a zero of L_n and also a zero of L'_n; thus, $L_n(w) = L'_n(w) = 0$ which would imply that w is a zero of multiplicity 2 of L_n. This is a contradiction since the zeros of L_n are all simple; thus, the multiplicity of w must be at most 2.

Statement 3 is a direct consequence of fundamental formula (8) since

$$\left|z_0^2 - 1\right| \, |L'_n(z_0)| = \left|\zeta^2 - 1\right| \, |L'_n(\zeta)| . \tag{20}$$

Finally, statement 4 follows by considering the factorization of $(z^2 - 1)L'_n(z)$. $\square$

Remark 1. The following example shows that zeros of $P_n(z)$ do not have to be simple. Let $\zeta = (2\sqrt{3})/3$ (or $\zeta = -(2\sqrt{3})/3$); hence, the corresponding polar Legendre polynomial of degree two is

$$P_2(z) = z^2 + \frac{2\sqrt{3}}{3}z + \frac{1}{3}\Big(\text{or } P_2(z) = z^2 - \frac{2\sqrt{3}}{3}z + \frac{1}{3}\Big).$$

Notice that $z = -\sqrt{3}/3$ (or $z = \sqrt{3}/3$) is a zero of multiplicity two of $P_2(z)$.

With the boundedness condition for the zeros of the polar Legendre polynomials $\{P_n\}$ we have the following result

Lemma 2. *Given $\zeta \in \mathbf{C}$ let us define $\Delta_\zeta = \sup_{x\in[-1,1]} |\zeta - x|$ and $\delta_\zeta = \inf_{x\in[-1,1]} |\zeta - x|$. Then*

1. *All zeros of the polar Legendre polynomials $\{P_n\}$ with pole $\zeta \in \mathbf{C}$ are contained in $|z| \le \Delta_\zeta + 1$.*

2. *If $\delta_\zeta > 1$, zeros of the polar Legendre polynomials $\{P_n\}$ with pole $\zeta \in \mathbf{C}$ are simple and contained in the exterior of the ellipse $|z+1| + |z-1| = 2\alpha$, where $1 < \alpha < \delta_\zeta$.*

Proof. 1. By (19) we already know that zeros of $P_n(z)$ are on the lemniscate $\Lambda_n(\zeta)$. Since $\rho_n(\zeta) < \Delta_\zeta^{n+1}$, they are contained in the interior of the lemniscate $\prod_{k=0}^{n} |z - x_{n,k}| = \Delta_\zeta^{n+1}$, where $x_{n,0} = -1$, $x_{n,n} = 1$ and $x_{n,1}, x_{n,2}, \dots, x_{n,n-1}$ are the $(n-1)$ critical points of the Legendre polynomial L_n, and therefore $|x_{n,k}| \le 1$. Now, for any z^*, such that $|z^*| > 1 + \Delta_\zeta$, we have

$$\prod_{k=0}^{n} |z^* - x_{n,k}| \ge \prod_{k=0}^{n} \big||z^*| - |x_{n,k}|\big| > \Delta_\zeta^{n+1},$$

so assertion 1 is obtained.

2. Let z be such that $|z+1| + |z-1| = 2\alpha$. From the well known arithmetic-geometric mean inequality we get

$$\prod_{k=0}^{n} |z - x_{n,k}| \le \left(\frac{1}{n+1}\sum_{k=0}^{n} |z - x_{n,k}|\right)^{n+1} < \alpha^{n+1}.$$

If z is a zero of P_n, from (19) we get

$$\prod_{k=0}^{n} |z - x_{n,k}| = \prod_{k=0}^{n} |\zeta - x_{n,k}| > \delta_\zeta^{n+1} > \alpha^{n+1}.$$

This and assertion 2 of Lemma 1 allows us to get the conclusion. $\square$

Finally, let us study the asymptotic behavior of the zeros of the polar Legendre polynomials,

Theorem 3. *Let $\{P_n\}$ be the polar Legendre polynomials with pole $\zeta \in \mathbf{C}\setminus[-1,1]$, such that $\delta_\zeta > 1$. Then the accumulation points of zeros of $\{P_n\}$ are located on the ellipse*

(21)
$$\Lambda(\zeta) := \left\{ z \in \mathbf{C} : z = \frac{\rho^2(\zeta)+1}{2\rho(\zeta)} \cos\theta + i\,\frac{\rho^2(\zeta)-1}{2\rho(\zeta)} \sin\theta,\ 0 \le \theta < 2\pi \right\},$$

where $\rho(\zeta) := |\zeta + \sqrt{\zeta^2-1}|$ and the branch of the square root is chosen so that $|z+\sqrt{z^2-1}| > 1$ for $z \in \mathbf{C}\setminus[-1,1]$.

Proof. From (20) zeros of the nth polar Legendre polynomial satisfy the equation

$$\left|\frac{z^2-1}{n}\right|^{1/n} |L_n'(z)|^{1/n} = \left|\frac{\zeta^2-1}{n}\right|^{1/n} |L_n'(\zeta)|^{1/n}. \tag{22}$$

On the other hand, from the asymptotic properties of the Legendre polynomials it is well known that

$$\lim_{n\to\infty} |L_n'(z)|^{1/n} = \frac{|z+\sqrt{z^2-1}|}{2}, \tag{23}$$

uniformly on compact subsets of $\mathbf{C}\setminus[-1,1]$. Taking limit as $n\to\infty$, from 2 of Lemma 2 and using (23) in both sides of (22), we have that set of accumulation points of zeros of the sequence of polynomials $\{P_n\}$ are contained in the curve

$$\Lambda(\zeta) = \left\{ z \in \mathbf{C} : |z+\sqrt{z^2-1}| = \rho(\zeta) \right\}.$$

Hence,

$$\begin{aligned} z + \sqrt{z^2 - 1} &= \rho(\zeta)\, e^{i\theta}, \quad 0 \le \theta < 2\pi, \\ z - \sqrt{z^2 - 1} &= \rho(\zeta)^{-1}\, e^{-i\theta}, \\ 2z &= \rho(\zeta) e^{i\theta} + \rho(\zeta)^{-1} e^{-i\theta}. \qquad \square \end{aligned}$$

Finally, we will study the relative asymptotics of the polar Legendre polynomials $\{P_n\}$ with respect to the Legendre polynomials $\{L_n\}$ and their derivatives $\{L'_n\}$.

Theorem 4. *Let ζ be a fixed complex number, and let L_n and P_n y Δ_ζ be as above. Then*

1.

$$\frac{n\, P_n(z)}{L'_n(z)} \underset{n\to\infty}{\rightrightarrows} \frac{z^2-1}{z-\zeta}, \tag{24}$$

uniformly on compact subsets of the set $\{z \in \mathbf{C} : |z| > \Delta_\zeta + 1\}$.

2.

$$\frac{P_n(z)}{L_n(z)} \underset{n\to\infty}{\rightrightarrows} \frac{\sqrt{z^2-1}}{z-\zeta}, \tag{25}$$

uniformly on compact subsets of the set $\{z \in \mathbf{C} : |z| > \Delta_\zeta + 1\}$.

Proof. Let K be a compact subset of $\{z \in \mathbf{C} : |z| > \Delta_\zeta + 1\}$. From formula (8) we have

$$\frac{n\, P_n(z)}{L'_n(z)} = \frac{1-\zeta^2}{z-\zeta} \frac{L'_n(\zeta)}{L'_n(z)} + \frac{z^2-1}{z-\zeta}.$$

Hence, in order to prove (21) it is sufficient to show that

$$\frac{L'_n(\zeta)}{L'_n(z)} \underset{n\to\infty}{\rightrightarrows} 0 \tag{26}$$

uniformly on a compact subset $K \subset \{z \in \mathbf{C} : |z| > \Delta_\zeta + 1\}$. Let $x_{n,1}, x_{n,2}, \dots, x_{n,n-1}$ be the $n-1$ zeros of $L'_n(z)$ and $d_{\zeta,K} =$

$\inf_{\substack{z\in K \\ |w|\geq \Delta_\zeta+1}} |z-w|$; hence,

$$\left|\frac{L_n'(\zeta)}{L_n'(z)}\right| = \frac{\prod_{k=1}^{n-1}|\zeta - x_{n,k}|}{\prod_{k=1}^{n-1}|z - x_{n,k}|} < \left(\frac{\Delta_\zeta}{d_{\zeta,K}+\Delta_\zeta}\right)^{n-1} < 1, \quad z\in K.$$

This inequality is equivalent to the uniform convergence of (26) on a compact subset K of $\{z\in \mathbf{C} : |z| > \Delta_\zeta + 1\}$.

The statement (25) is a direct consequence of (24) and the well-known asymptotic behavior of the Legendre polynomials (see [**8**, Corollary 1.6])

$$\frac{L_n'(z)}{n\, L_n(z)} \underset{n\to\infty}{\rightrightarrows} \frac{1}{\sqrt{z^2-1}},$$

uniformly on compact subsets of $\mathbf{C}\setminus[-1,1]$. $\square$

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