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A NOTE ABOUT $\{K_i(z)\}_{i=1}^{\infty}$ FUNCTIONS

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ABSTRACT. In the article [**11**], Petojević verified useful properties of $K_i(z)$ functions which generalize Kurepa's [**2**] left factorial function. In this note, we present simplified proofs of two of these results and we answer the open question stated in [**11**]. Finally, we discuss the differential transcendency of $K_i(z)$ functions.

1. Introduction. Petojević [**10**, page 3] considered the family of functions:

$$(1)\quad {}_vM_m(s;a,z) = \sum_{k=1}^{v}(-1)^{k-1}\binom{z+m+1-k}{m+1}\mathcal{L}[s;{}_2F_1(a,k-z,m+2;1-t)],$$

for $\Re(z) > v-m-2$, where $v \in \mathbf{N}$ is a positive integer; $m \in \{-1,0,1,2,\dots\}$ is an integer; s,a,z are complex variables; $\mathcal{L}[s;F(t)]$ is a *Laplace* transform and ${}_2F_1(a,b,c;x)$ is the hypergeometric function ($|x|<1$). Kurepa has considered in the articles [**2**, page 151] and [**3**, page 297] a complex function defined by the integral:

$$(2)\qquad K(z) = \int_0^{\infty} e^{-t}\frac{t^z-1}{t-1}\,dt,$$

for $\Re(z) > 0$. In particular, for Kurepa's function $K(z)$, it is true that $K(z) = {}_1M_0(1;1,z)$, for $\Re(z)>0$, according to [**11**]. For various values of parameters v,m,s,a,z from (1), different special functions, as presented in [**11**], are obtained. Petojević has considered in the article [**11**, page 1640] the following sequence of functions:

$$(3)\qquad K_i(z) = \frac{{}_1M_0(1;1,z+i-1) - {}_1M_0(1;1,i-1)}{{}_1M_{-1}(1;1,i)},$$

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for $i \in \mathbf{N}$ and $\Re(z) > -i$. On the basis of the definition in (3), the following representation via Kurepa's function is true:

$$K_i(z) = \frac{1}{(i-1)!}\Big(K(z+i-1) - K(i-1)\Big), \tag{4}$$

for $i \in \mathbf{N}$ and $\Re(z) > -i+1$. Note that $K(0) = 0$ [**3**, page 297] and therefore $K_1(z) = K(z)$ for $\Re(z) > 0$. Analytical and differential-algebraic properties of Kurepa's function $K(z)$ are considered in articles [**1**–**12**] and in many other articles. On the basis of well-known statements for Kurepa's function $K(z)$, using representation (4), in many cases we can get simple proofs for analogous statements for $K_i(z)$ functions. For example, it is a well-known fact that it is possible to analytically continue Kurepa's function to a meromorphic function with simple poles at integer points $z = -1$ and $z = -m$, $(m \geq 3)$ [**3**, page 303], [**12**, page 474]. Residues of Kurepa's function at these poles have the following form [**3**]:

$$\operatorname*{res}_{z=-1} K(z) = -1 \quad \text{and} \quad \operatorname*{res}_{z=-m} K(z) = \sum_{k=2}^{m-1} \frac{(-1)^{k-1}}{k!}, \ (m \geq 3). \tag{5}$$

For Kurepa's function $K(z)$ the infinite point is an essential singularity [**12**]. Hence, on the basis of (4), each function $K_i(z)$ is meromorphic with simple poles at integer points $z = -i$ and $z = -(i+m)$, $(m \geq 2)$. On the basis of (4), we have:

$$\begin{aligned} \operatorname*{res}_{z=-(i+m)} K_i(z) &= \frac{1}{(i-1)!} \cdot \operatorname*{res}_{z=-(i+m)} K(z+i-1) \\ &= \frac{1}{(i-1)!} \cdot \operatorname*{res}_{z=-(m+1)} K(z), \end{aligned} \tag{6}$$

where $m = 0$ or $m \geq 2$. Hence:

$$\begin{aligned} &\operatorname*{res}_{z=-i} K_i(z) = -\frac{1}{(i-1)!} \quad \text{and} \\ &\operatorname*{res}_{z=-(i+m)} K_i(z) = \frac{1}{(i-1)!} \cdot \sum_{k=2}^{m} \frac{(-1)^{k-1}}{k!}, \ (m \geq 2). \end{aligned} \tag{7}$$

For each $K_i(z)$ function the infinite point is an essential singularity. Therefore, we get Theorem 3.3 from [**11**]. Next, it is a well-known fact that for Kurepa's function the following asymptotic relation $K(x) \sim \Gamma(x)$ is true for real x such that $x \to \infty$ and where $\Gamma(x)$ is the gamma function [**3**, page 299]. Hence, for fixed $i \in \mathbf{N}$ and real $x > -i+1$, on the basis of (4), we get:

$$\frac{K_i(x)}{\Gamma(x+i-1)} = \frac{1}{(i-1)!} \cdot \frac{K(i+x-1) - K(i-1)}{\Gamma(x+i-1)} \underset{x\to\infty}{\longrightarrow} \frac{1}{(i-1)!} \tag{8}$$

and

$$\frac{K_i(x)}{\Gamma(x+i)} = \frac{1}{(i-1)!} \cdot \frac{K(i+x-1) - K(i-1)}{(x+i-1)\Gamma(x+i-1)} \underset{x\to\infty}{\longrightarrow} 0. \tag{9}$$

Therefore, we get Theorem 3.6 from [**11**]. Next we give a solution to the open problem stated in Question 3.7 in [**11**]. Namely, the following formula in the article [**4**, page 35] is given:

$$K(z) = \frac{\mathrm{Ei}\,(1) + \mathbf{i}\,\pi}{e} + \frac{(-1)^z \Gamma(1+z)\Gamma(-z,-1)}{e}, \tag{10}$$

for values $z \in \mathbf{C} \backslash \{-1,-2,-3,-4,\dots\}$ and $\mathbf{i} = \sqrt{-1}$. In the previous formula $\mathrm{Ei}\,(z)$ and $\Gamma(z,a)$ are exponential integral and incomplete gamma function, respectively, [**4**]. Then, for fixed $i \in \mathbf{N}$ and values $z \in \mathbf{C} \backslash \{-i,-i-1,-i-2,-i-3,\dots\}$, on the basis of (4) and (10), we get:

(11)
$$\begin{aligned} K_i(z) &= \frac{1}{(i-1)!}\Big(K(z+i-1) - K(i-1)\Big) \\ &= \frac{\mathrm{Ei}\,(1) + \mathbf{i}\,\pi}{e(i-1)!} + \frac{(-1)^{z+i-1}\Gamma(1+z+i-1)\Gamma(-z-i+1,-1)}{e(i-1)!} \\ &\quad - \frac{\mathrm{Ei}\,(1) + \mathbf{i}\,\pi}{e(i-1)!} - \frac{(-1)^{i-1}\Gamma(i)\Gamma(-i+1,-1)}{e(i-1)!} \\ &= (-1)^i e^{-1}\left(\Gamma(1-i,-1) - (-1)^z \frac{\Gamma(1-i-z,-1)\Gamma(i+z)}{(i-1)!}\right). \end{aligned}$$

Therefore, the affirmative answer for Question 3.7 from [**11**] is true for complex values $z \in \mathbf{C} \backslash \{-i,-i-1,-i-2,-i-3,\dots\}$.

Finally, at the end of this note let us emphasize one differential-algebraic fact for the sequence of functions $K_i(z)$. On the basis of formula (17) from the article [**11**], we can conclude that each $K_i(z)$ function satisfies the following recurrence relation $(i-1)!\,K_i(z+1) - (i-1)!\,K_i(z) = \Gamma(z+i)$. The previous relation can be used to verify the differential transcendency of these functions as discussed in [**6, 7**]. Therefore, we can conclude that each $K_i(z)$ function is a differential transcendental function, i.e., it satisfies no algebraic differential equation over the field of complex rational functions.

REFERENCES

1. A. Ivić and Ž. Mijajlović, *On Kurepa problems in number theory*, Publ. l'Institut Math., SANU Beograd, **57** (1995), 19–28.

2. D. Kurepa, *On the left factorial function !n*, Math. Balkan. **1** (1971), 147–153.

3. ———, *Left factorial function in complex domain*, Math. Balkan. **3** (1973), 297–307.

4. B. Malešević, *Some considerations in connection with Kurepa's function*, Univerzitet u Beogradu, Publ. Elektroteh. Fak. Ser. Mat. **14** (2003), 26–36.

5. ———, *Some inequalities for Kurepa's function*, J. Inequal. Pure Appl. Math., **5** (2004), 1–5.

6. Ž. Mijajlović and B. Malešević, *Analytical and differential–algebraic properties of Gamma function*, Internat. J. Appl. Math. Stat. **11** (2007), 118–129 (Special volume dedicated to the Tricentennial birthday anniversary of L. Euler, 2007).

7. ———, *Differentially transcendental functions*, The Bulletin of the Belgian Mathematical Society–Simon Stevin **15** (2008), 193–201.

8. G.V. Milovanović, *Expansions of the Kurepa function*, Publ. l'Institut Math., SANU Beograd **57** (1995), 81–90.

9. G.V. Milovanović and A. Petojević, *Generalized factorial functions, numbers and polynomials*, Math. Balkan. **16** (2002), 113–130.

10. A. Petojević, *The function ${}_vM_m(s;a,z)$ and some well-known sequences*, J. Integer Sequences, Article 02.1.6, Vol. **5** (2002), 1–16.

11. ———, *The $\{K_i(z)\}_{i=1}^{\infty}$ functions*, Rocky Mountain J. Mathematics **36** (2006), 1637–1650.

12. D. Slavić, *On the left factorial function of the complex argument*, Mathematica Balkanica **3** (1973), 472–477.

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