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ON THE CONNECTIVITY OF ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. The aim of the paper is to give sufficient and necessary conditions when the components of a fixed order of the attractor of an iterated function system are connected sets. By a component of order n of the attractor of an IFS we understand the image of the attractor through the composition of n functions from the IFS. This gives sufficient conditions when the attractor of an iterated function system is a finite union of connected sets. If these conditions are fulfilled the attractor of the iterated function system will be locally arcwise connected and every connected component of the attractor of the iterated function system will be arcwise connected.

1. Introduction. We start with a brief presentation of iterated function systems, IFSs for short. We will also fix the notations. Iterated function systems were conceived in the present form by Hutchinson [**5**] and popularized by Barnsley [**2**] and are one of the most common and most general ways to generate fractals. Many of the important examples of functions and sets with special and unusual properties turn out to be fractal sets, and a great portion of them are attractors of IFSs. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems (IIFSs) or, more generally, to multifunction systems and to study them (see for example [**1, 7–10**]). A recent such example can be found in [**8**] where the Lipscomb's space, which was an important example in dimension theory, can be obtained as an attractor of an IIFS defined in a very general setting. In this setting the attractor can be a closed and bounded set in contrast with the classical theory where only compact sets are considered. Although fractal sets are defined with measure theory, being sets with noninteger Hausdorff dimension [**3, 4**], it turns out that they have interesting topological properties as we can see from

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the above example [**8**]. The topological properties of fractal sets have a great importance in analysis on fractals, for example (see [**6**]). One of the most important result in these direction is Theorem 1.2 below [**6, 11**] which states when the attractor of an IFS is a connected set. We want to find sufficient and necessary conditions when the components of a fixed order (see Definition 2.2 below) of the attractor of an iterated function system are connected sets. This gives sufficient conditions when the attractor of an iterated function system is a finite union of connected sets. If these conditions are fulfilled, the attractor of the iterated function system will be locally arcwise connected, and every connected component of the attractor of the iterated function system will be arcwise connected.

The paper is divided into four parts. The first part is the introduction. In the second part the description of the shift space of an iterated function system is given. The main result, Theorem 3.1, is contained in the third part. The last part contains some examples.

For a metric space (X,d), $K(X)$ denotes the set of nonvoid compact subsets of X.

Definition 1.1. Let (X,d) be a metric space. $K(X)$ with the Hausdorff-Pompeiu distance $h : K(X) \times K(X) \to [0,+\infty)$ defined by

$$\begin{aligned} h(A,B) &= \max(d(A,B), d(B,A)) \\ &= \min\{r / A \subset B(B,r) \text{ and } B \subset B(A,r)\} \end{aligned}$$

where

$$d(A,B) = \sup_{x\in A} d(x,B) = \sup_{x\in A}(\inf_{y\in B} d(x,y))$$

is a metric space.

$(K(X),h)$ is a complete metric space if (X,d) is a complete metric space, compact if (X,d) is compact and separable if (X,d) is separable (see [**1, 2, 3, 10**]).

Definition 1.2. Let (X,d) be a metric space. For a function $f : X \to X$ let us denote by $\operatorname{Lip}(f) \in [0,+\infty]$ the Lipschitz constant associated to f that is

$$\mathrm{Lip}\,(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{d(f(x),f(y))}{d(x,y)}.$$

f is a Lipschitz function if $\mathrm{Lip}\,(f) < +\infty$ and a contraction if $\mathrm{Lip}\,(f) < 1$.

Definition 1.3. An iterated function system on a metric space (X,d) consists in a finite family of contractions $(f_k)_{k=\overline{1,n}}$ on X and is denoted by $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$. For an IFS $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$, $F_{\mathcal{S}}$ is the function $F_{\mathcal{S}} : K(X) \to K(X)$ defined by $F_{\mathcal{S}}(B) = \cup_{k=1}^{n} f_k(B)$.

The function $F_{\mathcal{S}}$ is a contraction with $\mathrm{Lip}\,(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} \mathrm{Lip}\,(f_k)$, see [**1, 2, 5, 10**].

Using Banach contraction theorem there exists, for an IFS $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$, a unique set $A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$. More precisely we have the following well-known result (see [**1, 2, 10**]).

Theorem 1.1. *Let (X,d) be a complete metric space, and let $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$ be an* IFS *with $c = \max_{k=\overline{1,n}} \mathrm{Lip}\,(f_k) < 1$. Then there exists a unique $A(\mathcal{S}) \in K(X)$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$. Moreover, for any $H_0 \in K(X)$, the sequence $(H_n)_{n\geq 1}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to $A(\mathcal{S})$. For the speed of the convergence we have the following estimation*

$$h(H_n, A(\mathcal{S})) \leq \frac{c^n}{1-c} h(H_0, H_1).$$

In particular we obtain $h(H_0, A(\mathcal{S})) \leq (1/(1-c))h(H_0,H_1)$.

Definition 1.4. The set $A(S)$ from the above theorem is named the attractor of the IFS $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$.

Definition 1.5. Let X be a set, and let $(A_i)_{i\in I}$ be a family of nonvoid subsets of X. The family $(A_i)_{i\in I}$ is said to be connected if, for every $i,j \in I$, there exists an $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \varnothing$ for every $k \in \{1,2,\dots,n-1\}$.

Definition 1.6. Let X be a set, and let $(A_i)_{i\in I}$ be a family of nonvoid subsets of X. On the family of sets $(A_i)_{i\in I}$, we consider the following equivalent relation, $A_i \sim A_j$ if and only if there exists an $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \varnothing$ for every $k \in \{1, 2, \dots, n-1\}$. A class of equivalence is named a component of the family of sets $(A_i)_{i\in I}$. If there is not any danger of confusion through a component of the family of sets $(A_i)_{i\in I}$ we will understand the component or the union of the elements of the component.

A family of sets is connected if and only if it has only one component.

Definition 1.7. A metric space (X, d) is arcwise connected if for every $x, y \in X$ there exists a continuous function $\varphi : [0, 1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

Concerning the connectivity of the attractor of an IFS we have the following theorem (see [**6, 11**]).

Theorem 1.2. *Let (X, d) be a complete metric space, let $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be an* IFS *with $c = \max_{k=\overline{1,n}} \mathrm{Lip}\,(f_k) < 1$, and let $A(\mathcal{S})$ be the attractor of $\mathcal{S}$. The following are equivalent*:

1) *The family $(A_i)_{i=\overline{1,n}}$ is connected where $A_i = f_i(A(\mathcal{S}))$.*

2) *$A(\mathcal{S})$ is arcwise connected.*

3) *$A(\mathcal{S})$ is connected.*

We want to find sufficient conditions when the attractor of an IFS has a finite number of components.

The next result is well known.

Lemma 1.1. *Let (X, d) be a metric space, and let $(A_i)_{i\in I}$ be a connected family of connected subsets of X. Then $\cup_{i\in I} A_i$ is connected. If the sets A_i are arcwise connected, then $\cup_{i\in I} A_i$ is arcwise connected.*

The next result is a converse of the previous one.

Lemma 1.2. *Let (X,d) be a metric space, and let $(A_i)_{i\in I}$ be a finite family of nonvoid, closed, connected subsets of X such that $\cup_{i\in I}A_i$ is connected. Then the family of sets $(A_i)_{i\in I}$ is connected.*

Proof. Set $A=\cup_{i\in I}A_i$. Let $l\in I$, $M=\{j\in I|$ there exists $(i_k)_{k=\overline{1,m}}\subset I$ such that $i_1=l$, $i_m=j$ and $A_{i_k}\cap A_{i_{k+1}}\neq\varnothing$ for every $k\in\{1,2,\dots,m-1\}\}$. Set $V_1=\cup_{j\in M}A_j$ and $V_2=\cup_{j\notin M}A_j$. Then $V_1\cup V_2=A$, V_1 and V_2 are closed sets. Also $V_1\cap V_2=\varnothing$. Indeed, let us suppose by reduction ad absurdum that $V_1\cap V_2\neq\varnothing$. Then there exists an $a\in V_1\cap V_2$, $j_1\in M$ and $j_2\notin M$ such that $a\in A_{j_1}\cap A_{j_2}$. Because $j_1\in M$, there exist $(i_k)_{k=\overline{1,m}}\subset I$ such that $i_1=l$, $i_m=j_1$ and $A_{i_k}\cap A_{i_{k+1}}\neq\varnothing$ for every $k\in\{1,2,\dots,m-1\}$. Because $a\in A_{j_1}\cap A_{j_2}$, it follows that $j_2\in M$, which is a contradiction.

Because V_1 is nonvoid and A is connected, it follows that $V_2=\varnothing$. So $M=I$. This ends the proof. □

2. The shift space for an IFS. In this section we present, in short, the shift space of an IFS (see [**2, 10**] for more details).

We start with set notations. $\mathbf{N}$ denotes the natural numbers, $\mathbf{N}^*=\mathbf{N}-\{0\}$, $\mathbf{N}_n^*=\{1,2,\dots,n\}$.

For two nonvoid sets A and B, B^A denotes the set of functions from A to B.

By $\Lambda=\Lambda(B)$ we will understand the set $B^{\mathbf{N}^*}$ and by $\Lambda_n=\Lambda_n(B)$ we will understand the set $B^{\mathbf{N}_n^*}$. The elements of $\Lambda=\Lambda(B)=B^{\mathbf{N}^*}$ will be written as infinite words $\omega=\omega_1\omega_2\cdots\omega_m\omega_{m+1}\cdots$ where $\omega_m\in B$, and the elements of $\Lambda_n=\Lambda_n(B)=B^{\mathbf{N}_n^*}$ will be written as words $\omega=\omega_1\omega_2\cdots\omega_n$. $\Lambda(B)$ is the set of infinite words with letters from the alphabet B and $\Lambda_n(B)$ is the set of words of length n. By $\Lambda^*=\Lambda^*(B)$ we will understand the set of all finite words $\Lambda^*=\Lambda^*(B)=\cup_{n\geq 1}\Lambda_n(B)$. An element of $\Lambda=\Lambda(B)$ is said to have length $+\infty$.

We denote by $|\omega|$ the length of the word ω.

If $\omega=\omega_1\omega_2\cdots\omega_m\omega_{m+1}\cdots$ or if $\omega=\omega_1\omega_2\cdots\omega_n$ and $n\geq m$, then $[\omega]_m=\omega_1\omega_2\cdots\omega_m$. More generally if $l<m$, $[\omega]_m^l=\omega_{l+1}\omega_{l+2}\cdots\omega_m$. We have $[\omega]_m=[\omega]_l[\omega]_m^l$ for $\omega\in\Lambda_n(B)$ if $n\geq m>l\geq 1$ and for $\omega\in\Lambda(B)$ if $m>l\geq 1$. For words $\alpha,\beta\in\Lambda^*(B)\cup\Lambda(B)$, $\alpha\prec\beta$ means $|\alpha|\leq|\beta|$ and $[\beta]_{|\alpha|}=\alpha$.

For two words $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$ by $\alpha\beta$ we will understand the concatenation of the words α and β, namely, $\alpha\beta = \alpha_1\alpha_2 \cdots \alpha_n\beta_1\beta_2 \cdots \beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2 \cdots \alpha_n\beta_1\beta_2 \cdots \beta_m\beta_{m+1} \cdots$.

On $\Lambda = \Lambda(\mathbf{N}_n^*) = (\mathbf{N}_n^*)^{\mathbf{N}^*}$ we consider the metric

$$d_s(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1 - \delta_{\alpha_k}^{\beta_k}}{3^k}$$

where

$$\delta_x^y = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

Definition 2.1. The pair $(\Lambda(\mathbf{N}_n^*) = (\mathbf{N}_n^*)^{\mathbf{N}^*}, d_s)$ is a compact metric space and it is named the shift space with n letters. This is the shift space for an IFS which contains n functions.

Let $F_k : \Lambda(\mathbf{N}_n^*) \to \Lambda(\mathbf{N}_n^*)$ be defined by $F_k(\omega) = k\omega$ for $k = \overline{1, n}$. The functions F_k are continuous functions and are named the right shift functions. Then

$$d_s(F_k(\alpha), F_k(\beta)) = \frac{d_s(\alpha, \beta)}{3}.$$

The function $R : \Lambda(\mathbf{N}_n^*) \to \Lambda(\mathbf{N}_n^*)$ defined by $R(\omega = \omega_1\omega_2 \cdots \omega_m \omega_{m+1} \cdots) = \omega_2\omega_3 \cdots \omega_m\omega_{m+1} \cdots$ is also continuous and is named the left shift function. Then

$$d_s(R(\alpha), R(\beta)) = 3d_s(\alpha, \beta) - (1 - \delta_{\alpha_1}^{\beta_1}) \leq 3d_s(\alpha, \beta).$$

Remark 2.1. With the above notations we have: 1) $R \circ F_k(\omega) = \omega$ and $F_k \circ R(\omega) = k\omega_2\omega_3 \cdots \omega_m\omega_{m+1} \cdots$ for $\omega \in \Lambda(\mathbf{N}_n^*)$.

2) $\Lambda(\mathbf{N}_n^*) = \cup_{k=1}^n F_k(\Lambda(\mathbf{N}_n^*))$ and so $\Lambda(\mathbf{N}_n^*)$ is the attractor of the IFS $\mathcal{S} = (\Lambda(\mathbf{N}_n^*), (F_k)_{k=\overline{1,n}})$.

Notation 2.1. Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an IFS on X and $A = A(\mathcal{S})$ the attractor of the IFS

$\mathcal{S}$. For $\omega = \omega_1\omega_2\cdots\omega_m \in \Lambda_m(\mathbf{N}_n^*)$, $f_\omega = f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_m}$ and $H_\omega = f_\omega(H)$ for a set $H \subset X$. In particular, $A_\omega = f_\omega(A)$.

Definition 2.2. The sets $A_\omega = f_\omega(A(\mathcal{S}))$, for $|\omega| = p$ and $p \in \mathbf{N}^*$, are named the components of order p of the attractor of the IFS $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

Notation 2.2. Let (X, d) be a complete metric space, and let $f : X \to X$ be a contraction. We denote by e_f the fixed point of f. If $f = f_\omega$ then we denote by e_{f_ω} or by e_ω the fixed point of $f = f_\omega$.

Notation 2.3. Let (X, d) be a metric space and $A \subset X$. Then $d(A)$ is the diameter of A, that is, $d(A) = \sup_{x,y\in A} d(x, y)$.

The main results concerning the relation between the attractor of an IFS and the shift space is contained in the following theorem (see [**2, 10**]).

Theorem 2.1. *If $A = A(\mathcal{S})$ is the attractor of the* IFS $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ *and $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$, then we have*

1) *for $\omega \in \Lambda = \Lambda(\mathbf{N}_n^*)$, $A_{[\omega]_{m+1}} \subset A_{[\omega]_m}$ and $d(A_{[\omega]_m}) \to 0$ when $m \to \infty$; more precisely,*

$$d(A_{[\omega]_m}) \leq c^m d(A).$$

2) *If a_ω is defined by $\{a_\omega\} = \cap_{m\geq 1} A_{[\omega]_m}$, then $d(e_{[\omega]_m}, a_\omega) \to 0$ when $m \to \infty$.*

3) *$A = A(\mathcal{S}) = \cup_{\omega\in\Lambda}\{a_\omega\}$, $A_\alpha = \cup_{\omega\in\Lambda}\{a_{\alpha\omega}\}$ for every $\alpha \in \Lambda^*$, $A = \cup_{\omega\in\Lambda_m} A_\omega$ for every $m \in \mathbf{N}^*$ and more general $A_\alpha = \cup_{\omega\in\Lambda_m} A_{\alpha\omega}$ for every $\alpha \in \Lambda^*$ and every $m \in \mathbf{N}^*$.*

4) *The set $\{e_{[\omega]_m}/\omega \in \Lambda$ and $m \in \mathbf{N}^*\}$ is dense in A.*

5) *The function $\pi : \Lambda \to A$ defined by $\pi(\omega) = a_\omega$ is a continuous and surjective function.*

6) *$\pi \circ F_k = f_k \circ \pi$ for every $k \in \{1, 2, \ldots, n\}$.*

Definition 2.3. The function $\pi : \Lambda \to A = A(\mathcal{S})$ from the above theorem is named the canonical projection from the shift space on the attractor of the IFS $\mathcal{S}$.

3. The main result. For the proof of the main result (Theorem 3.1) we need the following lemma.

Lemma 3.1. *Let (X,d) be a complete metric space and $(a_n)_{n\geq 1}$ be a sequence of positive numbers convergent to 0. Let $(\Delta_l)_{l\geq 0}$ be a sequence of divisions of the unit interval $[0,1]$ (i.e. $\Delta_l = (y_0^l = 0 < y_1^l < \cdots < y_{n_l}^l = 1)$) such that $\Delta_l \subset \Delta_{l+1}$ and $\lim_{l\to+\infty} \|\Delta_l\| = 0$, where $\|\Delta_l\| = \max_{i=1}^{n_l}(y_i^l - y_{i-1}^l)$. Let $(g_l)_{l\geq 0}$ be a sequence of functions $g_l : \Delta_l \to X$ such that $g_{l+1}|_{\Delta_l} = g_l$ and for every $m \geq n$ and every $y_i^m \in \Delta_m$ $\max\{d(g_m(y_i^m), g_n(y_j^n)), d(g_m(y_i^m), g_n(y_{j+1}^n))\} \leq a_n$ if $y_i^m \in [y_j^n, y_{j+1}^n]$. Then there exists a continuous function $g : [0,1] \to X$ such that $g|_{\Delta_l} = g_l$.*

Proof. Let $A = \cup_{n\geq 1}\Delta_n$ and $\widetilde{g} : A \to X$ be the function defined by $\widetilde{g}(x) = g_l(x)$ if $x \in \Delta_l$. The function $\widetilde{g}$ is well defined because $g_m|_{\Delta_l} = g_l$ for every $m \geq l$. We want to prove that $\widetilde{g}$ is a uniform continuous function. Let $\varepsilon > 0$ be fixed. Because the sequence $(a_n)_{n\geq 1}$ is convergent to 0 there exists an n_ε such that for every $n \geq n_\varepsilon$, $a_n < \varepsilon/2$. Set $\delta_l = \min_{i=1}^{n_l}(y_i^l - y_{i-1}^l)$ and $\delta = \delta_{n_\varepsilon}$. We have $0 < \delta_{l+1} \leq \delta_l \leq \|\Delta_l\|$.

Let $c, d \in [0,1] \cap A$ be such that $c < d$ and that $d - c < \delta$. There exists an $m_0 \geq n_\varepsilon$ such that $c, d \in \Delta_{m_0}$. The set $(c,d) \cap \Delta_{n_\varepsilon}$ has at most one element.

If $(c,d)\cap\Delta_{n_\varepsilon} = \varnothing$, then there exists a j such that $y_j^{n_\varepsilon} \leq c < d \leq y_{j+1}^{n_\varepsilon}$. In this case we have

$$\begin{aligned} d(\widetilde{g}(c), \widetilde{g}(d)) &= d(g_{m_0}(c), g_{m_0}(d)) \\ &\leq d(g_{m_0}(c), g_{n_\varepsilon}(y_j^{n_\varepsilon})) + d(g_{n_\varepsilon}(y_j^{n_\varepsilon}), g_{m_0}(d)) \leq 2a_{n_\varepsilon} < \varepsilon. \end{aligned}$$

If $(c,d) \cap \Delta_{n_\varepsilon} = \{y_j^{n_\varepsilon}\}$, we have $y_{j-1}^{n_\varepsilon} < c < y_j^{n_\varepsilon} < d < y_{j+1}^{n_\varepsilon}$ and

$$\begin{aligned} d(\widetilde{g}(c), \widetilde{g}(d)) &= d(g_{m_0}(c), g_{m_0}(d)) \\ &\leq d(g_{m_0}(c), g_{n_\varepsilon}(y_j^{n_\varepsilon})) + d(g_{n_\varepsilon}(y_j^{n_\varepsilon}), g_{m_0}(d)) \leq 2a_{n_\varepsilon} < \varepsilon. \end{aligned}$$

It follows that $\widetilde{g}$ is a uniform continuous function. Then there exists a unique continuous function $g : [0,1] \to X$ such that $g|_A = \widetilde{g}$. We also have $g|_{\Delta_l} = g_l$. $\square$

Theorem 3.1. *Let (X, d) be a complete metric space, $p \in \mathbf{N}^*$, let $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be an* IFS, *$c = \max_{k=\overline{1,n}} \operatorname{Lip}(f_k) < 1$ and let $A = A(\mathcal{S})$ be the attractor of $\mathcal{S}$. The following are equivalent*:

1) *For every $\omega \in \Lambda_p = \Lambda_p(\mathbf{N}_n^*)$ the family $(A_{\omega i})_{i=\overline{1,n}}$ is connected.*

2) *Each component of order p of $A(\mathcal{S})$ is arcwise connected.*

3) *Each component of order p of $A(\mathcal{S})$ is connected.*

4) *Each component of order m with $m \geq p$ of $A(\mathcal{S})$ is arcwise connected.*

5) *Each component of order m with $m \geq p$ of $A(\mathcal{S})$ is connected.*

In these cases we also have:

6) *$A(\mathcal{S})$ has at most n^p connected components, more precisely $A(\mathcal{S})$ has the same number of connected components as the number of components of the family of sets $(A_\omega)_{\omega \in \Lambda_p}$.*

7) *Each connected component of $A(\mathcal{S})$ is arcwise connected.*

8) *$A(\mathcal{S})$ is a local arcwise connected set.*

Proof. The implications 4) $\Rightarrow$ 2) $\Rightarrow$ 3) and 4) $\Rightarrow$ 5) $\Rightarrow$ 3) are obvious.

2) $\Rightarrow$ 4). Let $m \geq p$ be fixed. If $m = p$ the result is obvious. We can suppose that $m > p$. Let $\omega \in \Lambda_m$, $\alpha = [\omega]_{m-p}$ and $\beta = [\omega]_m^{m-p} \in \Lambda_p$. Then $\omega = [\omega]_{m-p}[\omega]_m^{m-p} = \alpha\beta$ and $A_\omega = f_\omega(A) = f_{\alpha\beta}(A) = f_\alpha \circ f_\beta(A) = f_\alpha(A_\beta)$. Because A_β is arcwise connected and f_α is continuous it results that A_ω is arcwise connected.

3) $\Rightarrow$ 5). Let $m \geq p$ be fixed. If $m = p$ the result is obvious. We can suppose that $m > p$. Let $\omega \in \Lambda_m$, $\alpha = [\omega]_{m-p}$ and $\beta = [\omega]_m^{m-p} \in \Lambda_p$. Then $\omega = \alpha\beta$ and $A_\omega = f_\alpha(A_\beta)$. Because A_β is connected and f_α is continuous it results that A_ω is connected.

5) $\Rightarrow$ 1). Let $\omega \in \Lambda_p$. Then the sets A_ω and $A_{\omega i}$ for $i \in \{1, 2, \dots, n\}$ are compact and connected. It follows from Lemma 1.2 that the family $(A_{\omega i})_{i=\overline{1,n}}$ is connected.

What remains to be proved for the equivalence of the first 5 sentences is 1) $\Rightarrow$ 2).

1) $\Rightarrow$ 2). For every $\omega \in \Lambda_p$ and each index $i, j \in \{1, 2, \dots, n\}$ such that $A_{\omega i} \cap A_{\omega j} \neq \varnothing$, let us fix $x_{i,j}^\omega \in A_{\omega i} \cap A_{\omega j}$. Also, for each index

$i, j \in \{1, 2, \ldots, n\}$, let us fix $(i^\omega_k)_{k=\overline{0,m(\omega,i,j)}}$ such that $i^\omega_0 = i$, $i^\omega_{m(\omega,i,j)} = j$ and $A_{\omega i^\omega_k} \cap A_{\omega i^\omega_{k+1}} \neq \varnothing$ for every $k \in \{0, 1, \ldots, m(\omega, i, j) - 1\}$. The sequence $(i^\omega_k = i^\omega_k(i, j))_{k=\overline{0,m(\omega,i,j)}}$ can be taken without repetition. In this case $m(\omega, i, j) \leq n - 1$. We can suppose that $m(\omega, i, j) = n - 1$ by taking $i^\omega_{n-1} = \cdots = i^\omega_{m(\omega,i,j)+1} = i^\omega_{m(\omega,i,j)}$ (and so $A_{\omega i^\omega_{n-1}} = \cdots = A_{\omega i^\omega_{m(\omega,i,j)+1}} = A_{\omega i^\omega_{m(\omega,i,j)}}$).

Since $A_\omega = \cup^n_{i=1} A_{\omega i}$ it follows that for every $z \in A_\omega$ there exists an $i_\omega(z)$ such that $z \in A_{\omega i_\omega(z)} = f_\omega(A_{i_\omega(z)})$. Let $i_\omega : A_\omega \to \{1, 2, \ldots, n\}$ be a fixed function such that $z \in A_{\omega i_\omega(z)}$.

We consider an $\omega \in \Lambda_p$. Then, for every two elements z_0 and z_1 from A_ω, we have fixed $A_{\omega i_\omega(z_0)}$ and $A_{\omega i_\omega(z_1)}$, a sequence $(i^\omega_k = i^\omega_k(i_\omega(z_0), i_\omega(z_1)))_{k=\overline{0,n-1}} \subset I$ such that $i^\omega_0 = i_\omega(z_0)$, $i^\omega_{n-1} = i_\omega(z_1)$ and $A_{\omega i^\omega_k} \cap A_{\omega i^\omega_{k+1}} \neq \varnothing$ for every $k \in \{0, 1, \ldots, n-2\}$ and elements $x^\omega_{i^\omega_k, i^\omega_{k+1}} \in A_{\omega i^\omega_k} \cap A_{\omega i^\omega_{k+1}}$. Set $i^\omega_k(z_0, z_1) = i^\omega_k$ for $k \in \{0, 1, \ldots, n-1\}$, $w^\omega_0(z_0, z_1) = z_0$, $w^\omega_n(z_0, z_1) = z_1$ and $w^\omega_k(z_0, z_1) = x^\omega_{i^\omega_{k-1}, i^\omega_k}$ for $k \in \{1, 2, \ldots, n - 1\}$. We remark that $i^\omega_0(z_0, z_1) = i_\omega(z_0)$, $i^\omega_{n-1}(z_0, z_1) = i_\omega(z_1)$ and $w^\omega_k(z_0, z_1), w^\omega_{k+1}(z_0, z_1) \in A_{\omega i^\omega_k(z_0,z_1)}$ for every $k \in \{0, 1, \ldots, n - 1\}$.

Let $\alpha \in \Lambda_p$ be fixed. We want to prove that A_α is an arcwise connected set.

Let x_0 and x_1 be two fixed different elements from A_α. We will define inductively after l a sequence $(\Delta_l)_{l\geq 0}$ of divisions of the unit interval $[0, 1]$, i.e., $\Delta_l = (y^l_0 = 0 < y^l_1 < \cdots < y^l_{n^l} = 1)$, elements $\omega^l_k \in \Lambda_{p+l}$, where $k \in \{0, 1, \ldots, n^l - 1\}$, and a sequence $(g_l)_{l\geq 0}$ of functions $g_l : \Delta_l \to A_\alpha$ such that $\Delta_l \subset \Delta_{l+1}$, $g_l|_{\Delta_l} = g_l$, $g_l(y^l_k), g_l(y^l_{k+1}) \in A_{\omega^l_k}$ for every l and $k \in \{0, 1, \ldots, n^l - 1\}$ and $\omega^l_k \prec \omega^{l'}_{k'}$ if $l' \geq l$ and $y^{l'}_{k'} \in [y^l_k, y^l_{k+1})$.

Set $\Delta_0 = (y^0_0 = 0 < y^0_1 = 1)$, $g_0(0) = x_0$ and $g_0(1) = x_1$. We have $x_0, x_1 \in A_\alpha$.

Set $\Delta_1 = (y^1_0 = 0 < y^1_1 < \cdots < y^1_n = 1)$, where $y^1_k = k/n$ and $g_1(y^1_k) = w^\alpha_k(x_0, x_1)$.

In general we will take $y^l_k = (k/n^l)$. Then $y^l_k = y^{l+1}_{kn}$ and $\Delta_l \subset \Delta_{l+1}$.

The induction step. Let us suppose that we have defined g_l. Let $k \in \{0, 1, \ldots, n^l-1\}$. There exists an $\omega^l_k \in \Lambda_{p+l}$ such that $g_l(y^l_k), g_l(y^l_{k+1}) \in$

$A_{\omega_k^l}$. Set $\gamma_k^l = [\omega_k^l]_{|\omega_k^l|-p}$ and $\beta_k^l = [\omega_k^l]_{|\omega_k^l|}^{|\omega_k^l|-p} \in \Lambda_p$. We remark that $\omega_k^l = \gamma_k^l \beta_k^l$ and $A_{\omega_k^l} = A_{\gamma_k^l \beta_k^l} = f_{\gamma_k^l}(A_{\beta_k^l})$. It follows that $g_l(y_k^l) = f_{\gamma_k^l}(z)$ and $g_l(y_{k+1}^l) = f_{\gamma_k^l}(z')$ for some $z, z' \in A_{\beta_k^l}$. Set $g_{l+1}(y_{kn+j}^{l+1}) = f_{\gamma_k^l}(w_j^{\beta_k^l}(z,z'))$ for $j \in \{0, 1, \ldots, n\}$.

We have

$$g_{l+1}(y_{kn}^{l+1}) = f_{\gamma_k^l}(w_0^{\beta_k^l}(z,z')) = f_{\gamma_k^l}(z) = g_l(y_k^l)$$

and

$$g_{l+1}(y_{(k+1)n}^{l+1}) = f_{\gamma_k^l}(w_n^{\beta_k^l}(z,z')) = f_{\gamma_k^l}(z') = g_l(y_{k+1}^l).$$

This means that g_{l+1} is well defined and $g_{l+1}|_{\Delta_l} = g_l$.

$w_j^{\beta_k^l}(z,z'), w_{j+1}^{\beta_k^l}(z,z') \in A_{\beta_k^l i_j^{\beta_k^l}(z,z')}$ for every $j \in \{0, 1, \ldots, n-1\}$. Set $\omega_{kn+j}^{l+1} = \omega_k^l i_j^{\beta_k^l}(z,z') = \gamma_k^l \beta_k^l i_j^{\beta_k^l}(z,z')$ for $k \in \{0, 1, \ldots, n^l - 1\}$ and $j \in \{0, 1, \ldots, n-1\}$. Then

$$g_{l+1}(y_{kn+j}^{l+1}) = f_{\gamma_k^l}(w_j^{\beta_k^l}(z,z')) \in f_{\gamma_k^l}(A_{\beta_k^l i_j^{\beta_k^l}(z,z')}) = A_{\omega_k^l i_j^{\beta_k^l}(z,z')} = A_{\omega_{kn+j}^{l+1}}$$

and

$$g_{l+1}(y_{kn+j+1}^{l+1}) = f_{\gamma_k^l}(w_{j+1}^{\beta_k^l}(z,z')) \in f_{\gamma_k^l}(A_{\beta_k^l i_j^{\beta_k^l}(z,z')}) = A_{\omega_k^l i_j^{\beta_k^l}(z,z')} = A_{\omega_{kn+j}^{l+1}}.$$

Because $\omega_k^l \prec \omega_{kn+j}^{l+1}$ for $j \in \{0, 1, \ldots, n-1\}$, it follows that if $l' \geq l$ then $\omega_{[k/n^{l'-l}]}^l \prec \omega_k^{l'}$.

The induction hypotheses are now checked.

We want to apply Lemma 3.1 to the functions g_l and divisions Δ_l defined above. We have seen that $\Delta_l \subset \Delta_{l+1}$ and $g_{l+1}|_{\Delta_l} = g_l$. Also $\|\Delta_l\| = 1/n^l$ and so $\lim_{l\to+\infty} \|\Delta_l\| = 0$. Set $a_l = c^{p+l} d(A)$. Let $l' \geq l$ and $y_{k'}^{l'} \in \Delta_{l'}$ be fixed. We have to prove that, for every $l' \geq l$ and every $y_{k'}^{l'} \in \Delta_{l'}$,

$$\max\{d(g_{l'}(y_{k'}^{l'}), g_l(y_k^l)), d(g_{l'}(y_{k'}^{l'}), g_l(y_{k+1}^l))\} \leq a_l \text{ if } y_{k'}^{l'} \in [y_k^l, y_{k+1}^l].$$

We have two cases: $y^{l'}_{k'} \in \Delta_l$ and $y^{l'}_{k'} \notin \Delta_l$.

In the first case, $y^{l'}_{k'} \in \Delta_l$, we have $k' = [k'/n^{l'-l}]n^{l'-l}$ and $y^{l'}_{k'} = y^l_k$, where $k = [k'/n^{l'-l}]$. Because for every l and $k \in \{0, 1, \dots , n^l - 1\}$ there exists an $\omega^l_k \in \Lambda_{p+l}$ such that $g_l(y^l_k) = g_{l'}(y^{l'}_{k'}), g_l(y^l_{k+1}) \in A_{\omega^l_k}$, we have

$$d(g_{l'}(y^{l'}_{k'}), g_l(y^l_{k+1})) = d(g_l(y^l_k), g_l(y^l_{k+1})) \le d(A_{\omega^l_k}) \le c^{p+l}d(A) = a_l.$$

Also there exists an $\omega^l_{k-1} \in \Lambda_{p+l}$ such that $g_l(y^l_k) = g_l(y^{l'}_{k'}), g_l(y^l_{k-1}) \in A_{\omega^l_{k-1}}$ and we have

$$d(g_l(y^{l'}_{k'}), g_l(y^l_{k-1})) = d(g_l(y^l_k), g_l(y^l_{k-1})) \le d(A_{\omega^l_{k-1}}) \le c^{p+l}d(A) = a_l.$$

In the second case we have $y^{l'}_{k'} \in (y^l_k, y^l_{k+1})$, where $k = [k'/n^{l'-l}]$. Then $g_l(y^l_k), g_l(y^l_{k+1}) \in A_{\omega^l_k}$, $\omega^l_k \prec \omega^{l'}_{k'}$ and $g_{l'}(y^{l'}_{k'}) \in A_{\omega^{l'}_{k'}} \subset A_{\omega^l_k}$. It follows that

$$\max\{d(g_{l'}(y^{l'}_{k'}), g_l(y^l_k)), d(g_{l'}(y^{l'}_{k'}), g_l(y^l_{k+1}))\} \le d(A_{\omega^l_k}) \le c^{p+l}d(A) = a_l.$$

From Lemma 3.1 there exists a continuous function $g : [0, 1] \to X$ such that $g|_{\Delta_l} = g_l$. Because $g(\Delta_l) = g_l(\Delta_l) \subset A_\alpha$, g is a continuous function and A_α is a compact set we have that $g([0, 1]) \subset A_\alpha$. This proves that A_α is arcwise connected.

6) and 7). Because $A = \cup_{\omega\in\Lambda_p} A_\omega$, A_ω are arcwise connected for $\omega \in \Lambda_p$ and Λ_p has n^p elements it follows that A has at most n^p arcwise connected components.

Let us consider the family of sets $(A_\omega)_{\omega\in\Lambda_p}$. Because the sets A_ω are arcwise connected it results from Lemma 1.1 that every component of the family of sets $(A_\omega)_{\omega\in\Lambda_p}$ is arcwise connected. This implies that A has at most connected components as the number of components of the family $(A_\omega)_{\omega\in\Lambda_p}$. Let M be a connected component of A. If $\omega \in \Lambda_p$ and $A_\omega \cap M \neq \varnothing$, because $A_\omega \cup M$ is a connected set, it follows that $A_\omega \subset M$. We obtain that $M = \cup_{\omega\in\Lambda_p; A_\omega\subset M} A_\omega$. From Lemma 1.2 the family $(A_\omega)_{\omega\in\Lambda_p; A_\omega\subset M}$ is connected. It follows that A has the same number of connected components as the number of components of the family of sets $(A_\omega)_{\omega\in\Lambda_p}$.

8) Let $a \in A(\mathcal{S})$ and $\varepsilon > 0$. Let $n_0 \in \mathbf{N}$ be such that $c^{n_0} d(A) < \varepsilon$ and $n_0 > p$. Set $M = \cup_{\omega \in \Lambda_{n_0}; a \in A_\omega} A_\omega$ and $N = \cup_{\omega \in \Lambda_{n_0}; a \notin A_\omega} A_\omega$. Because A_ω are arcwise connected then M is arcwise connected. Also $M \subset B(a, \varepsilon)$. On the other side N is compact, $a \notin N$ and so $d(a, N) > 0$. This means that M is a neighborhood of a in A. □

4. Examples.

Example 4.1. Let us consider the function $\phi : [0,1] \to [0,1]$ defined by

$$\phi(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{1}{4}] \\ \frac{1}{8} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ \frac{x}{2} - \frac{1}{4} & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Then $\mathrm{Lip}\,(\phi) = 1/2$. Let $\psi : [0,1] \to [0,1]$ be the function defined by $\psi(x) = 1 - \phi(1-x)$. Then $\mathrm{Lip}\,(\psi) = 1/2$. Let $\mathcal{S} = ([0,1], (\phi, \psi))$ be an IFS. We have $A(\mathcal{S}) = [0, (1/4)] \cup [(3/4), 1]$.

Indeed

$$\begin{aligned} \phi\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right) &= \phi\left(\left[0, \frac{1}{4}\right]\right) \cup \phi\left(\left[\frac{3}{4}, 1\right]\right) \\ &= \left[0, \frac{1}{8}\right] \cup \left[\frac{1}{8}, \frac{1}{4}\right] = \left[0, \frac{1}{4}\right] \end{aligned}$$

and

$$\begin{aligned} \psi\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right) &= \psi\left(\left[0, \frac{1}{4}\right]\right) \cup \psi\left(\left[\frac{3}{4}, 1\right]\right) \\ &= \left(1 - \phi\left(\left[\frac{3}{4}, 1\right]\right)\right) \cup \left(1 - \phi\left(\left[0, \frac{1}{4}\right]\right)\right) \\ &= \left[\frac{3}{4}, \frac{7}{8}\right] \cup \left[\frac{7}{8}, 1\right] = \left[\frac{3}{4}, 1\right]. \end{aligned}$$

So

$$\begin{aligned} F_{\mathcal{S}}\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right) &= \phi\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right) \cup \psi\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right) \\ &= \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]. \end{aligned}$$

This means that $A(\mathcal{S}) = [0, (1/4)] \cup [(3/4), 1]$.

We have $A_1 = \phi(A) = [0, (1/4)]$, $A_2 = \psi(A) = [(3/4), 1]$, $A_{11} = [0, (1/8)]$, $A_{12} = [(1/8), (1/4)]$, $A_{21} = [(3/4), (7/8)]$ and $A_{22} = [(7/8), 1]$. Then $A_1 \cap A_2 = \varnothing$, $A_{11} \cap A_{12} = \{1/8\}$ and $A_{21} \cap A_{22} = \{7/8\}$. This means that the families of sets (A_{11}, A_{12}) and (A_{21}, A_{22}) are connected. We also remark that A_1 and A_2 are connected sets but $A(\mathcal{S})$ is not a connected set.

Example 4.2. We consider the metric space $\mathbf{R}^3$ with the Euclidean distance. Set $T = \{(x, y, z) \in \mathbf{R}^3 | x + y + z = 1, x, y, z \in [0, 1]\}$, $R_1 = \{(x, y, z) \in T | x \in [(3/4), 1]\}$, $R_2 = \{(x, y, z) \in T | y \in [(3/4), 1]\}$, $R_3 = \{(x, y, z) \in T | z \in [(3/4), 1]\}$, $R_4 = \{(x, y, z) \in T | x, y, z \in [0, (1/2)]\}$, $R_5 = \{(x, y, z) \in T | x \in [(1/2), (3/4)]\}$, $R_6 = \{(x, y, z) \in T | y \in [(1/2), (3/4)]\}$ and $R_7 = \{(x, y, z) \in T | z \in [(1/2), (3/4)]\}$.

Let $f = (f_1, f_2, f_3) : T \to R_1 \subset T$ be the function defined by

$$f(x,y,z) = \begin{cases} \left(\frac{x+1}{2}, \frac{y}{2}, \frac{z}{2}\right) & \text{if } w \in R_1 \\ \left(\frac{2x+3}{4}, \frac{2y-1}{4}, \frac{z}{2}\right) & \text{if } w \in R_2 \\ \left(\frac{2x+3}{4}, \frac{y}{2}, \frac{2z-1}{4}\right) & \text{if } w \in R_3 \\ \left(\frac{x+3}{4}, \frac{y}{4}, \frac{z}{4}\right) & \text{if } w \in R_4 \\ \left(\frac{7}{8}, \frac{y}{8(1-x)}, \frac{z}{8(1-x)}\right) & \text{if } w \in R_5 \\ \left(\frac{3}{4} + \frac{x}{8(1-y)}, \frac{1}{8}, \frac{z}{8(1-y)}\right) & \text{if } w \in R_6 \\ \left(\frac{3}{4} + \frac{x}{8(1-z)}, \frac{y}{8(1-z)}, \frac{1}{8}\right) & \text{if } w \in R_7. \end{cases}$$

Let $g : T \to R_2 \subset T$ be the function defined by

$$g(x, y, z) = (f_2(y, x, z), f_1(y, x, z), f_3(y, x, z)).$$

In a similar way let $h : T \to R_3 \subset T$ be the function defined by $h(x, y, z) = (f_1(z, y, x), f_2(z, y, x), f_3(z, y, x))$. We consider the IFS

$\mathcal{S} = (T, (f, g, h))$. Let $A = A(\mathcal{S})$ be the attractor of $\mathcal{S}$. We have $A_f \subset R_1$, $A_g \subset R_2$ and $A_h \subset R_3$. Because the sets R_1, R_2 and R_3 are mutually disjoint it follows that the sets A_f, A_g and A_h are mutually disjoint. The family of sets (A_{ff}, A_{fg}, A_{fh}) is connected because

$$A_{ff} \cap A_{fg} = \left\{ \left(\frac{7}{8}, \frac{1}{8}, 0 \right) \right\},$$
$$A_{ff} \cap A_{fh} = \left\{ \left(\frac{7}{8}, 0, \frac{1}{8} \right) \right\}$$

and

$$A_{fg} \cap A_{fh} = \left\{ \left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8} \right) \right\}$$

and so are the families (A_{gf}, A_{gg}, A_{gh}) and (A_{hf}, A_{hg}, A_{hh}). This means that A_f, A_g and A_h are connected sets. In fact, A_f, A_g and A_h are isomorphic with the Sierpinsky triangle and A consists in three mutual disjoint copies of the Sierpinsky triangle.

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