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L-R-SMASH BIPRODUCTS, DOUBLE BIPRODUCTS AND A BRAIDED CATEGORY OF YETTER-DRINFELD-LONG BIMODULES

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ABSTRACT. Let H be a bialgebra and D an H-bimodule algebra and H-bicomodule coalgebra. We find sufficient conditions on D for the L-R-smash product algebra and coalgebra structures on $D \otimes H$ to form a bialgebra (in this case we say that (H, D) is an L-R-admissible pair), called L-R-smash biproduct. The Radford biproduct is a particular case, and so is, up to isomorphism, a double biproduct with trivial pairing. We construct a prebraided monoidal category $\mathcal{LR}(H)$, whose objects are H-bimodules H-bicomodules M endowed with left-left and right-right Yetter-Drinfeld module as well as left-right and right-left Long module structures over H, with the property that, if (H, D) is an L-R-admissible pair, then D is a bialgebra in $\mathcal{LR}(H)$.

1. Introduction. The L-R-smash product over a cocommutative Hopf algebra was introduced and studied in a series of papers [**1–4**], with motivation and examples coming from the theory of deformation quantization. This construction was generalized in [**13**] to the case of arbitrary bialgebras (even quasi-bialgebras), as follows: if H is a bialgebra and D is an H-bimodule algebra, the L-R-smash product $D\natural H$ is an associative algebra structure defined on $D\otimes H$ by the multiplication rule

$$(d\natural h)(d'\natural h') = (d \cdot h'_2)(h_1 \cdot d')\natural h_2h'_1, \quad \text{for all } d, d' \in D,\ h, h' \in H.$$

It was proved in [**13**] that, if H is moreover a Hopf algebra with bijective antipode, then $D\natural H$ is isomorphic to a diagonal crossed product $D \bowtie H$ as in [**5, 7**]; this result was used in [**12**] to give a very easy proof of the fact that two bialgebroids introduced independently in [**6, 8**] are actually isomorphic.

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The dual construction of the L-R-smash product was introduced also in [**13**] under the name L-R-smash coproduct; this is a coassociative coalgebra $D\natural H$, where D is an H-bicomodule coalgebra. A natural problem, not treated in [**13**], is to see under what conditions, for a given H-bimodule algebra H-bicomodule coalgebra D, the L-R-smash product and coproduct structures on $D \otimes H$ form a bialgebra. It seems to be difficult to obtain (nicely-looking) necessary and sufficient conditions on D for this to happen. The aim of the present paper is to present a list of *sufficient conditions*, looking reasonably nice and being general enough to cover some existing constructions from the literature.

More precisely, if D satisfies those conditions, we say that (H, D) is an *L-R-admissible pair* and the bialgebra $D\natural H$ is called an *L-R-smash biproduct*. The Radford biproduct is a particular case, corresponding to the situation when the right action and coaction are trivial. We prove that a *double biproduct* $A\#H\#B$ (as in [**10, 15**]) with trivial pairing is isomorphic to an L-R-smash biproduct $(A\otimes B)\natural H$. Also, we show that a construction introduced in [**16**] is a particular case of an L-R-smash biproduct.

It is known that the Radford biproduct has a categorical interpretation (due to Majid): (H, B) is an admissible pair (as in [**14**]) if and only if B is a bialgebra in the Yetter-Drinfeld category ${}_H^H\mathcal{YD}$. We give a similar interpretation for L-R-admissible pairs. Namely, we define a prebraided category $\mathcal{LR}(H)$ (which is braided if H has a skew antipode) consisting of H-bimodules H-bicomodules M which are left-left and right-right Yetter-Drinfeld modules as well as left-right and right-left Long modules over H (this category contains ${}_H^H\mathcal{YD}$ and $\mathcal{YD}_H^H$ as braided subcategories). We prove that all except one of the conditions for (H, D) to be an L-R-admissible pair are equivalent to D being a bialgebra in $\mathcal{LR}(H)$. The extra condition reads

$$c^{\langle 0\rangle} \cdot d^{(-1)} \otimes c^{\langle 1\rangle} \cdot d^{(0)} = c \otimes d, \quad \text{for all } c, d \in D,$$

and unfortunately does not seem to have a categorical interpretation inside $\mathcal{LR}(H)$.

1. The L-R-smash biproduct. We work over a field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. For a bialgebra H with comultiplication Δ we denote $\Delta(h) = h_1 \otimes h_2$, for $h \in H$. For terminology concerning bialgebras, Hopf algebras and monoidal categories we refer to [**9, 11**].

Let H be a bialgebra and let D be a vector space satisfying the following conditions:

(i) D is an H-bimodule, with actions $h \otimes d \mapsto h \cdot d$ and $d \otimes h \mapsto d \cdot h$, for $h \in H$ and $d \in D$;

(ii) D is an algebra, with unit 1_D and multiplication $c \otimes d \mapsto cd$, for $c, d \in D$;

(iii) D is an H-bimodule algebra, that is, $h \cdot 1_D = \varepsilon(h)1_D$, $1_D \cdot h = \varepsilon(h)1_D$, $h \cdot (cd) = (h_1 \cdot c)(h_2 \cdot d)$ and $(cd) \cdot h = (c \cdot h_1)(d \cdot h_2)$, for all $h \in H$ and $c, d \in D$;

(iv) D is an H-bicomodule, with structures (for all $d \in D$):

$$\rho : D \longrightarrow D \otimes H, \quad \rho(d) = d^{\langle 0 \rangle} \otimes d^{\langle 1 \rangle},$$
$$\lambda : D \longrightarrow H \otimes D, \quad \lambda(d) = d^{(-1)} \otimes d^{(0)};$$

(v) D is a coalgebra, with comultiplication $\Delta_D : D \rightarrow D \otimes D$, $\Delta_D(d) = d_1 \otimes d_2$, and counit $\varepsilon_D : D \rightarrow k$;

(vi) D is an H-bicomodule coalgebra, that is, for all $d \in D$:

$$\begin{aligned}
&d_1^{(-1)} d_2^{(-1)} \otimes d_1^{(0)} \otimes d_2^{(0)} = d^{(-1)} \otimes (d^{(0)})_1 \otimes (d^{(0)})_2, \\
&d^{(-1)} \varepsilon_D(d^{(0)}) = \varepsilon_D(d) 1_H, \\
&d_1^{\langle 0 \rangle} \otimes d_2^{\langle 0 \rangle} \otimes d_1^{\langle 1 \rangle} d_2^{\langle 1 \rangle} = (d^{\langle 0 \rangle})_1 \otimes (d^{\langle 0 \rangle})_2 \otimes d^{\langle 1 \rangle}, \\
&\varepsilon_D(d^{\langle 0 \rangle}) d^{\langle 1 \rangle} = \varepsilon_D(d) 1_H.
\end{aligned}$$

We denote the vector space $D \otimes H$ by $D \natural H$ and elements $d \otimes h$ by $d \natural h$. By [**13**], $D \natural H$ becomes an algebra (called L-R-smash product) with unit $1_D \natural 1_H$ and multiplication

$$(d \natural h)(d' \natural h') = (d \cdot h'_2)(h_1 \cdot d') \natural h_2 h'_1,$$
$$\text{for all } h, h' \in H, \ d, d' \in D,$$

and a coalgebra (called L-R-smash coproduct) with comultiplication and counit given by

$$\Delta : D \natural H \longrightarrow (D \natural H) \otimes (D \natural H), \ \varepsilon : D \natural H \rightarrow k,$$
$$\Delta(d \natural h) = (d_1^{\langle 0 \rangle} \natural d_2^{(-1)} h_1) \otimes (d_2^{(0)} \natural h_2 d_1^{\langle 1 \rangle}),$$
$$\varepsilon(d \natural h) = \varepsilon_D(d) \varepsilon_H(h).$$

We consider now the following list of conditions, for H and D as above, corresponding to elements $h \in H$ and $c, d \in D$:

$$\varepsilon_D(1_D) = 1, \quad \varepsilon_D(cd) = \varepsilon_D(c)\varepsilon_D(d), \tag{1.1}$$

$$\varepsilon_D(h \cdot d) = \varepsilon_D(d \cdot h) = \varepsilon_D(d)\varepsilon_H(h), \tag{1.2}$$

$$\rho(1_D) = 1_D \otimes 1_H, \quad \lambda(1_D) = 1_H \otimes 1_D, \tag{1.3}$$

$$\Delta_D(1_D) = 1_D \otimes 1_D, \tag{1.4}$$

$$\rho(cd) = c^{\langle 0 \rangle} d^{\langle 0 \rangle} \otimes c^{\langle 1 \rangle} d^{\langle 1 \rangle}, \tag{1.5}$$

$$\lambda(cd) = c^{(-1)} d^{(-1)} \otimes c^{(0)} d^{(0)}, \tag{1.6}$$

$$\Delta_D(h \cdot d) = h_1 \cdot d_1 \otimes h_2 \cdot d_2, \tag{1.7}$$

$$\Delta_D(d \cdot h) = d_1 \cdot h_1 \otimes d_2 \cdot h_2, \tag{1.8}$$

$$\Delta_D(cd) = c_1(c_2^{(-1)} \cdot d_1^{\langle 0 \rangle}) \otimes (c_2^{(0)} \cdot d_1^{\langle 1 \rangle}) d_2, \tag{1.9}$$

$$(h_1 \cdot d)^{(-1)} h_2 \otimes (h_1 \cdot d)^{(0)} = h_1 d^{(-1)} \otimes h_2 \cdot d^{(0)}, \tag{1.10}$$

$$(h \cdot d)^{\langle 0 \rangle} \otimes (h \cdot d)^{\langle 1 \rangle} = h \cdot d^{\langle 0 \rangle} \otimes d^{\langle 1 \rangle}, \tag{1.11}$$

$$(d \cdot h_2)^{\langle 0 \rangle} \otimes h_1 (d \cdot h_2)^{\langle 1 \rangle} = d^{\langle 0 \rangle} \cdot h_1 \otimes d^{\langle 1 \rangle} h_2, \tag{1.12}$$

$$(d \cdot h)^{(-1)} \otimes (d \cdot h)^{(0)} = d^{(-1)} \otimes d^{(0)} \cdot h, \tag{1.13}$$

$$c^{\langle 0 \rangle} \cdot d^{(-1)} \otimes c^{\langle 1 \rangle} \cdot d^{(0)} = c \otimes d. \tag{1.14}$$

If all these conditions hold, for all $h \in H$ and $c, d \in D$, by analogy with [**14**] we will say that (H, D) is an *L-R-admissible pair.*

Theorem 1.1. *If (H, D) is an L-R-admissible pair, then $D \natural H$ with structures as above is a bialgebra, called the* L-R-smash biproduct *of D and H.*

Proof. It is very easy to see that $\varepsilon_{D \natural H}$ is an algebra map and $\Delta_{D \natural H}$ is unital, so we will only prove that $\Delta_{D \natural H}$ is multiplicative. We will prove first two auxiliary relations:

$$[c(h \cdot d)]_1 \otimes [c(h \cdot d)]_2 = c_1(c_2^{(-1)} h_1 \cdot d_1^{\langle 0 \rangle}) \otimes (c_2^{(0)} \cdot d_1^{\langle 1 \rangle})(h_2 \cdot d_2), \tag{1.15}$$

$$(1.16)\quad [c(h_1\cdot d)]_1\otimes[c(h_1\cdot d)]_2^{(-1)}h_2\otimes[c(h_1\cdot d)]_2^{(0)} \\ = c_1(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle})\otimes c_2^{(0)(-1)}h_2d_2^{(-1)} \\ \otimes(c_2^{(0)(0)}\cdot d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)}),$$

for all $h\in H$, $c,d\in D$; we compute:

$$\begin{aligned}[c(h\cdot d)]_1\otimes[c(h\cdot d)]_2 &\overset{(1.9)}{=} c_1(c_2^{(-1)}\cdot(h\cdot d)_1^{\langle 0\rangle}) \\ &\quad\otimes(c_2^{(0)}\cdot(h\cdot d)_1^{\langle 1\rangle})(h\cdot d)_2 \\ &\overset{(1.7)}{=} c_1(c_2^{(-1)}\cdot(h_1\cdot d_1)^{\langle 0\rangle}) \\ &\quad\otimes(c_2^{(0)}\cdot(h_1\cdot d_1)^{\langle 1\rangle})(h_2\cdot d_2) \\ &\overset{(1.11)}{=} c_1(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle}) \\ &\quad\otimes(c_2^{(0)}\cdot d_1^{\langle 1\rangle})(h_2\cdot d_2),\end{aligned}$$

$$\begin{aligned}&[c(h_1\cdot d)]_1\otimes[c(h_1\cdot d)]_2^{(-1)}h_2\otimes[c(h_1\cdot d)]_2^{(0)} \\ &\overset{(1.15)}{=} c_1(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle})\otimes[(c_2^{(0)}\cdot d_1^{\langle 1\rangle})(h_2\cdot d_2)]^{(-1)}h_3 \\ &\quad\otimes[(c_2^{(0)}\cdot d_1^{\langle 1\rangle})(h_2\cdot d_2)]^{(0)} \\ &\overset{(1.6)}{=} c_1(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle})\otimes(c_2^{(0)}\cdot d_1^{\langle 1\rangle})^{(-1)}(h_2\cdot d_2)^{(-1)}h_3 \\ &\quad\otimes(c_2^{(0)}\cdot d_1^{\langle 1\rangle})^{(0)}(h_2\cdot d_2)^{(0)} \\ &\overset{(1.10)}{=} c_1(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle})\otimes(c_2^{(0)}\cdot d_1^{\langle 1\rangle})^{(-1)}h_2d_2^{(-1)} \\ &\quad\otimes(c_2^{(0)}\cdot d_1^{\langle 1\rangle})^{(0)}(h_3\cdot d_2^{(0)}) \\ &\overset{(1.13)}{=} c_1(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle})\otimes c_2^{(0)(-1)}h_2d_2^{(-1)} \\ &\quad\otimes(c_2^{(0)(0)}\cdot d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)}), \qquad \square\end{aligned}$$

Let now $c,d\in D$ and $h,g\in H$; we compute:

$$\begin{aligned}\Delta((c\natural h)(d\natural g)) &= \Delta((c\cdot g_2)(h_1\cdot d)\natural h_2g_1) \\ &= ((c\cdot g_3)(h_1\cdot d))_1^{\langle 0\rangle}\otimes((c\cdot g_3)(h_1\cdot d))_2^{(-1)}h_2g_1 \\ &\quad\otimes((c\cdot g_3)(h_1\cdot d))_2^{(0)}\otimes h_3g_2((c\cdot g_3)(h_1\cdot d))_1^{\langle 1\rangle} \\ &\overset{(1.16)}{=} [(c\cdot g_3)_1((c\cdot g_3)_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle})]^{\langle 0\rangle}\end{aligned}$$

$$
\begin{aligned}
&\otimes (c\cdot g_3)_2^{(0)(-1)}h_2d_2^{(-1)}g_1\\
&\otimes ((c\cdot g_3)_2^{(0)(0)}\cdot d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)})\\
&\otimes h_4g_2[(c\cdot g_3)_1((c\cdot g_3)_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle})]^{\langle 1\rangle}\\
\overset{(1.8),\,(1.5)}{=}\ &(c_1\cdot g_3)^{\langle 0\rangle}((c_2\cdot g_4)^{(-1)}h_1\cdot d_1^{\langle 0\rangle})^{\langle 0\rangle}\\
&\otimes (c_2\cdot g_4)^{(0)(-1)}h_2d_2^{(-1)}g_1\\
&\otimes ((c_2\cdot g_4)^{(0)(0)}\cdot d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)})\\
&\otimes h_4g_2(c_1\cdot g_3)^{\langle 1\rangle}((c_2\cdot g_4)^{(-1)}h_1\cdot d_1^{\langle 0\rangle})^{\langle 1\rangle}\\
\overset{(1.11)}{=}\ &(c_1\cdot g_3)^{\langle 0\rangle}((c_2\cdot g_4)^{(-1)}h_1\cdot d_1^{\langle 0\rangle\langle 0\rangle})\\
&\otimes (c_2\cdot g_4)^{(0)(-1)}h_2d_2^{(-1)}g_1\\
&\otimes ((c_2\cdot g_4)^{(0)(0)}\cdot d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)})\\
&\otimes h_4g_2(c_1\cdot g_3)^{\langle 1\rangle}d_1^{\langle 0\rangle\langle 1\rangle}\\
\overset{(1.12)}{=}\ &(c_1^{\langle 0\rangle}\cdot g_2)((c_2\cdot g_4)^{(-1)}h_1\cdot d_1^{\langle 0\rangle\langle 0\rangle})\\
&\otimes (c_2\cdot g_4)^{(0)(-1)}h_2d_2^{(-1)}g_1\\
&\otimes ((c_2\cdot g_4)^{(0)(0)}\cdot d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)})\otimes h_4c_1^{\langle 1\rangle}g_3d_1^{\langle 0\rangle\langle 1\rangle}\\
\overset{(1.13)}{=}\ &(c_1^{\langle 0\rangle}\cdot g_2)(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle\langle 0\rangle})\otimes (c_2^{(0)}\cdot g_4)^{(-1)}h_2d_2^{(-1)}g_1\\
&\otimes ((c_2^{(0)}\cdot g_4)^{(0)}\cdot d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)})\otimes h_4c_1^{\langle 1\rangle}g_3d_1^{\langle 0\rangle\langle 1\rangle}\\
\overset{(1.13)}{=}\ &(c_1^{\langle 0\rangle}\cdot g_2)(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle\langle 0\rangle})\otimes c_2^{(0)(-1)}h_2d_2^{(-1)}g_1\\
&\otimes (c_2^{(0)(0)}\cdot g_4d_1^{\langle 1\rangle})(h_3\cdot d_2^{(0)})\otimes h_4c_1^{\langle 1\rangle}g_3d_1^{\langle 0\rangle\langle 1\rangle},
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c\natural h)\Delta(d\natural g) &= (c_1^{\langle 0\rangle}\natural c_2^{(-1)}h_1)(d_1^{\langle 0\rangle}\natural d_2^{(-1)}g_1)\\
&\quad\otimes (c_2^{(0)}\natural h_2c_1^{\langle 1\rangle})(d_2^{(0)}\natural g_2d_1^{\langle 1\rangle})\\
&= (c_1^{\langle 0\rangle}\cdot (d_2^{(-1)})_2g_2)((c_2^{(-1)})_1h_1\cdot d_1^{\langle 0\rangle})\\
&\quad\natural (c_2^{(-1)})_2h_2(d_2^{(-1)})_1g_1\\
&\quad\otimes (c_2^{(0)}\cdot g_4(d_1^{\langle 1\rangle})_2)(h_3(c_1^{\langle 1\rangle})_1\cdot d_2^{(0)})\\
&\quad\natural h_4(c_1^{\langle 1\rangle})_2g_3(d_1^{\langle 1\rangle})_1\\
&= (c_1^{\langle 0\rangle\langle 0\rangle}\cdot d_2^{(0)(-1)}g_2)(c_2^{(-1)}h_1\cdot d_1^{\langle 0\rangle\langle 0\rangle})\\
&\quad\otimes c_2^{(0)(-1)}h_2d_2^{(-1)}g_1
\end{aligned}
$$

$$\begin{aligned}
&\otimes (c_2^{(0)(0)} \cdot g_4 d_1^{\langle 1\rangle})(h_3 c_1^{\langle 0\rangle\langle 1\rangle} \cdot d_2^{(0)(0)}) \\
&\otimes h_4 c_1^{\langle 1\rangle} g_3 d_1^{\langle 0\rangle\langle 1\rangle} \\
\overset{(1.14)}{=}\ & (c_1^{\langle 0\rangle} \cdot g_2)(c_2^{(-1)} h_1 \cdot d_1^{\langle 0\rangle\langle 0\rangle}) \otimes c_2^{(0)(-1)} h_2 d_2^{(-1)} g_1 \\
&\otimes (c_2^{(0)(0)} \cdot g_4 d_1^{\langle 1\rangle})(h_3 \cdot d_2^{(0)}) \\
&\otimes h_4 c_1^{\langle 1\rangle} g_3 d_1^{\langle 0\rangle\langle 1\rangle},
\end{aligned}$$

and we see that the two terms are equal. □

Remark 1.2. Obviously, the Radford biproduct (cf. [**14**]) is a particular case of the L-R-smash biproduct, corresponding to the case when the right action and coaction are trivial.

We recall now from [**10, 15**] the construction of the so-called *double biproduct*, more precisely a particular case of it (corresponding to a trivial pairing, in the terminology of [**10**]). Let H be a bialgebra, A a bialgebra in the Yetter-Drinfeld category ${}_H^H\mathcal{YD}$ and B a bialgebra in the Yetter-Drinfeld category $\mathcal{YD}_H^H$, with the following notation for the structure maps: counits ε_A and ε_B, comultiplications $\Delta_A(a) = a_1 \otimes a_2$ and $\Delta_B(b) = b_1 \otimes b_2$, and actions and coactions

$$\begin{aligned}
H \otimes A &\longrightarrow A, \quad h \otimes a \longmapsto h \triangleright a, \\
A &\longrightarrow H \otimes A, \quad a \longmapsto a^1 \otimes a^2, \\
B \otimes H &\longrightarrow B, \quad b \otimes h \longmapsto b \triangleleft h, \\
B &\longrightarrow B \otimes H, \quad b \longmapsto b^1 \otimes b^2,
\end{aligned}$$

for all $h \in H$, $a \in A$, $b \in B$. We denote by $A\#H\#B$ the vector space $A \otimes H \otimes B$ (the element $a \otimes h \otimes b$ is denoted by $a\#h\#b$), which becomes an algebra (called two-sided smash product) with unit $1_A\#1_H\#1_B$ and multiplication

$$(a\#h\#b)(a'\#h'\#b') = a(h_1 \triangleright a')\#h_2 h_1'\#(b \triangleleft h_2')b',$$

and a coalgebra (called two-sided smash coproduct) with counit defined by $\varepsilon(a\#h\#b) = \varepsilon_A(a)\varepsilon_H(h)\varepsilon_B(b)$ and comultiplication

$$\begin{aligned}
&\Delta : A\#H\#B \longrightarrow (A\#H\#B) \otimes (A\#H\#B), \\
&\Delta(a\#h\#b) = (a_1\#a_2^1 h_1\#b_1^1) \otimes (a_2^2\#h_2 b_1^2\#b_2).
\end{aligned}$$

Proposition 1.3 [**10, 15**]. *Assume that moreover the following condition holds*:

$$b^2 \triangleright a^2 \otimes b^1 \triangleleft a^1 = a \otimes b, \quad \text{for all } a \in A, \ b \in B. \tag{1.17}$$

Then $A\#H\#B$ is a bialgebra, called the double biproduct.

Proposition 1.4. *Let $A\#H\#B$ be a double biproduct bialgebra. Define $D = A \otimes B$, with tensor product algebra and coalgebra structures and with two-sided actions and coactions given by*

$$H \otimes (A \otimes B) \to A \otimes B, \ h \otimes (a \otimes b) \mapsto h \cdot (a \otimes b) := h \triangleright a \otimes b,$$
$$A \otimes B \to H \otimes (A \otimes B), \quad a \otimes b \mapsto (a \otimes b)^{(-1)} \otimes (a \otimes b)^{(0)} := a^1 \otimes (a^2 \otimes b),$$
$$(A \otimes B) \otimes H \to A \otimes B, \ (a \otimes b) \otimes h \mapsto (a \otimes b) \cdot h := a \otimes b \triangleleft h,$$
$$A \otimes B \to (A \otimes B) \otimes H, \quad a \otimes b \mapsto (a \otimes b)^{\langle 0 \rangle} \otimes (a \otimes b)^{\langle 1 \rangle} := (a \otimes b^1) \otimes b^2.$$

Then (H, D) is an L-R-admissible pair and we have a bialgebra isomorphism

$$\phi : (A \otimes B) \natural H \simeq A\#H\#B, \quad (a \otimes b) \natural h \mapsto a\#h\#b.$$

Proof. The fact that (H, D) is an L-R-admissible pair follows by direct computation; let us only check (1.14), for $a, a' \in A$ and $b, b' \in B$:

$$\begin{aligned}
&(a \otimes b)^{\langle 0 \rangle} \cdot (a' \otimes b')^{(-1)} \otimes (a \otimes b)^{\langle 1 \rangle} \cdot (a' \otimes b')^{(0)} \\
&\qquad = (a \otimes b^1) \cdot a'^1 \otimes b^2 \cdot (a'^2 \otimes b') \\
&\qquad = (a \otimes b^1 \triangleleft a'^1) \otimes (b^2 \triangleright a'^2 \otimes b') \\
&\qquad \overset{(1.17)}{=} (a \otimes b) \otimes (a' \otimes b'), \qquad \square
\end{aligned}$$

We know from [**13**, Proposition 2.4], that ϕ is an algebra isomorphism, and an easy computation shows that ϕ is also a coalgebra map. $\square$

We recall now the following result from [**16**]. Let H be a bialgebra and D an H-bimodule bialgebra (i.e., D is a bialgebra which is an H-bimodule algebra and an H-bimodule coalgebra). Consider the L-R-smash product algebra $D \natural H$, together with the tensor product coalgebra structure on it (i.e., $\Delta(d\natural h)=(d_1\natural h_1)\otimes(d_2\natural h_2)$ and $\varepsilon(d\natural h)=\varepsilon_D(d)\varepsilon_H(h)$). Then $D\natural H$ with these structures is a bialgebra if and only if the following conditions are satisfied, for all $h\in H$, $d\in D$:

$$h_1\cdot d\otimes h_2=h_2\cdot d\otimes h_1, \tag{1.18}$$

$$d\cdot h_1\otimes h_2=d\cdot h_2\otimes h_1. \tag{1.19}$$

This result is a particular case of Theorem 1.1. Indeed, consider on D the left and right trivial H-coactions (i.e. $d^{(-1)}\otimes d^{(0)}=1_H\otimes d$ and $d^{\langle 0\rangle}\otimes d^{\langle 1\rangle}=d\otimes 1_H$, for $d\in D$). Then one can easily check that (H,D) is an L-R-admissible pair ((1.18) and (1.19) are precisely (1.10) and respectively (1.12)) and the L-R-smash coproduct coalgebra structure in this case coincides with the tensor product coalgebra structure.

2. A braided category related to L-R-smash biproducts. Let H be a bialgebra. We will introduce a prebraided category associated to H, denoted by $\mathcal{LR}(H)$. The objects of $\mathcal{LR}(H)$ are vector spaces M endowed with H-bimodule and H-bicomodule structures (denoted by $h\otimes m\mapsto h\cdot m$, $m\otimes h\mapsto m\cdot h$, $m\mapsto m^{(-1)}\otimes m^{(0)}$, $m\mapsto m^{\langle 0\rangle}\otimes m^{\langle 1\rangle}$, for all $h\in H$, $m\in M$), such that M is a left-left Yetter-Drinfeld module, a left-right Long module, a right-right Yetter-Drinfeld module and a right-left Long module, i.e.,

$$(h_1\cdot m)^{(-1)}h_2\otimes(h_1\cdot m)^{(0)}=h_1m^{(-1)}\otimes h_2\cdot m^{(0)}, \tag{2.1}$$

$$(h\cdot m)^{\langle 0\rangle}\otimes(h\cdot m)^{\langle 1\rangle}=h\cdot m^{\langle 0\rangle}\otimes m^{\langle 1\rangle}, \tag{2.2}$$

$$(m\cdot h_2)^{\langle 0\rangle}\otimes h_1(m\cdot h_2)^{\langle 1\rangle}=m^{\langle 0\rangle}\cdot h_1\otimes m^{\langle 1\rangle}h_2, \tag{2.3}$$

$$(m\cdot h)^{(-1)}\otimes(m\cdot h)^{(0)}=m^{(-1)}\otimes m^{(0)}\cdot h, \tag{2.4}$$

for all $h\in H$, $m\in M$. The morphisms in $\mathcal{LR}(H)$ are the H-bilinear H-bicolinear maps.

One can check that $\mathcal{LR}(H)$ becomes a strict monoidal category, with unit k endowed with usual H-bimodule and H-bicomodule structures, and tensor product given as follows: if $M,N\in\mathcal{LR}(H)$ then $M\otimes N\in\mathcal{LR}(H)$ with structures (for all $m\in M$, $n\in N$, $h\in H$):

$$h\cdot(m\otimes n)=h_1\cdot m\otimes h_2\cdot n,$$
$$(m\otimes n)\cdot h=m\cdot h_1\otimes n\cdot h_2,$$
$$(m\otimes n)^{(-1)}\otimes(m\otimes n)^{(0)}=m^{(-1)}n^{(-1)}\otimes(m^{(0)}\otimes n^{(0)}),$$
$$(m\otimes n)^{\langle 0\rangle}\otimes(m\otimes n)^{\langle 1\rangle}=(m^{\langle 0\rangle}\otimes n^{\langle 0\rangle})\otimes m^{\langle 1\rangle}n^{\langle 1\rangle}.$$

Proposition 2.1. *The monoidal category $\mathcal{LR}(H)$ is a prebraided category, with braiding defined, for all $M, N \in \mathcal{LR}(H)$, $m \in M$, $n \in N$, by*

$$c_{M,N} : M \otimes N \longrightarrow N \otimes M,$$
$$c_{M,N}(m \otimes n) = m^{(-1)} \cdot n^{\langle 0\rangle} \otimes m^{(0)} \cdot n^{\langle 1\rangle}.$$

If H has a skew antipode S^{-1}, then $\mathcal{LR}(H)$ is braided, the inverse of c being given by

$$c^{-1}_{M,N} : N \otimes M \longrightarrow M \otimes N,$$
$$c^{-1}_{M,N}(n \otimes m) = m^{(0)} \cdot S^{-1}(n^{\langle 1\rangle}) \otimes S^{-1}(m^{(-1)}) \cdot n^{\langle 0\rangle}.$$

Proof. We only check that c is left H-linear, right H-colinear and satisfies one of the two hexagonal equations, and leave the rest to the reader. For $M, N, P \in \mathcal{LR}(H)$ and $h \in H$, $m \in M$, $n \in N$, $p \in P$, we compute:

$$\begin{aligned} c_{M,N}(h \cdot (m \otimes n)) &= c_{M,N}(h_1 \cdot m \otimes h_2 \cdot n) \\ &= (h_1 \cdot m)^{(-1)} \cdot (h_2 \cdot n)^{\langle 0\rangle} \otimes (h_1 \cdot m)^{(0)} \cdot (h_2 \cdot n)^{\langle 1\rangle} \\ &\overset{(2.2)}{=} (h_1 \cdot m)^{(-1)} h_2 \cdot n^{\langle 0\rangle} \otimes (h_1 \cdot m)^{(0)} \cdot n^{\langle 1\rangle} \\ &\overset{(2.1)}{=} h_1 m^{(-1)} \cdot n^{\langle 0\rangle} \otimes h_2 \cdot m^{(0)} \cdot n^{\langle 1\rangle} \\ &= h_1 \cdot (m^{(-1)} \cdot n^{\langle 0\rangle}) \otimes h_2 \cdot (m^{(0)} \cdot n^{\langle 1\rangle}) \\ &= h \cdot c_{M,N}(m \otimes n), \end{aligned}$$

$$\begin{aligned} &(\rho_{N\otimes M} \circ c_{M,N})(m \otimes n) \\ &\quad = \rho_{N\otimes M}(m^{(-1)} \cdot n^{\langle 0\rangle} \otimes m^{(0)} \cdot n^{\langle 1\rangle}) \\ &\quad = (m^{(-1)} \cdot n^{\langle 0\rangle})^{\langle 0\rangle} \otimes (m^{(0)} \cdot n^{\langle 1\rangle})^{\langle 0\rangle} \\ &\qquad \otimes (m^{(-1)} \cdot n^{\langle 0\rangle})^{\langle 1\rangle} (m^{(0)} \cdot n^{\langle 1\rangle})^{\langle 1\rangle} \\ &\quad \overset{(2.2)}{=} m^{(-1)} \cdot n^{\langle 0\rangle\langle 0\rangle} \otimes (m^{(0)} \cdot n^{\langle 1\rangle})^{\langle 0\rangle} \\ &\qquad \otimes n^{\langle 0\rangle\langle 1\rangle} (m^{(0)} \cdot n^{\langle 1\rangle})^{\langle 1\rangle} \\ &\quad = m^{(-1)} \cdot n^{\langle 0\rangle} \otimes (m^{(0)} \cdot (n^{\langle 1\rangle})_2)^{\langle 0\rangle} \otimes (n^{\langle 1\rangle})_1 (m^{(0)} \cdot (n^{\langle 1\rangle})_2)^{\langle 1\rangle} \\ &\quad \overset{(2.3)}{=} m^{(-1)} \cdot n^{\langle 0\rangle} \otimes m^{(0)\langle 0\rangle} \cdot (n^{\langle 1\rangle})_1 \otimes m^{(0)\langle 1\rangle} (n^{\langle 1\rangle})_2 \end{aligned}$$

$$
\begin{aligned}
&= m^{\langle 0\rangle(-1)} \cdot n^{\langle 0\rangle\langle 0\rangle} \otimes m^{\langle 0\rangle(0)} \cdot n^{\langle 0\rangle\langle 1\rangle} \otimes m^{\langle 1\rangle} n^{\langle 1\rangle} \\
&= c_{M,N}(m^{\langle 0\rangle} \otimes n^{\langle 0\rangle}) \otimes m^{\langle 1\rangle} n^{\langle 1\rangle} \\
&= (c_{M,N} \otimes id_H) \circ \rho_{M\otimes N}(m \otimes n),
\end{aligned}
$$

$$
\begin{aligned}
&(id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P)(m \otimes n \otimes p) \\
&\quad = (id_N \otimes c_{M,P})(m^{(-1)} \cdot n^{\langle 0\rangle} \otimes m^{(0)} \cdot n^{\langle 1\rangle} \otimes p) \\
&\quad = m^{(-1)} \cdot n^{\langle 0\rangle} \otimes (m^{(0)} \cdot n^{\langle 1\rangle})^{(-1)} \cdot p^{\langle 0\rangle} \otimes (m^{(0)} \cdot n^{\langle 1\rangle})^{(0)} \cdot p^{\langle 1\rangle} \\
&\quad \overset{(2.4)}{=} m^{(-1)} \cdot n^{\langle 0\rangle} \otimes m^{(0)(-1)} \cdot p^{\langle 0\rangle} \otimes m^{(0)(0)} \cdot n^{\langle 1\rangle} p^{\langle 1\rangle} \\
&\quad = (m^{(-1)})_1 \cdot n^{\langle 0\rangle} \otimes (m^{(-1)})_2 \cdot p^{\langle 0\rangle} \otimes m^{(0)} \cdot n^{\langle 1\rangle} p^{\langle 1\rangle} \\
&\quad = m^{(-1)} \cdot (n \otimes p)^{\langle 0\rangle} \otimes m^{(0)} \cdot (n \otimes p)^{\langle 1\rangle} \\
&\quad = c_{M,N\otimes P}(m \otimes n \otimes p).
\end{aligned}
$$

Also, the bijectivity of c in the presence of a skew antipode follows by a direct computation which is left to the reader. □

Remark 2.2. We denote as usual by ${}_H^H\mathcal{YD}$ and $\mathcal{YD}_H^H$ the categories of left-left and respectively right-right Yetter-Drinfeld modules over H. One can check that, if $V \in {}_H^H\mathcal{YD}$ and $W \in \mathcal{YD}_H^H$, then $V \otimes W \in \mathcal{LR}(H)$, with structures as in Proposition 1.4. In particular, for $W = k$ and respectively $V = k$, we obtain that ${}_H^H\mathcal{YD}$ and $\mathcal{YD}_H^H$ are subcategories of $\mathcal{LR}(H)$, and one can see that they are actually braided subcategories, i.e., the braiding of $\mathcal{LR}(H)$ restricts to the usual braidings of ${}_H^H\mathcal{YD}$ and $\mathcal{YD}_H^H$.

We can state now the categorical interpretation of L-R-admissible pairs:

Proposition 2.3. *Let H be a bialgebra and D a vector space. Then (H, D) is an L-R-admissible pair if and only if D is a bialgebra in $\mathcal{LR}(H)$ satisfying* (1.14).

Proof. A straightforward verification; we only note that (1.9) expresses the fact that the comultiplication of D is an algebra map inside the category $\mathcal{LR}(H)$. □

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