ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 6, 2010

COMPOSITION OPERATORS WHICH IMPROVE INTEGRABILITY BETWEEN WEIGHTED DIRICHLET SPACES

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ABSTRACT. The weighted Dirichlet space D_α, $(\alpha > -1)$, is the space of analytic functions with derivatives in the weighted Bergman space A^2_α. Let Φ be an analytic self-map of the disc. The composition operator C_Φ is said to improve integrability if, for some α, β with $\alpha > \beta$, $C_\Phi(f) = f \circ \Phi \in D_\beta$ for all $f \in D_\alpha$. For $\beta \geq 0$, Carleson-type conditions on a measure related to the generalized Nevanlinna counting function are shown to be necessary and sufficient for the operator $C_\Phi : D_\alpha \to D_\beta$ to be bounded or compact. A simpler characterization is given in the case $\alpha > \beta > 0$ for functions Φ of finite valence. For $\beta < 0$ and $\alpha > \beta$, $C_\Phi : D_\alpha \to D_\beta$ is compact if and only if $\Phi \in D_\beta$ and $\|\Phi\|_\infty < 1$. Examples are given to show that for $\alpha > \beta \geq 0$, the condition $\|\Phi\|_\infty < 1$ is not necessary for $C_\Phi : D_\alpha \to D_\beta$ to be compact.

1. The weighted Bergman space A^p_α, $(\alpha > -1)$, is the set of functions f analytic in the unit disc D such that

$$\|f\|^p_{A^p_\alpha} = \int_D |f(z)|^p \,(\log(1/|z|))^\alpha \, dA(z) < \infty.$$

Here A denotes normalized area measure on the disc. The measure defined by $dA_\alpha(z) = (\log(1/|z|))^\alpha \, dA(z)$ can be replaced by $(1 - |z|^2)^\alpha \, dA(z)$. This results in the same space of functions and an equivalent norm [**13**]. As noted in [**13**], the appropriate definition of A^2_α when $\alpha = -1$ is the Hardy space H^2.

An analytic function belongs to the weighted Dirichlet space D_α if its derivative belongs to A^2_α. The space D_α is normed by

$$\|f\|^2_{D_\alpha} = |f(0)|^2 + \|f'\|^2_{A^2_\alpha}.$$

Note that point evaluation functionals are bounded on D_α. For $\beta < \alpha$, the inclusion $D_\beta \subset D_\alpha$ is bounded.

Received by the editors on January 26, 2008, and in revised form on June 19, 2008.

DOI:10.1216/RMJ-2010-40-6-1887

Let Φ be an analytic self-map of the disc. The composition operator C_Φ is defined by $C_\Phi(f) = f \circ \Phi$ for functions f analytic in the disc. The focus of this work is on composition operators that map D_α into D_β where $-1 < \beta < \alpha$. In this case $C_\Phi : D_\alpha \to D_\beta$ is bounded, by the Closed Graph theorem, and C_Φ is said to improve integrability.

In his dissertation, Riedl [**8**] characterized the self-maps Φ for which $C_\Phi : H^p \to H^q$ is bounded or compact in the case $0 < p \leq q$. More generally, Smith [**13**] characterized the functions Φ for which $C_\Phi : A^p_\alpha \to A^q_\beta$ is bounded (or compact) in the case $0 < p \leq q$, $\alpha \geq -1$, $\beta \geq -1$. Since $D_\beta = A^2_{\beta-2}$ for $\beta \geq 1$, the focus of this work is on the case $\beta < 1$.

The main results of this work are given next, with proofs to follow in Sections 2 and 3. The first result characterizes the maps Φ for which $C_\Phi : D_\alpha \to D_\beta$ is bounded or compact in the case $0 \leq \beta < \alpha$. The result is stated in terms of the generalized Nevanlinna counting function N_β and the behavior of a related measure. Precise definitions appear in the next section.

Theorem. *Let $0 \leq \beta < \alpha$, and let $\Phi \in D_\beta$. Let ν_β be the measure defined by $d\nu_\beta(w) = N_\beta(w)\, dA(w)$.*

(1) *$C_\Phi : D_\alpha \to D_\beta$ is bounded if and only if ν_β is α-Carleson.*

(2) *$C_\Phi : D_\alpha \to D_\beta$ is compact if and only if ν_β is compact α-Carleson.*

The function Φ is said to be of finite valence if there exists an M such that no point $w \in D$ has more than M preimages (counting multiplicities) under Φ. A simpler version of the previous theorem will be given for functions Φ of finite valence in the case $0 < \beta < \alpha$.

A standard argument shows that if $\Phi \in D_\beta$ and $\|\Phi\|_\infty < 1$, then $C_\Phi : D_\alpha \to D_\beta$ is compact for every $\alpha > \beta$. For negative β the converse is true.

Theorem. *Suppose that $-1 < \beta < 0$, $\beta < \alpha$ and $\Phi \in D_\beta$. Then $C_\Phi : D_\alpha \to D_\beta$ is compact if and only if $\|\Phi\|_\infty < 1$.*

Theorem. *For each pair α, β with $\alpha > \beta \geq 0$, there is an analytic function Φ with $\|\Phi\|_\infty = 1$ and such that $C_\Phi : D_\alpha \to D_\beta$ is compact.*

Riedl [**8**] showed that if $0 < p \leq q$ then $C_\Phi : H^p \to H^q$ is bounded (respectively, compact) if and only if $N_1(w) = O((\log(1/|w|))^{q/p})$ (respectively, little oh) as $|w| \to 1$. For fixed $\beta \geq 1$, it follows that if $C_\Phi : H^p \to H^{\beta p}$ is bounded (or compact) for some $p > 0$, then $C_\Phi : H^p \to H^{\beta p}$ is bounded (compact) for all $p > 0$. The next result is an analogous statement for the weighted Dirichlet spaces.

Theorem. *Let $P > 1$, and let $\alpha > 0$.*

(1) *If $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded (respectively, compact) and if $\widehat{\alpha} > \alpha$, then $C_\Phi : D_{\widehat{\alpha}} \to D_{\widehat{\alpha}/P}$ is bounded (compact).*

(2) *Assume that Φ is of finite valence. If $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded (respectively, compact) for some $\alpha > 0$, then $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded (compact) for all $\alpha > 0$.*

Various authors have discussed composition operators that improve integrability. In [**3**], Hunziker and Jarchow studied functions Φ for which $C_\Phi : H^p \to H^q$ is bounded, when $p \leq q$. Related results appear in [**13**], where Smith characterized functions Φ for which $C_\Phi : A^p_\alpha \to A^q_\beta$ is bounded or compact in the case $p \leq q$. This study was continued by Smith and Yang [**14**] in the case $q < p$.

2. In this section we dispose of the cases where boundedness of $C_\Phi : D_\alpha \to D_\beta$ $(\beta < \alpha)$ requires $\|\Phi\|_\infty < 1$. Also we state a theorem of Smith [**13**], which provides necessary and sufficient conditions for $C_\Phi : D_\alpha \to D_\beta$ to be bounded or compact in the case $\alpha > \beta \geq 1$.

For $-1 < \beta < 0$, we will use techniques similar to those of MacCluer and Shapiro [**7**]. Let $S(\zeta, \delta) = \{z \in D : |z - \zeta| < \delta\}$ and let ν be a finite positive Borel measure on the disc. The measure ν is said to be β-Carleson if

$$\sup_{|\zeta|=1, 0<\delta\leq 2} \frac{\nu(S(\zeta,\delta))}{\delta^{\beta+2}} < \infty.$$

The measure ν is compact β-Carleson if

$$\lim_{\delta\to 0} \sup_{|\zeta|=1} \frac{\nu(S(\zeta,\delta))}{\delta^{\beta+2}} = 0.$$

Let $\Phi \in D_\beta$ and define the measure μ_β by $d\mu_\beta(z) = |\Phi'(z)|^2(1-|z|^2)^\beta\, dA(z)$. The measure $\mu_\beta\Phi^{-1}$ is the measure assigning mass $\mu_\beta(\Phi^{-1}(E))$ to Borel sets $E \subset D$. MacCluer and Shapiro's characterization states that $C_\Phi : D_\beta \to D_\beta$ is bounded (respectively, compact) if and only if $\mu_\beta\Phi^{-1}$ is β-Carleson (respectively, compact β-Carleson). See [**7**] for further details.

Throughout the rest of this work, C will denote a positive constant that may vary from one appearance to the next.

Lemma 2.1. *Suppose that $C_\Phi : D_\alpha \to D_\beta$ is bounded, where $-1 < \beta < \alpha$. Then C_Φ is compact on D_β.*

Proof. The hypothesis implies that $\Phi \in D_\beta$, and thus the measure μ_β is finite. Since $C_\Phi : D_\alpha \to D_\beta$ is bounded, there is a constant C with $\|C_\Phi(f)\|_{D_\beta} \le C\|f\|_{D_\alpha}$ for all $f \in D_\alpha$. Let $g \in A^2_\alpha$, and let $f(z) = \int_0^z g(w)\, dw$. Then $f \in D_\alpha$ and

$$\|(g \circ \Phi)\Phi'\|_{A^2_\beta} \le \|f \circ \Phi\|_{D_\beta} \le C\|f\|_{D_\alpha} = C\|g\|_{A^2_\alpha}.$$

Since the measure $(\log(1/|z|))^\beta\, dA(z)$ is comparable to $(1-|z|^2)^\beta\, dA(z)$, the change of variable formula from measure theory (see, for example, [**7**, page 891]) yields

$$\int_D |g(\Phi(z))|^2 d\mu_\beta(z) = \int_D |g|^2 d(\mu_\beta\Phi^{-1}) \le C\|g\|^2_{A^2_\alpha}.$$

Thus the identity map $I : A^2_\alpha \to L^2(\mu_\beta\Phi^{-1})$ is bounded. By [**7**, Theorem 4.3] the measure $\mu_\beta\Phi^{-1}$ is α-Carleson, that is, there is a constant C such that

$$\mu_\beta\Phi^{-1}(S(\zeta,\delta)) < C\delta^{\alpha+2}$$

for all $|\zeta| = 1$ and $0 < \delta \le 2$. Thus

$$\sup_{|\zeta|=1} \frac{\mu_\beta\Phi^{-1}(S(\zeta,\delta))}{\delta^{\beta+2}} < C\delta^{\alpha-\beta},$$

and it follows that $\mu_\beta\Phi^{-1}$ is compact β-Carleson. Proposition 5.1 in [**7**] implies that $C_\Phi : D_\beta \to D_\beta$ is compact. $\square$

Theorem 2.2. *Suppose that* $-1 < \beta < 0$. *Let* $\Phi \in D_\beta$. *The following are equivalent.*

(1) $\|\Phi\|_\infty < 1$.

(2) $C_\Phi : D_\alpha \to D_\beta$ *is compact for all* $\alpha > \beta$.

(3) $C_\Phi : D_\alpha \to D_\beta$ *is bounded for all* $\alpha > \beta$.

Proof. Standard arguments yield (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Suppose that $C_\Phi : D_\alpha \to D_\beta$ is bounded for some $\alpha > \beta$. By Lemma 2.1, $C_\Phi : D_\beta \to D_\beta$ is compact on D_β. Shapiro ([**9**, Theorem 2.1]) proved that for negative β, C_Φ is compact on D_β if and only if $\Phi \in D_\beta$ and $\|\Phi\|_\infty < 1$. □

In the remainder of this work we consider the case $0 \leq \beta < \alpha$.

Let $0 < |\lambda| < 1$, and let $\sigma_\lambda(z) = (\lambda - z)/(1 - \overline{\lambda} z)$. It is easy to show that C_{σ_λ} is bounded on D_α for all $\alpha \geq 0$. Thus we may assume that $\Phi(0) = 0$ in the rest of this work.

For $w \in D$ and Φ a self-map of the disc, $N_0(w)$ denotes the number of preimages, counting multiplicities, of w. The generalized Nevanlinna counting function N_γ $(\gamma > 0)$ was introduced by Shapiro in [**11**].

Definition 2.3. Let Φ be an analytic self-map of the disc, and let $\gamma > 0$. For $w \in D$ with $w \neq \Phi(0)$,

$$N_\gamma(w) = \sum (\log(1/|z|))^\gamma$$

where the sum extends over all zeros of $\Phi - w$, repeated by multiplicity.

Smith ([**13**, Corollary 4.4]) used the generalized counting function to characterize self-maps Φ for which $C_\Phi : A^p_\alpha \to A^q_\beta$ is bounded or compact in the case $0 < p \leq q$ and $\alpha \geq -1$, $\beta \geq -1$. His result is stated here for $p = q = 2$. Since $D_\beta = A^2_{\beta-2}$ when $\beta \geq 1$, Corollary 2.5 follows immediately.

Theorem 2.4 [**13**]. *Let* Φ *be an analytic self-map of the disc. Let* $\alpha \geq -1$ *and* $\beta \geq -1$. *The following are equivalent.*

(1) $C_\Phi : A^2_\alpha \to A^2_\beta$ *is bounded* (*respectively, compact*).

(2) $N_{\beta+2}(w) = O((1 - |w|^2)^{\alpha+2})$ *as* $|w| \to 1$ (*respectively, little oh*).

Corollary 2.5. *Let $\alpha > \beta \geq 1$. The following are equivalent.*

(1) $C_\Phi : D_\alpha \to D_\beta$ *is bounded* (*respectively, compact*).

(2) $N_\beta(w) = O((1-|w|^2)^\alpha)$ *as* $|w| \to 1$ (*respectively, little oh*).

Suppose that $\gamma > \beta > 0$. Smith [**13**] proved that $N_\gamma(w) \leq (N_\beta(w))^{\gamma/\beta}$ for $w \neq \Phi(0)$. If Φ is of finite valence, that is, there exists an $M < \infty$ such that $\Phi^{-1}(w)$ has at most M elements for all $w \in D$, then $N_\beta(w) \leq M(N_\gamma(w))^{\beta/\gamma}$, and thus $(N_\beta(w))^\gamma \approx (N_\gamma(w))^\beta$ [**13**]. Corollary 2.5 yields the following results.

Corollary 2.6. *Fix $P > 1$, and let $\alpha \geq P$. If $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded* (*respectively, compact*), *then $C_\Phi : D_{\widehat{\alpha}} \to D_{\widehat{\alpha}/P}$ is bounded* (*compact*) *for all $\widehat{\alpha} > \alpha$.*

Proof. Suppose that $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded. Since $\alpha > \alpha/P \geq 1$, Corollary 2.5 yields $N_{\alpha/P}(w) = O((1-|w|^2)^\alpha)$ as $|w| \to 1$. Since $\widehat{\alpha} > \alpha$, the previous remarks show that $N_{\widehat{\alpha}/P}(w) = O((1-|w|^2)^{\widehat{\alpha}})$. Thus Corollary 2.5 implies that $C_\Phi : D_{\widehat{\alpha}} \to D_{\widehat{\alpha}/P}$ is bounded. The proof of the statement about compactness is similar. $\square$

Corollary 2.7. *Fix $P > 1$, and suppose that Φ is finite valent. The following are equivalent.*

(1) $C_\Phi : D_\alpha \to D_{\alpha/P}$ *is bounded* (*respectively, compact*) *for some* $\alpha \geq P$.

(2) $C_\Phi : D_\alpha \to D_{\alpha/P}$ *is bounded* (*respectively, compact*) *for all* $\alpha \geq P$.

Proof. Suppose that $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded for some $\alpha \geq P$. By Corollary 2.5,

$$N_{\alpha/P}(w) = O((1-|w|^2)^\alpha)$$

as $|w| \to 1$. Assume $\widehat{\alpha} \geq P$. Since Φ is finite valent,

$$N_{\widehat{\alpha}/P}(w) \approx (N_{\alpha/P}(w))^{\widehat{\alpha}/\alpha},$$

and thus $N_{\widehat{\alpha}/P}(w) = O((1-|w|^2)^{\widehat{\alpha}})$. Corollary 2.5 implies that $C_\Phi : D_{\widehat{\alpha}} \to D_{\widehat{\alpha}/P}$ is bounded. The proof of the statement about compactness is similar. $\square$

More general versions of Corollaries 2.6 and 2.7 will be given in the next section.

3. Theorem 2.2 and Corollary 2.5 characterize the maps Φ for which $C_\Phi : D_\alpha \to D_\beta$ is bounded (or compact) for $\alpha > \beta$ in the cases $\beta < 0$ and $\beta \geq 1$. In this section we assume that $\alpha > \beta$ and $0 \leq \beta < 1$.

The argument for Theorem 3.1 is based on [**7**, Theorem 4.3], where MacCluer and Shapiro prove that a finite positive Borel measure ν is α-Carleson if and only if $A^2_\alpha \subset L^2(\nu)$. In this case, there exists $C > 0$ with $\|f\|_{L^2(\nu)} \leq C\|f\|_{A^2_\alpha}$ for all $f \in A^2_\alpha$. Furthermore, if ν is α-Carleson, then the identity map $I : A^2_\alpha \to L^2(\nu)$ is compact if and only if ν is compact α-Carleson. See [**7**] for further details.

Theorem 3.1. *Let $0 \leq \beta < \alpha$, and let $\Phi \in D_\beta$. Let ν_β be the measure defined by $d\nu_\beta(w) = N_\beta(w)\, dA(w)$.*

(1) *$C_\Phi : D_\alpha \to D_\beta$ is bounded if and only if ν_β is α-Carleson.*

(2) *$C_\Phi : D_\alpha \to D_\beta$ is compact if and only if ν_β is compact α-Carleson.*

Proof. Since $\Phi \in D_\beta$, a change of variable as in Shapiro [**11**] implies that ν_β is a finite measure. For details, see [**11**, page 398] or, for $\beta = 0$, [**1**, page 36].

Suppose that $C_\Phi : D_\alpha \to D_\beta$ is bounded. Thus there is a positive constant C such that $\|C_\Phi(f)\|_{D_\beta} \leq C\|f\|_{D_\alpha}$ for all $f \in D_\alpha$. Let $g \in A^2_\alpha$, and let $f(z) = \int_0^z g(w)\, dw$. Then $f \in D_\alpha$ and $\|f\|_{D_\alpha} = \|g\|_{A^2_\alpha}$. Since $\Phi(0) = 0$, $\|(f \circ \Phi)'\|_{A^2_\beta} = \|C_\Phi(f)\|_{D_\beta}$, and thus

$$\int_D |g(\Phi(z))|^2 |\Phi'(z)|^2 (\log(1/|z|))^\beta dA(z) \leq C\|g\|^2_{A^2_\alpha}.$$

The change of variables formula (see [**11**, page 398]) yields

$$\begin{aligned}\int_D |g(\Phi(z))|^2 |\Phi'(z)|^2 \,(\log(1/|z|))^\beta dA(z) &= \widehat{C}\int_D |g(w)|^2 N_\beta(w) dA(w) \\ &= \widehat{C}\int_D |g|^2 d\nu_\beta\end{aligned}$$

where $\widehat{C}$ is a constant depending only on β. It follows that

$$\|g\|^2_{L^2(\nu_\beta)} \le C\|g\|^2_{A^2_\alpha}.$$

Thus, $A^2_\alpha \subset L^2(\nu_\beta)$. Theorem 4.3 [**7**] implies that ν_β is α-Carleson.

For the converse of the first statement, assume that ν_β is α-Carleson. Thus $\|g\|^2_{L^2(\nu_\beta)} \le C\|g\|^2_{A^2_\alpha}$ for a positive constant C and for all $g \in A^2_\alpha$. In particular, let $f \in D_\alpha$, and let $g = f'$. A change of variable as in the previous argument yields

$$\|(f\circ\Phi)'\|^2_{A^2_\beta} \le C\|f\|^2_{D_\alpha}.$$

Since $|f(\Phi(0))|^2 = |f(0)|^2 \le \|f\|^2_{D_\alpha}$, it follows that $C_\Phi : D_\alpha \to D_\beta$ is bounded.

If $C_\Phi : D_\alpha \to D_\beta$ is compact, then $\|f_n\circ\Phi\|_{D_\beta} \to 0$ as $n\to\infty$ for any sequence in D_α with $\|f_n\|_{D_\alpha} \le C$ and $f_n \to 0$ uniformly on compact subsets as $n\to\infty$. Assume that $\|g_n\|_{A^2_\alpha} \le C$ and $g_n \to 0$ uniformly on compact sets. Let f_n be the antiderivative of g_n with $f_n(0) = 0$. The change of variables yields

$$\|g_n\|^2_{L^2(\nu_\beta)} = \widehat{C}\|(f_n'\circ\Phi)\Phi'\|^2_{A^2_\beta} \le \widehat{C}\|f_n\circ\Phi\|^2_{D_\beta}.$$

Thus $\|g_n\|_{L^2(\nu_\beta)} \to 0$ for any sequence g_n as described. Theorem 4.3 [**7**] implies that ν_β is compact α-Carleson.

Finally, if ν_β is compact α-Carleson, $\|f_n\|_{D_\alpha} \le C$ and $f_n \to 0$ uniformly on compact sets, then an argument as above shows that $\|(f_n\circ\Phi)'\|_{A^2_\beta} \to 0$. Since $|f_n(\Phi(0))| \to 0$, this yields $\|f_n\circ\Phi\|_{D_\beta} \to 0$ for any sequence f_n as described. Thus $C_\Phi : D_\alpha \to D_\beta$ is compact. The proof is complete. $\square$

Suppose that $0<\beta<1$, $\Phi\in D_\beta$ and $N_\beta(w) = O((1-|w|^2)^\alpha)$ as $|w|\to 1$ for some $\alpha>\beta$. It follows that there exists a constant $C>0$ with

$$\nu_\beta(S(\zeta,\delta)) = \int_{S(\zeta,\delta)} N_\beta(w)\,dA(w) \le C\delta^{\alpha+2}$$

for all $|\zeta| = 1$ and $0<\delta\le 2$. Theorem 3.1 implies that $C_\Phi : D_\alpha \to D_\beta$ is bounded. Thus the conditions in Corollary 2.5 remain sufficient to imply boundedness even when $0 < \beta < 1$. A similar argument holds for compactness. Lemma 3.2 and Theorem 3.3 will show that the conditions in Corollary 2.5 are necessary if Φ is of finite valence. Thus Theorem 3.1 can be simplified in this case. First note that if Φ is of finite valence, then Φ belongs to D_0 and thus $\Phi \in D_\beta$ for all $\beta > 0$.

Lemma 3.2. *Let $\beta > 0$, $f \in D_\beta$ and $z \in D$. There is a positive constant C_β depending only on β such that*

$$|f(z)| \leq C_\beta \|f\|_{D_\beta} (1 - |z|^2)^{-\beta/2}.$$

Proof. Smith [**13**, Lemma 2.5] proved that there is a constant C depending only on β such that

$$|g(w)| \leq C\|g\|_{A^2_\beta} \ (1 - |w|)^{-(\beta+2)/2}$$

for all $g \in A^2_\beta$ and all $w \in D$. Let $f \in D_\beta$ and fix $z \in D$. Then $f' \in A^2_\beta$ and Smith's estimate yield

$$\begin{aligned} |f(z)| &\leq \int_0^1 |f'(tz)||z| \, dt + |f(0)| \\ &\leq C\|f\|_{D_\beta} \int_0^1 |z|(1 - t|z|)^{-(\beta+2)/2} \, dt + |f(0)| \\ &\leq \frac{2}{\beta} C\|f\|_{D_\beta} (1 - |z|)^{-\beta/2} + \|f\|_{D_\beta}. \end{aligned}$$

The result follows. $\square$

Let $\alpha > 0$ and let $a \in D$. Let

$$k_a(z) = \frac{(1 - |a|^2)^{(\alpha+2)/2}}{(1 - \overline{a}z)^{\alpha+2}}$$

for $|z| < 1$. Smith [**13**, page 2340] noted that, for each α, there are positive constants C_1 and C_2 depending only on α with

$$C_1 \leq \|k_a\|^2_{A^2_\alpha} \leq C_2$$

for all $a \in D$. For $a \neq 0$, let

$$f_a(z) = \int_0^z k_a(w)\, dw.$$

Then $f_a(0) = 0$, $\|f_a\|_{D_\alpha}^2 = \|k_a\|_{A_\alpha^2}^2$, and

$$f_a(z) = \frac{(1-|a|^2)^{(\alpha+2)/2}}{(\alpha+1)\overline{a}}\{(1-\overline{a}z)^{-(\alpha+1)} - 1\}.$$

Theorem 3.3. *Let $0 < \beta < \alpha$.*

(1) *Fix r_0 and $0 < r_0 < 1$. If $C_\Phi : D_\alpha \to D_\beta$ is bounded, then there is a positive constant C depending only on α, β and r_0 such that*

$$1 - |z|^2 \leq C(1 - |\Phi(z)|^2)^{\alpha/\beta}$$

for all $z \in D$ with $|\Phi(z)| > r_0$.

(2) *Suppose that $C_\Phi : D_\alpha \to D_\beta$ is compact. If $\varepsilon > 0$, then there exists r_1, $0 < r_1 < 1$, such that*

$$1 - |z|^2 < \varepsilon(1 - |\Phi(z)|^2)^{\alpha/\beta}$$

for all $z \in D$ with $|\Phi(z)| > r_1$.

Proof. We may assume that $\|\Phi\|_\infty = 1$. To prove the first assertion, assume that $C_\Phi : D_\alpha \to D_\beta$ is bounded and r_0 is fixed. Thus there exists a $C > 0$ with $\|C_\Phi(f)\|_{D_\beta} \leq C\|f\|_{D_\alpha}$ for all $f \in D_\alpha$, and Lemma 3.2 implies that

$$|C_\Phi(f)(z)| \leq CC_\beta \|f\|_{D_\alpha}(1-|z|^2)^{-\beta/2}$$

for all $z \in D$ and all $f \in D_\alpha$. Fix $z \in D$ with $|\Phi(z)| > r_0$, and let $w = \Phi(z)$. A calculation shows that

$$\begin{aligned} |C_\Phi(f_w)(z)| &> \frac{(1-|w|^2)^{(\alpha+2)/2}}{\alpha+1}\{(1-|w|^2)^{-(\alpha+1)} - 1\} \\ &\geq \frac{(1-|w|^2)^{-\alpha/2}}{\alpha+1}\left\{1 - (1-r_0^2)^{\alpha+1}\right\}. \end{aligned}$$

The argument shows that

$$(1-|w|^2)^{-\alpha/2} \le CC_\beta \|f_w\|_{D_\alpha}(1-|z|^2)^{-\beta/2}\frac{\alpha+1}{1-(1-r_0^2)^{\alpha+1}}$$

for all z with $|\Phi(z)| > r_0$. Since $\|f_w\|_{D_\alpha} \approx 1$ for $w \in D$ $(w \neq 0)$, this yields a constant K such that

$$(1-|z|^2)^{\beta/2} \le K(1-|\Phi(z)|^2)^{\alpha/2}$$

for all $z \in D$ with $|\Phi(z)| > r_0$, where the constant K depends only on α, β and r_0. This completes the proof of the first assertion.

Next suppose that $C_\Phi : D_\alpha \to D_\beta$ is compact. By way of contradiction, suppose that $\varepsilon_0 > 0$ and that there is no r_1 as described. Thus there is a sequence $z_n \subset D$ with $|\Phi(z_n)| > 1 - 1/(n+1)$ and

$$1-|z_n|^2 \ge \varepsilon_0(1-|\Phi(z_n)|^2)^{\alpha/\beta}$$

for $n = 1, 2, \dots$. Let $w_n = \Phi(z_n)$. Since $|w_n| \to 1$, it follows that $f_{w_n} \to 0$ uniformly on compact subsets of D as $n \to \infty$. Since $\|f_{w_n}\|_{D_\alpha} \le C$ for all n and since $C_\Phi : D_\alpha \to D_\beta$ is compact, there is a subsequence which we continue to call w_n for which $\|f_{w_n} \circ \Phi\|_{D_\beta} \to 0$. Lemma 3.2 yields

$$(1-|z_n|^2)^{\beta/2}|f_{w_n}(\Phi(z_n))| \le C_\beta \|f_{w_n} \circ \Phi\|_{D_\beta}.$$

Estimates as in the proof of part (1) imply that the left-hand side of the previous inequality is bounded away from 0. This contradiction completes the proof. $\square$

Theorem 3.4. *Suppose that Φ is of finite valence and $0 < \beta < \alpha$.*

(1) $C_\Phi : D_\alpha \to D_\beta$ *is bounded* $\Leftrightarrow N_\beta(w) = O((1-|w|^2)^\alpha)$ *as* $|w| \to 1$.

(2) $C_\Phi : D_\alpha \to D_\beta$ *is compact* $\Leftrightarrow N_\beta(w) = o((1-|w|^2)^\alpha)$ *as* $|w| \to 1$.

Proof. Sufficiency of the conditions involving N_β was noted previously in this section.

Assume that each $w \in D$ has no more than M preimages, counting multiplicities, under Φ. There exists R, $0 < R < 1$, such that $\log(1/|z|) < 1 - |z|^2$ for all z with $R < |z| < 1$.

Now let $w = \Phi(z)$, and assume that $|w| > R$. Since $\Phi(0) = 0$, the Schwarz lemma implies that $|z| \geq |w| > R$, and thus

$$N_\beta(w) = \sum (\log(1/|z|))^\beta \leq \sum (1-|z|^2)^\beta$$

where both sums extend over all preimages of w. Since $C_\Phi : D_\alpha \to D_\beta$ is bounded, part (1) of Theorem 3.3 yields

$$N_\beta(w) \leq MC(1-|w|^2)^\alpha$$

for all w with $|w| > R$. Thus $N_\beta(w) = O((1-|w|^2)^\alpha)$ as $|w| \to 1$. This completes the proof of the first assertion.

Next assume that $C_\Phi : D_\alpha \to D_\beta$ is compact. Let $\varepsilon > 0$, and let $\gamma = (\varepsilon/M)^{1/\beta}$. Since $\gamma > 0$, part (2) of Theorem 3.3 provides r_1 such that

$$1-|z|^2 < \gamma(1-|\Phi(z)|^2)^{\alpha/\beta}$$

for all z with $|\Phi(z)| > r_1$. We may assume that $r_1 > R$, so that the previous estimate on $\log(1/|z|)$ holds. Thus

$$N_\beta(w) \leq \sum (1-|z|^2)^\beta \leq \gamma^\beta (1-|w|^2)^\alpha M = \varepsilon(1-|w|^2)^\alpha$$

for all $w \in D$ with $|w| > r_1$. The proof is complete. $\square$

Corollary 3.5. *Let $P > 1$, and let $\alpha > 0$.*

(1) *If $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded (respectively, compact) and if $\widehat{\alpha} > \alpha$, then $C_\Phi : D_{\widehat{\alpha}} \to D_{\widehat{\alpha}/P}$ is bounded (compact).*

(2) *Suppose that Φ is finite valent. If $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded (respectively, compact) for some $\alpha > 0$, then $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded (compact) for all $\alpha > 0$.*

Proof. Suppose that $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded, and let $\widehat{\alpha} > \alpha$. Suppose $w \in S(\zeta, \delta) \cap \Phi(D)$ where $\delta < 1 - R$ and R is as described in the previous proof. Then

$$R < 1 - \delta < |\Phi(z)| \leq |z|$$

for all z with $\Phi(z) = w$ and part (1) of Theorem 3.3 yields

$$\begin{aligned} N_{\widehat{\alpha}/P}(w) &\le \sum (\log 1/|z|)^{\alpha/P}(1-|z|^2)^{(\widehat{\alpha}-\alpha)/P} \\ &\le C(1-|w|^2)^{\widehat{\alpha}-\alpha} N_{\alpha/P}(w) \\ &< C\,\delta^{\widehat{\alpha}-\alpha} N_{\alpha/P}(w). \end{aligned}$$

It follows that

$$\nu_{\widehat{\alpha}/P}(S(\zeta,\delta)) \le C\delta^{\widehat{\alpha}-\alpha}\nu_{\alpha/P}(S(\zeta,\delta))$$

for $|\zeta| = 1$ and $\delta < 1 - R$. Since $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded, Theorem 3.1 yields $\nu_{\alpha/P}(S(\zeta,\delta)) < C\delta^{\alpha+2}$ for all $|\zeta| = 1$ and $0 < \delta \le 2$, where C depends only on α and P. Thus,

$$\nu_{\widehat{\alpha}/P}(S(\zeta,\delta)) \le C\,\delta^{\widehat{\alpha}+2}$$

for all $|\zeta| = 1$ and all small δ. This is sufficient to establish that $\nu_{\widehat{\alpha}/P}$ is $\widehat{\alpha}$-Carleson. Part (1) of Theorem 3.1 yields that $C_\Phi : D_{\widehat{\alpha}} \to D_{\widehat{\alpha}/P}$ is bounded. The proof of the statement about compactness is similar. This completes the proof of part (1).

Next suppose that Φ is finite valent and $C_\Phi : D_\alpha \to D_{\alpha/P}$ is bounded for some $\alpha > 0$. Theorem 3.4 yields $N_{\alpha/P}(w) = O((1-|w|^2)^\alpha)$ as $|w| \to 1$. Let $\widehat{\alpha} > 0$. Since $N_{\widehat{\alpha}/P}(w) \approx (N_{\alpha/P}(w))^{\widehat{\alpha}/\alpha}$ [**13**], it follows that

$$N_{\widehat{\alpha}/P}(w) = O((1-|w|^2)^{\widehat{\alpha}})$$

as $|w| \to 1$. Theorem 3.4 implies that $C_\Phi : D_{\widehat{\alpha}} \to D_{\widehat{\alpha}/P}$ is bounded.

The proof of the statement about compactness is omitted. □

The next two results are in comparison with Theorem 2.2.

Theorem 3.6. *The condition $\|\Phi\|_\infty < 1$ is not necessary for $C_\Phi : D_\alpha \to D_\beta$ to be compact in the case $\alpha > \beta > 0$.*

Proof. Fix $\alpha > \beta > 0$, and fix η with $\eta > \alpha/\beta$. Let $P \subset \overline{D}$ be a polygon with $P \cap \partial D = \{1\}$ and with angular aperture π/η at 1. Let

Φ be a Riemann map from D onto the interior of P. Then Φ extends to a homeomorphism of $\overline{D}$ onto P, and we may assume that $\Phi(1) = 1$. As in [**13**] it follows that

$$N_1(w) = O((1 - |w|^2)^\eta)(|w| \to 1).$$

Since Φ is univalent, $N_\beta(w) \approx (N_1(w))^\beta$. Thus,

$$N_\beta(w) = O((1 - |w|^2)^{\eta\beta})(|w| \to 1)$$

and

$$\frac{N_\beta(w)}{(1 - |w|^2)^\alpha} \le C(1 - |w|^2)^{\eta\beta - \alpha} \longrightarrow 0$$

as $|w| \to 1$. By Theorem 3.4, $C_\Phi : D_\alpha \to D_\beta$ is compact. $\square$

The proof of the next theorem is a minor adaptation of a result of Jovović and MacCluer [**4**, page 232].

Theorem 3.7. *Let $\alpha > 0$. The condition $\|\Phi\|_\infty < 1$ is not necessary for $C_\Phi : D_\alpha \to D_0$ to be compact.*

Proof. An example will be given in which $\|\Phi\|_\infty = 1$ and $C_\Phi : D_1 \to D_0$ is compact. Let Ω be a simply connected region in $\overline{D}$ with $\Omega \cap \partial D = \{1\}$. In a neighborhood of 1, assume that the region Ω is symmetric about the x-axis, and the upper boundary of Ω is given by $y = (1 - x^{1/2})^3$. Let Φ be a univalent map carrying D onto the interior of Ω. A calculation yields

$$\nu_0(S(1, \delta)) = 2 \int_{1-\delta}^1 (1 - x^{1/2})^3 \, dx = o(\delta^3).$$

Since it suffices to verify the Carleson condition at $\zeta = 1$, Theorem 3.1 now implies that $C_\Phi : D_1 \to D_0$ is compact, and it follows easily that $C_\Phi : D_\alpha \to D_0$ is compact for $0 < \alpha \le 1$. The example can be adapted for any natural number $n \ge 2$, using the function $y = (1 - x^{1/2})^{n+2}$. Calculations as above show that $C_\Phi : D_n \to D_0$ is compact. $\square$

The final result relates this work to the work of Smith [**13**] and Riedl [**8**]. In what follows, A^p denotes the standard unweighted Bergman space. Smith proved that, for $0 < p \leq q$, $C_\Phi : A^p \to H^q$ is bounded (respectively, compact) if and only if $N_1(w) = O((1-|w|)^{2q/p})$ (respectively, little oh). Riedl proved that, for $0 < p \leq q$, $C_\Phi : H^p \to H^q$ is bounded if and only if $N_1(w) = O((1-|w|)^{q/p})$, with the analogous statement for compactness. Smith [**13**] obtained the corollary that for $\eta \geq 2$, $C_\Phi : H^p \to H^{p\eta}$ is bounded for some (for all) $p > 0$ if and only if $C_\Phi : A^p \to H^{p\eta/2}$ is bounded for some (for all) $p > 0$. The analogous statement holds for compactness.

Fix $\eta \geq 2$ and assume that Φ is of finite valence and $p > 0$. By Theorem 3.4, $C_\Phi : D_p \to D_{p/\eta}$ is bounded if and only if $N_{p/\eta}(w) = O((1-|w|)^p)(|w| \to 1)$. Since $N_1(w) \approx (N_{p/\eta}(w))^{\eta/p}$, Smith's corollary leads to the following result. The proof is omitted.

Corollary 3.8. *Let Φ be of finite valence, and let $\eta \geq 2$. The following are equivalent.*

(1) $C_\Phi : D_p \to D_{p/\eta}$ *is bounded for some* (*for all*) $p > 0$.

(2) $C_\Phi : A^p \to H^{p\eta/2}$ *is bounded for some* (*for all*) $p \geq 0$.

(3) $C_\Phi : H^p \to H^{p\eta}$ *is bounded for some* (*for all*) $p > 0$.

The statements remain equivalent if "bounded" is replaced by "compact".

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