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THE NUMBER OF KHALIMSKY-CONTINUOUS FUNCTIONS ON INTERVALS

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ABSTRACT. We determine the number of Khalimsky-continuous functions defined on an interval and with values in an interval.

1. Introduction. Digital geometry was developed as a geometry for the computer screen, the elements of which are pixels organized in a grid. It is natural to use pairs of integers as addresses of the pixels; hence the use of $\mathbf{Z}^2$, or more generally $\mathbf{Z}^n$, as the basic space.

Discretization in general is important in many other branches of mathematics, one of them being analysis with a focus on continuous functions. To define a continuous function we need a topological structure on $\mathbf{Z}^n$. Khalimsky et al. [**5**] defined a connected topology on $\mathbf{Z}^n$. We shall define here the Khalimsky topology in Section 1 in a simple way just by using open subsets of $\mathbf{Z}$ and then going to higher dimensions using a product topology. We shall discuss more about the Khalimsky topology and Khalimsky-continuous function in Section 1 (for more information on these subjects, see Kiselman [**6**] and Melin [**10, 11**]).

A subject which has been studied extensively is the digital straight line segment. (For more information about this topic see Kiselman [**6**], Klette and Rosenfeld [**7, 8**], Melin [**9, 10**] and Samieinia [**12**].) The pioneering combinatorial study on digital straight line segment was made by Berenstein and Lavine [**2**]. They described in their common work the number of discrete segments of slope $0 \leq \alpha \leq 1$ of length L. Bédaride et al. [**1**] worked on the number of digital segments with given length and height. Other combinatorial aspect in digital geometry is the digital disc, i.e., the set of all integer points inside some given disc. Huxley and Zunić [**4**] studied the number of different digital discs consisting of N points and showed an upper bound for it.

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In this paper we shall study the Khalimsky-continuous functions from a combinatorial point of view. We shall determine the number of continuous functions which are defined on an interval of the digital line $\mathbf{Z}$ equipped with the Khalimsky topology and with values in that line. We begin by recalling the definition and first properties of the Khalimsky topology and then consider Khalimsky-continuous functions. In Section 2, we consider these functions when they have two points in the codomain. In this section we shall present a new example of the classical Fibonacci sequence. In Sections 3 and 4, we study the Khalimsky-continuous functions with three or four points in their codomain, and as a consequence of these studies we find some new sequences, the asymptotic behavior of which we investigate.

The Khalimsky topology. There are several different ways to introduce the Khalimsky topology on the integers. We present the Khalimsky topology using a topological basis. For every even integer m, the set $\{m-1, m, m+1\}$ is open, and for every odd integer n, the singleton $\{n\}$ is open. A basis is given by

$$\{\{2n+1\}, \{2n-1, 2n, 2n+1\};\ n \in \mathbf{Z}\}.$$

It follows that even points are closed. A digital interval $[a,b]_{\mathbf{Z}} = [a,b] \cap \mathbf{Z}$ with the subspace topology is called a *Khalimsky interval*, and a homeomorphic image of a Khalimsky interval into a topological space is called a *Khalimsky arc*. On the digital plane $\mathbf{Z}^2$, the Khalimsky topology is given by the product topology. A point with both coordinates odd is open. If both coordinates are even, the point is closed. These types of points are called *pure*. Points with one even and one odd coordinate are neither open nor closed; these are called *mixed*. Note that a mixed point $m = (m_1, m_2)$ is connected only to its 4-neighbors,

$$(m_1 \pm 1, m_2) \quad \text{and} \quad (m_1, m_2 \pm 1),$$

whereas a pure point $p = (p_1, p_2)$ is connected to all its 8-neighbors,

$$(p_1 \pm 1, p_2), (p_1, p_2 \pm 1), (p_1 + 1, p_2 \pm 1) \text{ and } (p_1 - 1, p_2 \pm 1).$$

More information on the Khalimsky plane and the Khalimsky topology can be found in [**6**].

Khalimsky-continuous functions. When we equip $\mathbf{Z}$ with the Khalimsky topology, we may speak of continuous functions $\mathbf{Z} \to \mathbf{Z}$, i.e., functions for which the inverse image of open sets are open. It is easily proved that a continuous function f is Lipschitz with constant 1. This is however not sufficient for continuity. It is not hard to prove that $f: \mathbf{Z} \to \mathbf{Z}$ is continuous if and only if (i) f is Lipschitz with constant 1 and (ii) for every x, $x \not\equiv f(x) \pmod 2$ implies $f(x \pm 1) = f(x)$. For more information see [**9, 10**].

For example, we observe that the following functions are continuous:

(1) $\mathbf{Z} \ni x \mapsto a \in \mathbf{Z}$, where a is constant;

(2) $\mathbf{Z} \ni x \mapsto \pm x + c \in \mathbf{Z}$, where c is an even constant;

(3) $\max(f, g)$ and $\min(f, g)$ if f and g are continuous.

Actually every continuous function on a bounded Khalimsky interval can be obtained by a finite succession of the rules (1), (2), (3); see [**6**].

2. Continuous functions with a two-point codomain. We now look at the functions which take their values in an interval consisting of two points. It turns out that the number of such functions is given by the Fibonacci sequence.

Theorem 2.1. *Let a_n be the number of Khalimsky-continuous functions $[0, n-1]_{\mathbf{Z}} \to [0, 1]_{\mathbf{Z}}$. Then $a_n = F_{n+2}$, where $(F_n)_0^\infty$ is the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.*

Proof. Let $a_n^i = \operatorname{card}(\{f: [0, n-1]_{\mathbf{Z}} \to [0, 1]_{\mathbf{Z}}; f(n-1) = i\})$ for $i = 0, 1$, so that

$$a_n = a_n^0 + a_n^1. \tag{2.1}$$

By the definition of the Khalimsky topology, we see that

$$\begin{aligned} a_{2k+1}^0 &= a_{2k}, \qquad k \geq 1, \\ a_{2k+1}^1 &= a_{2k}^1, \qquad k \geq 1. \end{aligned} \tag{2.2}$$

Moreover,

$$\begin{aligned} a_{2k}^1 &= a_{2k-1}, \qquad k \geq 1, \\ a_{2k}^0 &= a_{2k-1}^0, \qquad k \geq 1. \end{aligned} \tag{2.3}$$

Hence, using in turn (2.1), (2.2) and (2.3),

$$a_{2k+1} = a^0_{2k+1} + a^1_{2k+1} = a_{2k} + a^1_{2k} = a_{2k} + a_{2k-1},$$

which is the Fibonacci relation. Similarly, by using (2.1), (2.3) and (2.2), we get

$$a_{2k} = a^0_{2k} + a^1_{2k} = a^0_{2k-1} + a_{2k-1} = a_{2k-2} + a_{2k-1}.$$

Now we need only observe that $a_1 = 2 = F_3$ and $a_2 = 3 = F_4$. $\square$

We notice that Theorem 2.1 leads us to a new example of the classical Fibonacci sequence. We list the number a_n of Khalimsky-continuous functions for $n = 1, \dots, 14$ in the next table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_n	2	3	5	8	13	21	34	55	89	144	233	377	610	987

The asymptotic behavior of the number of continuous functions with a two-point codomain. We consider two frequencies

$$P^0_n = \frac{a^0_n}{a_n},$$

and

$$P^1_n = \frac{a^1_n}{a_n}.$$

By (2.1),

$$P^0_n + P^1_n = 1. \tag{2.4}$$

We shall determine these frequencies asymptotically. First, we recall the interesting property of the Fibonacci sequence: the fraction F_{n+1}/F_n tends to α as $n \to \infty$, where α denotes the Golden Ratio $(\sqrt{5}+1)/2$. Therefore F_{n+1}/F_{n-1} tends to α^2. In the following theorem we consider the frequencies for odd and even indices separately.

Theorem 2.2. *Let a_n and a_n^i be as in Theorem 2.1 and define $P_n^i = a_n^i/a_n$ for $i = 0, 1$. Then as $k \to +\infty$ we have that*

$$P_{2k-1}^0 \longrightarrow \frac{1}{\alpha}, P_{2k}^0 \longrightarrow \frac{1}{\alpha^2}$$

and

$$P_{2k-1}^1 \longrightarrow \frac{1}{\alpha^2}, P_{2k}^1 \longrightarrow \frac{1}{\alpha}$$

where $\alpha = (\sqrt{5}+1)/2$.

Proof. By (2.3) and (2.1),

$$a_{2k}^1 = a_{2k-1}^1 + a_{2k-1}^0,$$

therefore we obtain another relation between frequencies and the values of a_{2k} and a_{2k-1} as

$$P_{2k}^1 a_{2k} = P_{2k-1}^1 a_{2k-1} + P_{2k-1}^0 a_{2k-1}. \tag{2.5}$$

Then using (2.4) leads us to

$$P_{2k}^1 a_{2k} = a_{2k-1}.$$

Thus,

$$P_{2k}^1 = \frac{a_{2k-1}}{a_{2k}} \longrightarrow \frac{1}{\alpha} \quad \text{as } k \to +\infty.$$

By Theorem 2.1,

$$a_{2k} - P_{2k}^0 a_{2k} = a_{2k-1},$$

so

$$P_{2k}^0 = \frac{a_{2k} - a_{2k-1}}{a_{2k}}. \tag{2.6}$$

By using (2.1), (2.2) and (2.3),

$$a_{2k} - a_{2k-1} = a_{2k-2}, \tag{2.7}$$

thus by (2.6) and (2.7),

$$P_{2k}^0 = \frac{a_{2k-2}}{a_{2k}},$$

and so

$$P_{2k}^0 \longrightarrow \frac{1}{\alpha^2} \quad \text{as } k \to +\infty.$$

As before, we can find

$$P_{2k+1}^0 = \frac{a_{2k}}{a_{2k+1}},$$

implying that

$$P_{2k+1}^0 \longrightarrow \frac{1}{\alpha} \quad \text{as } k \to +\infty.$$

Also,

$$P_{2k+1}^1 = \frac{a_{2k-1}}{a_{2k+1}},$$

which implies that

$$P_{2k+1}^1 \longrightarrow \frac{1}{\alpha^2} \quad \text{as } k \to +\infty. \qquad \square$$

In the next table we can see the values of a_n^0, a_n^1, a_n, P_n^0, and P_n^1 for $n = 6, \dots, 13$.

n	6	7	8	9	10	11	12	13
a_n^0	5	13	13	34	34	89	89	233
a_n^1	8	8	21	21	55	55	144	144
a_n	13	21	34	55	89	144	233	377
P_n^0	0.3846	0.6190	0.3824	0.6182	0.3820	0.6181	0.382	0.618
P_n^1	0.6154	0.381	0.6176	0.382	0.618	0.382	0.618	0.382

3. Continuous functions with a three-point codomain. We summarize the results for functions with up to three values.

Theorem 3.1. *Let b_n be the number of Khalimsky-continuous functions $[0, n-1]_{\mathbf{Z}} \to [0, 2]_{\mathbf{Z}}$. Then $b_1 = 3$, $b_2 = 5$, and*

$$\begin{aligned} b_{2k} &= b_{2k-1} + b_{2k-2} + b_{2k-3} = 2b_{2k-2} + 3b_{2k-3}, && k \geq 2, \\ b_{2k-1} &= b_{2k-2} + 2b_{2k-3}, && k \geq 2. \end{aligned} \tag{3.1}$$

Proof. Let $b_n^i = \operatorname{card}(\{f\colon [0,n-1]_{\mathbf{Z}} \to [0,2]_{\mathbf{Z}}; f(n-1) = i\})$ for $i = 0,1,2$. Therefore it is clear that

$$b_n = b_n^0 + b_n^1 + b_n^2. \tag{3.2}$$

From the properties of the Khalimsky topology we see that

$$\begin{aligned} b_{2k}^0 &= b_{2k-1}^0, && k \geq 1,\\ b_{2k}^1 &= b_{2k-1}^0 + b_{2k-1}^1 + b_{2k-1}^2, && k \geq 1,\\ b_{2k}^2 &= b_{2k-1}^2, && k \geq 1, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} b_{2k-1}^0 &= b_{2k-2}^0 + b_{2k-2}^1, && k \geq 2,\\ b_{2k-1}^1 &= b_{2k-2}^1, && k \geq 2,\\ b_{2k-1}^2 &= b_{2k-2}^2 + b_{2k-2}^1, && k \geq 2. \end{aligned} \tag{3.4}$$

We assume that $n = 2k-1$ in equation (3.2); then using in turn (3.4) and (3.3) we obtain the equalities

$$b_{2k-1} = b_{2k-2} + 2b_{2k-2}^1 = b_{2k-2} + 2b_{2k-3}. \tag{3.5}$$

Now we need to do the same for $n = 2k$ in equation (3.2); then using in turn (3.3) and (3.4),

$$b_{2k} = b_{2k-1} + b_{2k-1}^0 + b_{2k-1}^2 = b_{2k-1} + b_{2k-2} + b_{2k-2}^1. \tag{3.6}$$

Now if we use equation (3.3) in (3.6) we can see the result for b_{2k}, i.e.,

$$b_{2k} = b_{2k-1} + b_{2k-2} + b_{2k-3}. \tag{3.7}$$

Another result for b_{2k} will be obvious if we put equation (3.5) into equation (3.7), i.e.,

$$\begin{aligned} b_{2k} &= b_{2k-1} + b_{2k-2} + b_{2k-3}\\ &= b_{2k-2} + 2b_{2k-3} + b_{2k-2} + b_{2k-3}\\ &= 2b_{2k-2} + 3b_{2k-3}. \qquad \square \end{aligned}$$

The Jacobsthal sequence is defined by $J_n = J_{n-1} + 2J_{n-2}$ with $J_1 = 0$ and $J_2 = 1$ (this is sequence number A001045 in Sloane's On-line

Encyclopedia of Integer Sequences), and the Tribonacci sequence is defined by the formula $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with initial values $1, 1, 1$ (sequence number A000213); by Theorem 3.1 we see that b_n is a mixture between the Tribonacci and Jacobsthal sequences.

We give below the sequence (b_n) for $n = 1, \ldots, 12$.

n	1	2	3	4	5	6	7	8	9	10	11	12
b_n	3	5	11	19	41	71	153	265	571	989	2131	3691

The asymptotic behavior of the number of continuous functions with a three-point codomain. We shall now determine how the number of continuous functions grows with the number of points in the domain.

Theorem 3.2. *Let b_n be the number of Khalimsky-continuous functions $[0, n-1]_{\mathbf{Z}} \to [0, 2]_{\mathbf{Z}}$. Then there is a sequence (t_n) tending to a positive limit $t = 1/2 + \sqrt{3}/6 \approx 0.788675$ as $k \to +\infty$ and such that*

$$\begin{aligned} b_{2k} &= t_{2k}\sqrt{3}\left(2+\sqrt{3}\right)^k, \qquad k \geq 2, \\ b_{2k-1} &= t_{2k-1}\left(2+\sqrt{3}\right)^k, \quad k \geq 2. \end{aligned} \tag{3.8}$$

Proof. We define a sequence (t_n) by the following equations,

$$\begin{aligned} t_{2k} &= b_{2k}\theta^{-1}\gamma^{-k}, \qquad k \geq 2, \\ t_{2k-1} &= b_{2k-1}\gamma^{-k}, \quad k \geq 2. \end{aligned} \tag{3.9}$$

Thus, using (3.1) and (3.9),

$$\begin{aligned} t_{2k} &= 2\gamma^{-1}t_{2k-2} + 3\gamma^{-1}\theta^{-1}t_{2k-3}, \quad k \geq 2, \\ t_{2k-1} &= \theta\gamma^{-1}t_{2k-2} + 2\gamma^{-1}t_{2k-3}, \qquad k \geq 2. \end{aligned} \tag{3.10}$$

With equation (3.10), we have the following equation for all $\theta, \gamma > 0$,

$$t_{2k} - t_{2k-1} = (2\gamma^{-1} - \theta\gamma^{-1})t_{2k-2} + (3\gamma^{-1}\theta^{-1} - 2\gamma^{-1})t_{2k-3}. \tag{3.11}$$

While this formula is true for all values of γ and θ, it is of interest mainly when the two coefficients in equation (3.11) sum up to zero. We therefore define γ and θ so that $2\gamma^{-1}-\theta\gamma^{-1}+3\gamma^{-1}\theta^{-1}-2\gamma^{-1}=0$. This implies $\theta=\sqrt{3}$.

Next we consider the equation for $t_{2k+1}-t_{2k}$,

$$\begin{aligned} t_{2k+1}-t_{2k} &= \theta\gamma^{-1}t_{2k}+2\gamma^{-1}t_{2k-1}-t_{2k} \\ &= (\theta\gamma^{-1}-1)t_{2k}+2\gamma^{-1}t_{2k-1}. \end{aligned} \tag{3.12}$$

In the same way we consider the special case of equation (3.12) when the coefficients have zero sum, and therefore we obtain $\gamma=2+\theta=2+\sqrt{3}$. By using induction in equation (3.11),

$$t_{2k}-t_{2k-1}=\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{k-1}(t_2-t_1), \tag{3.13}$$

and for equation (3.12),

$$\begin{aligned} t_{2k+1}-t_{2k} &= \left(\frac{-2}{2+\sqrt{3}}\right)(t_{2k}-t_{2k-1}) \\ &= \left(\frac{-2}{2+\sqrt{3}}\right)\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{k-1}(t_2-t_1). \end{aligned} \tag{3.14}$$

Since $\left|(2-\sqrt{3})/(2+\sqrt{3})\right|<1$, equations (3.13) and (3.14) lead us to the same limit t, $0<t<+\infty$ for the sequence (t_n) as k tends to infinity.

To determine the limit t, we shall use matrices, inspired by the treatment in Cull et al. [**3**].

Formula (3.1) can be written in matrix form as follows:

$$X_n=AX_{n-1} \quad \text{where } X_n=\begin{pmatrix} b_{2n} \\ b_{2n-1}\end{pmatrix} \text{ and } A=\begin{pmatrix} 2 & 3 \\ 1 & 2\end{pmatrix}.$$

With initial condition $X_1=\binom{5}{3}$ we have $X_n=A^{n-1}\binom{5}{3}$. The matrix A has characteristic polynomial

$$\mathrm{ch}_A(x)=\det\begin{pmatrix} 2-x & 3 \\ 1 & 2-x\end{pmatrix}=(2-x)^2-3$$

and has distinct eigenvalues, $\lambda_1 = 2+\sqrt{3}$ and $\lambda_2 = 2-\sqrt{3}$, and this implies that A is diagonalizable. With a simple computation, we can see that $A = PDP^{-1}$, where

$$D = \begin{pmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{pmatrix},$$
$$P = \begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{pmatrix},$$

and

$$P^{-1} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{pmatrix}.$$

Therefore

$$\begin{aligned} X_n &= A^{n-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = PD^{n-1}P^{-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix} \\ &= \frac{1}{2\sqrt{3}} \begin{pmatrix} (5\sqrt{3}+9)(2+\sqrt{3})^{n-1} + (5\sqrt{3}-9)(2-\sqrt{3})^{n-1} \\ (5+3\sqrt{3})(2+\sqrt{3})^{n-1} + (-5+3\sqrt{3})(2-\sqrt{3})^{n-1} \end{pmatrix}, \end{aligned}$$

so

(3.15)
$$b_{2n} = \frac{1}{2\sqrt{3}} \left((5\sqrt{3}+9)(2+\sqrt{3})^{n-1} + (5\sqrt{3}-9)(2-\sqrt{3})^{n-1} \right).$$

Inserting the values already found for θ and γ into (3.9),

$$t_{2n} = \frac{1}{6}\left((5\sqrt{3}+9)(2+\sqrt{3})^{-1} + (5\sqrt{3}-9)\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{n-1}(2+\sqrt{3})^{-1} \right),$$

proving that t_{2n} tends to $1/2+\sqrt{3}/6 \approx 0.7886751$, and so t_n converges to this number. $\square$

Proposition 3.3. *Let b_n be the number of Khalimsky-continuous functions $[0,n-1]_{\mathbf{Z}} \to [0,2]_{\mathbf{Z}}$, let b_n^i be the number of Khalimsky-continuous functions $[0,n-1]_{\mathbf{Z}} \to [0,2]_{\mathbf{Z}}$ satisfying $f(n-1) = i$ for $i = 0,1,2$, and define $P_n^i = b_n^i/b_n$ for $i = 0,1,2$. As k tends to infinity,*

$$P_{2k}^2 = P_{2k}^0 \longrightarrow \frac{1}{2} - \frac{1}{6}\sqrt{3}, \qquad P_{2k-1}^2 = P_{2k-1}^0 \longrightarrow \frac{1}{2}\sqrt{3} - \frac{1}{2},$$

also

$$P^1_{2k} \longrightarrow \frac{1}{\sqrt{3}}, \qquad P^1_{2k-1} \longrightarrow 2-\sqrt{3}.$$

Proof. Using the Khalimsky topology,

$$\begin{aligned} b^0_{2k} &= b^0_{2k-1}, & k &\geq 2,\\ b^1_{2k} &= 2b^0_{2k-1} + b^1_{2k-1}, & k &\geq 2,\\ b^2_{2k} &= b^0_{2k-1}, & k &\geq 2, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} b^0_{2k-1} &= b^0_{2k-2} + b^1_{2k-2}, & k &\geq 2,\\ b^1_{2k-1} &= b^1_{2k-2}, & k &\geq 2,\\ b^2_{2k-1} &= b^0_{2k-2} + b^1_{2k-2}, & k &\geq 2. \end{aligned} \tag{3.17}$$

Let

$$P^i_n = \frac{b^i_n}{b_n} \quad \text{for } i = 0, 1, 2.$$

Also we can see easily that $P^0_n = P^2_n$, so by using (3.2) we get

$$2P^0_n + P^1_n = 1. \tag{3.18}$$

It is obvious that the frequencies for odd and even indices are different but there is a relation between them. We shall study them separately. By (3.16),

$$\begin{cases} P^0_{2k} b_{2k} = P^0_{2k-1} b_{2k-1},\\ (1-2P^0_{2k}) b_{2k} = 2P^0_{2k-1} b_{2k-1} + (1-2P^0_{2k-1}) b_{2k-1}. \end{cases} \tag{3.19}$$

We solve equation (3.19) and obtain

$$P^0_{2k} = \frac{b_{2k} - b_{2k-1}}{2b_{2k}} \text{ and } P^0_{2k-1} = \frac{b_{2k} - b_{2k-1}}{2b_{2k-1}}.$$

Therefore, by Theorem 3.2, we see that, as $k \to \infty$,

$$P^0_{2k} \longrightarrow \frac{\theta - 1}{2\theta} = \frac{1}{2} - \frac{1}{6}\sqrt{3},$$

and

$$P^0_{2k-1} \longrightarrow \frac{\theta - 1}{2} = \frac{1}{2}\sqrt{3} - \frac{1}{2}.$$

Also, by using (3.18) and a simple calculation,

$$P^1_{2k} \longrightarrow \frac{1}{\sqrt{3}} \quad \text{and} \quad P^1_{2k-1} \longrightarrow 2 - \sqrt{3}. \qquad \square$$

In the following table we can see some values of P^i_n for $i = 0, 1, 2$.

n	6	7	8	9	10	11	12
b^0_n	15	56	56	209	209	780	780
b^1_n	41	41	153	153	571	571	2131
b^2_n	15	56	56	209	209	780	780
b_n	71	153	265	571	989	2131	3691
P^0_n	0.2113	0.36601	0.21132	0.36602	0.21132	0.36602	0.21132
P^1_n	0.57746	0.26797	0.57736	0.26795	0.57735	0.26795	0.57735
P^2_n	0.2113	0.36601	0.21132	0.36602	0.21132	0.36602	0.21132

4. Continuous functions with a four-point codomain.

Theorem 4.1. *Let c_n be the number of Khalimsky-continuous functions $f\colon [0, n-1]_{\mathbf{Z}} \to [0, 3]_{\mathbf{Z}}$, and let c^i_n be the number of Khalimsky-continuous functions $f\colon [0, n-1]_{\mathbf{Z}} \to [0, 3]_{\mathbf{Z}}$ such that $f(n-1) = i$ for $i = 0, 1, 2, 3$. Then $c^1_1 = c^2_1 = 1$, $c_2 = 7$, $c_3 = 15$ and*

$$c_n = c_{n-1} + 2c_{n-2} + c^1_{n-3} + c^2_{n-3}. \tag{4.1}$$

Formula (4.1) together with formulas (4.3) and (4.4) below determine the value of c_n.

Proof. We have by definition

$$c_n = c^0_n + c^1_n + c^2_n + c^3_n. \tag{4.2}$$

Using properties of the Khalimsky topology, we see that

$$\begin{aligned} c^0_{2k+1} &= c^0_{2k} + c^1_{2k}, && k \geq 1,\\ c^1_{2k+1} &= c^1_{2k}, && k \geq 1,\\ c^2_{2k+1} &= c^1_{2k} + c^2_{2k} + c^3_{2k}, && k \geq 1,\\ c^3_{2k+1} &= c^3_{2k}, && k \geq 1, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} c^0_{2k} &= c^0_{2k-1}, && k \geq 1,\\ c^1_{2k} &= c^0_{2k-1} + c^1_{2k-1} + c^2_{2k-1}, && k \geq 1,\\ c^2_{2k} &= c^2_{2k-1}, && k \geq 1,\\ c^3_{2k} &= c^2_{2k-1} + c^3_{2k-1}, && k \geq 1. \end{aligned} \tag{4.4}$$

If we insert (4.3) into (4.2),

$$c_{2k+1} = c_{2k} + 2c^1_{2k} + c^3_{2k}. \tag{4.5}$$

By using (4.4),

$$2c^1_{2k} + c^3_{2k} = 2c_{2k-1} + c^2_{2k-1} - c^3_{2k-1}. \tag{4.6}$$

But the equations in (4.3) give us

$$\begin{aligned} c^2_{2k-1} &= c^1_{2k-2} + c^2_{2k-2} + c^3_{2k-2},\\ c^3_{2k-1} &= c^3_{2k-2}. \end{aligned} \tag{4.7}$$

Now, we need just to consider the equations (4.5), (4.6) and (4.7) to have the result for odd n, $n = 2k + 1$. Next we proceed in the same way for $n = 2k$. Using properties of the Khalimsky topology we see that, if we add equation (4.4) to equation (4.2), we have

$$c_{2k} = c_{2k-1} + 2c^2_{2k-1} + c^0_{2k-1}. \tag{4.8}$$

Therefore, by (4.3) we have

$$2c^2_{2k-1} + c^0_{2k-1} = 2c_{2k-2} + c^1_{2k-2} - c^0_{2k-2}. \tag{4.9}$$

Also, (4.4) gives us

$$\begin{aligned} c^1_{2k-2} &= c^0_{2k-3} + c^1_{2k-3} + c^2_{2k-3}, \\ c^0_{2k-2} &= c^0_{2k-3}. \end{aligned} \tag{4.10}$$

We insert (4.10) and (4.9) into (4.8) to get the result for even n. □

We present in the following table the sequence with four values in the codomain and $n \leq 10$ points in the domain.

n	1	2	3	4	5	6	7	8	9	10
c_n	4	7	15	31	65	136	285	597	1251	2621

The asymptotic behavior of the number of continuous functions with a four-point codomain.

Theorem 4.2. *Let c^i_n be the number of Khalimsky-continuous functions $f\colon [0, n-1]_{\mathbf{Z}} \to [0,3]_{\mathbf{Z}}$ such that $f(n-1) = i$ for $i = 0, 1, 2, 3$, and let c_n be their sum. Then*

$$\frac{c^1_n + c^2_n}{c^1_{n-1} + c^2_{n-1}}, \quad \frac{c^0_n + c^3_n}{c^0_{n-1} + c^3_{n-1}} \text{ as well as } \frac{c_n}{c_{n-1}} \text{ tend to}$$

$$\frac{1}{2}\sqrt{7 + \sqrt{5} + \sqrt{38 + 14\sqrt{5}}} \approx 2.095293985.$$

Proof. Let us fix a positive number γ (to be determined later) and define sequence t^i_n for $i = 0, \ldots, 3$ by the following equation

$$c^i_n = t^i_n \gamma^n. \tag{4.11}$$

Let

$$t_n = t^0_n + t^1_n + t^2_n + t^3_n. \tag{4.12}$$

Then (4.3) and (4.11) yield

$$\begin{aligned} t^0_{2k+1} &= \gamma^{-1}(t^0_{2k} + t^1_{2k}), \\ t^1_{2k+1} &= \gamma^{-1} t^1_{2k}, \\ t^2_{2k+1} &= \gamma^{-1}(t^1_{2k} + t^2_{2k} + t^3_{2k}), \\ t^3_{2k+1} &= \gamma^{-1} t^3_{2k}. \end{aligned} \tag{4.13}$$

By (4.4) and (4.11) we obtain

$$
\begin{aligned}
t_{2k}^0 &= \gamma^{-1} t_{2k-1}^0, \\
t_{2k}^1 &= \gamma^{-1} (t_{2k-1}^0 + t_{2k-1}^1 + t_{2k-1}^2), \\
t_{2k}^2 &= \gamma^{-1} t_{2k-1}^2, \\
t_{2k}^3 &= \gamma^{-1} (t_{2k-1}^2 + t_{2k-1}^3).
\end{aligned} \tag{4.14}
$$

We now define a sequence (X_n) as follows:

$$
X_n = \begin{pmatrix} t_n^0 \\ t_n^1 \\ t_n^2 \\ t_n^3 \end{pmatrix}, \tag{4.15}
$$

and introduce the two matrices

$$
A_{2k} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad A_{2k-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.16}
$$

By using (4.13), (4.14), (4.15) and (4.16) we can see easily that

$$
X_n = \gamma^{-1} A_n X_{n-1} \quad \text{for } n \geq 2. \tag{4.17}
$$

Let B be equal to $A_{2k+1}A_{2k}$, which is independent of k. Then

$$
B = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.
$$

This matrix is symmetric, so there exists a diagonal matrix D whose diagonal entries are the eigenvalues of B and a matrix P such that each column of P is an eigenvector of B with $B = PDP^{\mathrm{T}}$. The columns of P form an orthogonal set, so $PP^{\mathrm{T}} = P^{\mathrm{T}}P$. We shall now determine the eigenvalues and eigenvectors of the matrix B. It has the following characteristic function:

$$
\det(B - xI) = x^4 - 7x^3 + 13x^2 - 7x + 1. \tag{4.18}
$$

The symmetry of the coefficients in this equation implies that, if λ is an eigenvalue then also $1/\lambda$ is an eigenvalue. Thus we can find the four eigenvalues of equation (4.18) by starting with $\alpha = \lambda_0 + (1/\lambda_0)$ and $\beta = \lambda_1 + (1/\lambda_1)$. Then we get $\alpha + \beta = 7$ and $\alpha\beta = 11$, so $\alpha = (7+\sqrt{5})/2$ and $\beta = (7-\sqrt{5})/2$ and therefore

$$\begin{aligned} \lambda_0 &= \tfrac{7+\sqrt{5}-\sqrt{38+14\sqrt{5}}}{4} \quad \text{and} \quad \lambda_3 = 1/\lambda_0 = \tfrac{7+\sqrt{5}+\sqrt{38+14\sqrt{5}}}{4}, \\ \lambda_1 &= \tfrac{7+\sqrt{5}-\sqrt{38-14\sqrt{5}}}{4} \quad \text{and} \quad \lambda_2 = 1/\lambda_1 = \tfrac{7+\sqrt{5}+\sqrt{38-14\sqrt{5}}}{4}. \end{aligned} \tag{4.19}$$

Let $P = (\,P_0\ P_1\ P_2\ P_3\,)$, where P_i is an eigenvector with respect to the eigenvalue λ_i for $i = 0, \dots, 3$. Therefore, $BP_i = \lambda_i P_i$. Now we shall solve the following system of equations:

$$\begin{cases} 2x + y + z = \lambda x, \\ x + y + z = \lambda y, \\ x + y + 3z + t = \lambda z, \\ z + t = \lambda t, \end{cases} \tag{4.20}$$

where λ is one of the eigenvalues λ_i, and where $P_i = (\,x\ y\ z\ t\,)^{\mathrm{T}}$ for $i = 0, \dots, 3$. Therefore,

$$y = \frac{\lambda - 1}{\lambda} x, \qquad z = \frac{\lambda^2 - 3\lambda + 1}{\lambda} x, \qquad t = \frac{\lambda^2 - 3\lambda + 1}{\lambda(\lambda - 1)} x.$$

We choose for convenience $x = \lambda(\lambda - 1)$; thus

$$y = (\lambda - 1)^2, \qquad z = (\lambda^2 - 3\lambda + 1)(\lambda - 1), \qquad t = \lambda^2 - 3\lambda + 1.$$

From now on, let $\lambda = \lambda_3$ and $(x, y, z, t)^{\mathrm{T}}$ be the eigenvectors related to λ_3. Since we need to consider B^k as $k \to \infty$, we need not consider the powers of λ_i for $i = 0, 1, 2$. Hence, the powers of B that we need to consider are as follows:

$$B^k = \begin{pmatrix} \lambda^k x^2 & \lambda^k xy & \lambda^k xz & \lambda^k xt \\ \lambda^k xy & \lambda^k y^2 & \lambda^k yz & \lambda^k yt \\ \lambda^k xz & \lambda^k yz & \lambda^k z^2 & \lambda^k zt \\ \lambda^k xt & \lambda^k yt & \lambda^k zt & \lambda^k t^2 \end{pmatrix}.$$

Equation (4.17) and the previous calculation lead us to

$$X_{2k-1} = (\gamma^{-2}\lambda)^{k-3} \begin{pmatrix} x^2t_5^0 + xyt_5^1 + xzt_5^2 + xtt_5^3 \\ xyt_5^0 + y^2t_5^1 + zyt_5^2 + tyt_5^3 \\ xzt_5^0 + yzt_5^1 + z^2t_5^2 + tzt_5^3 \\ xtt_5^0 + ytt_5^1 + ztt_5^2 + t^2t_5^3 \end{pmatrix}. \tag{4.21}$$

Let $\alpha = xt_5^0 + yt_5^1 + zt_5^2 + tt_5^3$. Thus by (4.14) and (4.21),

$$\begin{aligned} t_{2k}^1 &= \gamma^{-1}(t_{2k-1}^0 + t_{2k-1}^1 + t_{2k-1}^2) \\ &= (\gamma^{-2}\lambda)^{(k-3)}\gamma^{-1}(x+y+z)\alpha \\ t_{2k-1}^2 &= (\gamma^{-2}\lambda)^{(k-3)}z\alpha. \end{aligned}$$

We now define $\gamma = \sqrt{\lambda}$, the positive square root of the largest eigenvalue, and find that $(\gamma^{-2}\lambda)^{k-3}$ tends to 1 as $k \to +\infty$. We claim that $\gamma^{-1}(x+y+z) = z$, or, equivalently, that

$$\begin{aligned} 0 &= \gamma^{-1}(x+y+z) - z \\ &= x\left[\gamma^{-1}\left(1 + \frac{\lambda-1}{\lambda} + \frac{\lambda^2-3\lambda+1}{\lambda}\right) - \frac{\lambda^2-3\lambda+1}{\lambda}\right]. \end{aligned}$$

We need to show that

$$\gamma^{-1}(\lambda-1)\lambda - (\lambda^2 - 3\lambda + 1) = 0.$$

Since λ is the largest root of equation (4.18),

$$\begin{aligned} 0 &= \lambda^4 - 7\lambda^3 + 13\lambda^2 - 7\lambda + 1 \\ &= \lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda + 1 - \lambda^3 + 2\lambda^2 - \lambda \\ &= \left(\lambda^2 - 3\lambda + 1\right)^2 - \lambda\left(\lambda - 1\right)^2. \end{aligned} \tag{4.22}$$

The equations in (4.22) imply that

$$\lambda = \frac{\lambda^2\left(\lambda-1\right)^2}{\left(\lambda^2-3\lambda+1\right)^2}.$$

Therefore,

$$\gamma^{-1} = \frac{\lambda^2-3\lambda+1}{\lambda\left(\lambda-1\right)}.$$

This proves our claim. Hence the sequences t^1_{2k} and t^2_{2k-1} are identical, and therefore they tend to the same limit $z\alpha$ as $k \to \infty$. Similarly, we can prove the corresponding result for some other sequences as follows:

$$\begin{aligned} t^1_{2k} &= t^2_{2k-1} \longrightarrow z\alpha \quad \text{as } k \to \infty, \\ t^2_{2k} &= t^1_{2k-1} \longrightarrow y\alpha \quad \text{as } k \to \infty, \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} t^3_{2k} &= t^0_{2k-1} \longrightarrow x\alpha \quad \text{as } k \to \infty, \\ t^0_{2k} &= t^3_{2k-1} \longrightarrow t\alpha \quad \text{as } k \to \infty. \end{aligned} \tag{4.24}$$

If we sum the two limits in (4.23),

$$(t^1_n + t^2_n) \text{ tends to } (y+z)\alpha \text{ as } n \to \infty. \tag{4.25}$$

Analogously, (4.24) shows that

$$(t^0_n + t^3_n) \text{ tends to } (x+t)\alpha \text{ as } n \to \infty. \tag{4.26}$$

We now easily conclude that the sum of these two sequences, i.e., (t_n), converges to $(x+y+z+t)\alpha$. Since the sequence $(t^1_n + t^2_n)$ converges, we see easily that

$$\frac{c^1_n + c^2_n}{c^1_{n-1} + c^2_{n-1}} \longrightarrow \gamma \quad \text{as } n \to \infty,$$

and also the convergence of the sequence $(t^0_n + t^3_n)$ leads us to

$$\frac{c^0_n + c^3_n}{c^0_{n-1} + c^3_{n-1}} \longrightarrow \gamma \quad \text{as } n \to +\infty.$$

We have the same result for c_n/c_{n-1} because, as we found, the sequence (t_n) converges to some real number, so

$$\frac{c_n}{c_{n-1}} \longrightarrow \gamma \quad \text{as } n \to +\infty. \qquad \square$$

We shall now investigate frequencies in the case of a four-point codomain.

Proposition 4.3. *Let c_n be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbf{Z}} \to [0,3]_{\mathbf{Z}}$, and let c_n^i be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbf{Z}} \to [0,3]_{\mathbf{Z}}$ such that $f(n-1)=i$ for $i=0,1,2,3$. If $p_n^i = c_n^i/c_n$ for $i=0,1,2,3$, then*

$$
\begin{aligned}
&P_{2k}^3 \text{ and } P_{2k-1}^0 \longrightarrow \frac{x}{x+y+z+t} \approx 0.258582;\\
&P_{2k}^2 \text{ and } P_{2k-1}^1 \longrightarrow \frac{y}{x+y+z+t} \approx 0.199679;\\
&P_{2k}^1 \text{ and } P_{2k-1}^2 \longrightarrow \frac{z}{x+y+z+t} \approx 0.418335;\\
&P_{2k}^0 \text{ and } P_{2k-1}^3 \longrightarrow \frac{t}{x+y+z+t} \approx 0.123402;
\end{aligned} \tag{4.27}
$$

as $k \to \infty$, where x,y,z,t are the numbers which were defined in the proof of Theorem 4.2. As a consequence, if we add these numbers in pairs, the different parities play no role, and we obtain that

$$
\begin{aligned}
P_n^1 + P_n^2 &\longrightarrow \frac{y+z}{x+y+z+t} \approx 0.618014\\
P_n^0 + P_n^3 &\longrightarrow \frac{x+t}{x+y+z+t} \approx 0.381984
\end{aligned} \tag{4.28}
$$

as n tends to infinity.

Proof. By the proof of Theorem 4.2 we know that the sequence (t_n) converges to the number $(x+y+z+t)\alpha$. This fact and (4.23) imply that

$$
\begin{aligned}
&P_{2k}^2 \text{ and } P_{2k-1}^1 \longrightarrow \frac{y}{x+y+z+t} \approx 0.199679 \quad \text{as } k \to \infty;\\
&P_{2k}^1 \text{ and } P_{2k-1}^2 \longrightarrow \frac{z}{x+y+z+t} \approx 0.418335 \quad \text{as } k \to \infty.
\end{aligned} \tag{4.29}
$$

Analogously, by using (4.24) we conclude that

$$
\begin{aligned}
&P_{2k}^3 \text{ and } P_{2k-1}^0 \longrightarrow \frac{x}{x+y+z+t} \approx 0.258582 \quad \text{as } k \to \infty;\\
&P_{2k}^0 \text{ and } P_{2k-1}^3 \longrightarrow \frac{t}{x+y+z+t} \approx 0.123402 \quad \text{as } k \to \infty.
\end{aligned} \tag{4.30}
$$

It is obvious that if we sum up the limits in (4.29),

$$P_n^1 + P_n^2 \longrightarrow \frac{y+z}{x+y+z+t} \approx 0.618014 \quad \text{as } n \to \infty,$$

similarly if we sum the limits in (4.30),

$$P_n^0 + P_n^3 \longrightarrow \frac{x+t}{x+y+z+t} \approx 0.381984 \quad \text{as } n \to \infty. \qquad \square$$

In the next table we can see the values of P_n^i for $i = 0, \ldots, 3$ and the sums of some of the frequencies.

n	6	7	8	9	10
c_n^0	17	74	74	324	324
c_n^1	57	57	250	250	1097
c_n^2	27	119	119	523	523
c_n^3	35	35	154	154	677
c_n	136	285	597	1251	2621
P_n^0	0.125	0.259649	0.123953	0.258992	0.123616
P_n^1	0.419117	0.2	0.418760	0.199840	0.418542
P_n^2	0.1985294	0.4175439	0.19933	0.4180655	0.1995422
P_n^3	0.2573529	0.122807	0.2579564	0.1231015	0.2582984
$P_n^0 + P_n^3$	0.3823529	0.382456	0.3819094	0.3820935	0.3819144
$P_n^1 + P_n^2$	0.6176464	0.6175439	0.61809	0.6179055	0.6180842

Conclusion. In this work we studied Khalimsky-continuous functions from a combinatorial point of view. We considered the graphs of these functions as paths between two points with special properties depending on the topological structure. Based on these properties, we investigated some problems and determined the number of Khalimsky-continuous functions with arbitrary domain of definition and with codomain containing up to four points.

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