ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 5, 2010

PARTIALLY ORDERED GROUPS WHICH ACT ON ORIENTED ORDER TREES

MATTHEW HORAK AND MELANIE STEIN

ABSTRACT. It is well known that a countable group admits a left-invariant total order if and only if it acts faithfully on **R** by orientation preserving homeomorphisms. Such group actions are special cases of group actions on simply connected 1-manifolds, or equivalently, actions on oriented order trees. We characterize a class of left-invariant partial orders on groups which yield such actions, and show conversely that groups acting on oriented order trees by order preserving homeomorphisms admit such partial orders as long as there is an action with a point whose stabilizer is left-orderable.

1. Introduction. It is a well-known result that a countable group admits a left- (or right-)invariant total order if and only if it acts faithfully by orientation preserving homeomorphisms on **R** (see Theorem 6.8 of [**11**] for a proof). Many groups that arise naturally in topology are left-orderable, for example braid groups [**7**], certain mapping class groups of Riemann surfaces with boundary [**17**], and many 3-manifold groups. Boyer, Rolfsen and Wiest establish in [**5**] that there are compact connected manifolds modeled on each of the eight 3-dimensional geometries with both orderable and nonorderable fundamental groups. The first examples of nonorderable hyperbolic 3-manifold groups are given in [**14**], and the nonorderability of these groups is established by showing that they cannot act via faithful orientation preserving homeomorphisms on **R**. Now, **R** is a special case of a simply connected 1-manifold, and in fact the paper shows that these groups cannot act nontrivially on any oriented simply connected 1-manifold. For a simple example of a non Hausdorff simply connected 1-manifold, consider the disjoint union of two copies of $[0,+\infty)$ and a single copy of $(-\infty,0)$, in which both endpoints of the closed segments are limit points of the sequence $\{-1/n\}_{n=1}^{\infty}$ (see also [**2**]). The points

2010 AMS *Mathematics subject classification.* Primary 20E08, 20F60, 06F15, 20F65, 57M60.
Keywords and phrases. Partially ordered group, 3-manifold, tree, order tree, 1-manifold, group action.
Received by the editors on May 3, 2007, and in revised form on April 11, 2008.
DOI:10.1216/RMJ-2010-40-5-1527

of oriented simply-connected 1-manifolds have a natural order, just as the points of $\mathbf{R}$ do. In contrast to the total order of the points on the real line, the points of these more general manifolds are in general only partially ordered (see [**14**]). The goal of this paper is to characterize the left invariant partial orders on groups that correspond to group actions on oriented simply-connected 1-manifolds and their generalizations, oriented order trees, and to prove the analogue of the classical theorem relating left orderability of groups and faithful group actions on $\mathbf{R}$.

The paper is organized as follows. In Section 2, we recall some of the relevant background information on simply connected 1-manifolds and their generalizations, order trees. We also extend some basic ideas about ends from Hausdorff trees to the more general setting of order trees. In Section 3, we define the abstract partial orders we will be concerned with, which we call partial orders with rectifiable simply connected extensions (see Definitions 14 (simply connected posets), 15 (extensions) and 23 (rectifiable)). We establish certain properties of sets equipped with such partial orders, which allow us to define a notion of "betweenness" (see Definition 18) for arbitrary simply connected posets that agrees with the natural notion of "between" for 1-manifolds. We also show that, under a suitable convexity hypothesis on subgroups, these orders naturally pass to both subgroups and quotient groups (see Theorem 3). After Section 3, we have established the terminology necessary to state both classification theorems, one for groups acting on simply connected 1-manifolds and one for groups acting on order trees.

Main theorem 1. *If G is a countable group, the following are equivalent*:

(1) *The group G admits a nontrivial left-invariant rectifiable simply-connected partial order.*

(2) *The group G acts, minimally and without fixing a unique end, on an oriented simply-connected* 1*-manifold T by orientation-preserving homeomorphisms in such a way that some point of T has a trivial stabilizer.*

(3) *The group G acts, minimally and without fixing a unique end, on an oriented simply-connected* 1*-manifold T by orientation-preserving*

homeomorphisms in such a way that some point of T has a left-orderable stabilizer.

Main theorem 2. *If G is a countable group, the following are equivalent:*

(1) *The group G admits a partial order with nontrivial left-invariant rectifiable simply-connected extension.*

(2) *The group G acts, minimally and without fixing a unique end, on an oriented order tree T by orientation-preserving homeomorphisms in such a way that some point of T has a trivial stabilizer.*

(3) *The group G acts, minimally and without fixing a unique end, on an oriented order tree T by orientation-preserving homeomorphisms in such a way that some point of T has a left-orderable stabilizer.*

The next four sections are devoted to the proof of the various implications in the two main theorems. In Section 4, we prove the implications (1) implies (2) in both main theorems in Theorem 4 by using the extension of the partial order and its betweenness relation to construct a suitable 1-manifold T on which the group acts. In Section 5, we prove Theorem 5, which establishes that (2) implies (1) in Main theorem 1. In Section 6, we examine the necessity of the assumption that there be a point with trivial stabilizer in the hypothesis of Theorem 5. To motivate this section, consider the simple example $G = G_1 *_H G_2$, where G_1 and G_2 are nonisomorphic finite cyclic groups and H is a proper nontrivial subgroup of both. G acts faithfully without fixing a unique end on a simplicial tree where every point has a nontrivial stabilizer, but we will show in this section that it does not admit a nontrivial left-invariant rectifiable simply-connected partial order. Therefore, the trivial stabilizer condition in Theorem 5 cannot simply be dropped. By adapting the proof of Theorem 5, we prove in Theorem 6 that the conclusion of Theorem 5 still holds if we replace "trivial stabilizer" in the hypothesis by "left-orderable stabilizer," establishing (3) implies (1) of Main theorem 1. Since it is immediate that (2) implies (3), this completes the proof of Main theorem 1. Finally, the goal of Section 7 is to prove the analogues of Theorems 5 and 6 (the stronger version established in Section 6) for groups acting on general order trees, proving the remaining implications necessary to establish Main theorem 2.

The study of treelike structures is common throughout the literature, in for example the theories of **R**-trees, Λ-trees, protrees, dendrons, pretrees, etc. A certainly nonexhaustive survey of the literature developing the theories of such structures includes [**1–4, 6, 8, 13, 16**]. In particular, the notion of "betweenness" and the associated betweenness relation is central to the theory of pretrees. Bowditch and Crisp [**3, 4**] have successfully used betweenness relations to generalize results for topological actions on **R**-trees to actions on pretrees. After we completed the constructions of Section 4, it came to our attention that the constructions of Bowditch and Crisp are very similar to our constructions. Indeed, our betweenness relation on a set with a simply connected partial order does satisfy Bowditch's notion of betweenness, and hence a simply connected poset has the structure of a pretree. However, the constructions in [**3, 4**] applied to one of our groups would give an action on an **R**-tree without orientation, since the pretrees themselves have no notion of order. One could give a proof of Theorem 4 by strengthening the constructions of Bowditch and Crisp in the special case of a simply connected partial order to carry the full poset information (referring the reader to those constructions), and then orienting the resulting **R**-trees and proving that their actions preserve the orientations. However, including the extra structure on the poset from the beginning allows for a more straightforward construction. Hence, for reasons of both completeness and readability, we include in Section 4 a self-contained exposition of our construction.

2. Background information. An *order tree* T [**10**] is a set T together with a collection $\mathcal{S}$ of linearly ordered subsets called *segments*. If σ is a segment then $-\sigma$ denotes the same subset with reverse order. The segments satisfy:

- Each segment σ has distinct least and greatest elements, which we will denote by $i(\sigma)$ and $f(\sigma)$, respectively. (We also write $\sigma = [i(\sigma), f(\sigma)]$.)
- If σ is a segment, so is $-\sigma$.
- A closed nondegenerate (i.e., containing more than one element) subinterval of a segment is a segment.
- Any two elements of T can be joined by a sequence $\sigma_1, \dots, \sigma_k$ of segments such that $f(\sigma_j) = i(\sigma_{j+1})$ for all j.

• Given a word $\sigma_0\sigma_1\cdots\sigma_{k-1}$ with $i(\sigma_j) = f(\sigma_{j+1})$ for all $j, 0 \leq j \leq k-2$, and $f(\sigma_{k-1}) = i(\sigma_0)$, there is a subdivision of the σ's $\rho_0\cdots\rho_{n-1}$ so that when adjacent pairs $(\rho)(-\rho)$ are canceled, we have the trivial word.

• If $f(\sigma_1) = i(\sigma_2) = \sigma_1 \cap \sigma_2$, then $\sigma_1 \cup \sigma_2$ is a segment.

An **R**-*order tree* [**9**] is an order tree satisfying also:

• Each segment is order isomorphic to a closed interval in **R**.

• T is a countable union of segments.

An *orientation* of an order tree is a choice of subset $\mathcal{S}_+ \subset \mathcal{S}$ such that

• $\mathcal{S}_+ \cap (-\mathcal{S}_+) = \varnothing$, where $-\mathcal{S}_+ = \{-\sigma | \sigma \in \mathcal{S}_+\}$.

• A closed nondegenerate subinterval of a segment in $\mathcal{S}_+$ is in $\mathcal{S}_+$.

• Any two elements of T can be joined by a sequence $\sigma_1, \dots, \sigma_k$ of segments in $\mathcal{S}_+ \cup (-\mathcal{S}_+)$ such that $f(\sigma_j) = i(\sigma_{j+1})$ for all j.

• If $\sigma_1, \sigma_2 \in \mathcal{S}_+$ and $f(\sigma_1) = i(\sigma_2) = \sigma_1 \cap \sigma_2$, then $\sigma_1 \cup \sigma_2 \in \mathcal{S}_+$.

Since there are no nontrivial cyclic words, orientations always exist.

Remark. For simplicity of exposition, we will take all order trees to be **R**-order trees. Thus, unless otherwise noted, any order tree will be assumed to satisfy the above two **R** axioms.

Some special cases of order trees include **R**-trees with countably many branch points and simply-connected (not necessarily Hausdorff) 1-manifolds. In the case of a simply-connected 1-manifold there are exactly two possible orientations, whereas for a general order tree there may be many more. Any nontrivial orientation preserving group action on an oriented **R**-order tree canonically induces an action on a related simply connected 1-manifold. The full details of the construction of this 1-manifold (a Denjoy blow-up of the original) appear in [**14**, Section 5], but for ease in reference later on we summarize the construction here, including some details.

First, we recall the notion of *incidence* for order trees. We remark that the definition of incidence or branching degree extends to arbitrary order trees. Fix an orientation on T, and let $x \in T$. Define an equivalence relation $\approx_f$ on the set $S(x, f) = \{\sigma \in \mathcal{S}_+ | f(\sigma) = x\}$

by $\sigma_1 \approx_f \sigma_2$ if and only if $|\sigma_1 \cap \sigma_2| > 1$. For each $\sigma \in S(x,f)$, let $r_\sigma = \{\tau \in S(x,f) | \tau \approx_f \sigma\}$ and call r_σ an *incoming ray* at x. Let $R(x,f) = \{r_\sigma | \sigma \in S(x,f)\}$. Call $n_f(x) = |R(x,f)|$ the *in degree* at x. Similarly, define an equivalence relation $\approx_o$ on the set $S(x,o) = \{\sigma \in \mathcal{S}_+ | i(\sigma) = x\}$ by $\sigma_1 \approx_o \sigma_2$ if and only if both $i(\sigma_1) = i(\sigma_2) = x$ and $|\sigma_1 \cap \sigma_2| > 1$. For each $\sigma \in S(x,o)$, let $r_\sigma = \{\tau \in S(x,o) | \tau \approx_o \sigma\}$ and call r_σ an *outgoing ray* at x. Let $R(x,o) = \{r_\sigma | \sigma \in S(x,o)\}$. Call $n_o(x) = |R(x,o)|$ the *out degree* at x. We say that a segment σ is incident to x if $\sigma \in S(x,o) \cup S(x,f)$, and we say that an incoming or outgoing ray r_σ is incident to x if $r_\sigma \in R(x,o) \cup R(x,f)$. Call $x \in T$ *regular* if $n_o(x) = n_f(x) = 1$. Call $x \in T$ a *branch point* if it is not regular, and let $\mathcal{B}$ denote the set of branch points of T. Note that if $\mathcal{B} = \varnothing$, then T can also be given the structure of a simply connected 1-manifold.

Now consider an element x of $\mathcal{B}$. If the out-degree $n_o(x)$ is equal to 0 (in-degree $n_f(x)$ equals 0), call x a *sink* (respectively, *source*). If $n_o(x) = 1$ and $n_f(x) > 1$, call the single element $r_\sigma \in R(x,o)$ the *distinguished ray* at x. Symmetrically, if $n_f(x) = 1$ and $n_o(x) > 1$, call the single element $r_\sigma \in R(x,f)$ the *distinguished ray* at x.

Lemma 1 (see [**14**, Lemma 5.9]). *Let T_0 be an oriented order tree such that, at every $x \in \mathcal{B}$, there is a distinguished ray. Then any nontrivial orientation-preserving action on T_0 canonically induces a nontrivial orientation preserving action on a related oriented simply connected 1-manifold T'.*

Proof. The 1-manifold T' is obtained from the order tree T by blowing up each branch point of T into a set of endpoints for each ray except the distinguished ray, which is left open. See [**14**] for full details. □

Proposition 1 [**14**, Proposition 5.10]. *Any nontrivial orientation-preserving action on an oriented order tree T canonically induces a nontrivial orientation preserving action on a related oriented simply connected 1-manifold T'.*

Proof. Again, a full proof appears in [**14**]. At each branch point with in- and out-degree greater than 1, a linear Denjoy blowup (as

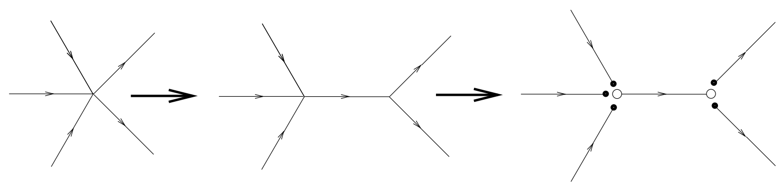

FIGURE 1. Blowing up at a branch point to construct a 1-manifold.

in Definition 9.1 of [**14**]) is performed to create a tree T_0 in which all branch points are sinks, sources or have distinguished rays. A distinguished ray is then added at each sink and source, and then Lemma 1 is applied to obtain the 1-manifold T'. We show that any nontrivial orientation-preserving action on an oriented $\mathbf{R}$-order tree T canonically induces a nontrivial orientation preserving action on an oriented $\mathbf{R}$-order tree T_0 such that at every $x \in \mathcal{B}$, there is a distinguished ray. Lemma 1 then applies. An example of this process is illustrated in Figure 1, which shows the two steps required in the case of a single branch point with in-degree and out-degree both greater than one. $\square$

In the construction described above, there is a natural map $\varphi : T' \to T$. This map collapses to a point each segment added during the Denjoy blow-up of T to T_0, collapses to the sink or source each segment added to these points to give them distinguished rays and collapses all the points in each set $\{x_{r_\sigma}\}$ added for each distinguished ray $\hat{r}_x$. We will need to keep track of an important implicit subtree of T' which maps surjectively onto T, called the core of T'.

Definition 1. The *core* of T' is the subset $\widehat{T}$ of T' consisting of all points not in the union of open rays which are added to T at the last step in forming T_0 (in order to transform branch points which are sources or sinks into branch points with a distinguished ray).

Next we recall from [**15**] some of the basics about the structure of order trees.

Definition 2. Let T be an order tree. A *path* from $x \in T$ to $y \in T$ is a sequence of segments $\sigma_1 \cdots \sigma_n$ with $f(\sigma_i) = i(\sigma_{i+1})$ for $1 \leq i < n$ and $i(\sigma_1) = x$ and $f(\sigma_n) = y$.

Definition 3. A *standard geodesic* from x to y is a path $\sigma_1 \cdots \sigma_n$ from x to y satisfying:

- $\sigma_i \cap \sigma_j = \varnothing$ if $|i - j| > 1$.
- For all j, either $\sigma_j \cap \sigma_{j+1} = i(\sigma_{j+1}) = f(\sigma_j)$ or $\sigma_j \cap \sigma_{j+1} = (i(\sigma_j), f(\sigma_j)] = (f(\sigma_{j+1}), i(\sigma_{j+1})]$.

Define the set S by

$$S = \{(\sigma, \tau) \mid \sigma \in \mathcal{S}, \tau \in \mathcal{S}, (i(\sigma), f(\sigma)] = (f(\tau), i(\tau)] \text{ and } i(\sigma) \neq f(\tau)\}.$$

We define a relation on S as follows: $([x, z], [z, y]) \equiv ([x, z'], [z', y])$ if there exists an element $r \in (x, z] \cap (x, z']$ so that the following hold, where the segments $[r, z]$ and/or $[r, z']$ are understood to be empty if $r = z$ or $r = z'$.

- $[x, z] = [x, r] \cup [r, z]$.
- $[z, y] = [z, r] \cup [r, y]$.
- $[x, z'] = [x, r] \cup [r, z']$.
- $[z', y] = [z', r] \cup [r, y]$.

The relation $\equiv$ is an equivalence relation.

Definition 4. A *cusp* is an equivalence class of pairs of segments in S under the above equivalence relation $\equiv$.

Notation. Note that a cusp represented by a pair $([x, z], [z, y]) \in S$ is determined by the pair of points x and y, for by axiom 5 in the definition of the order tree, any other pair in S of the form $([x, z'], [z', y])$ must be in the same equivalence class. Hence, we will use the symbol $[x, y]^c$ to denote the cusp. In this situation, we refer to the points x and y as *cusp points*.

Then we have the following existence theorem.

Theorem 1 (see [**15**, Theorem 3.4]). *Let T be an order tree. Given x and $y \in T$, there exists a standard geodesic from x to y.*

Remark. In particular, if a path satisfies the first condition in the definition of a standard geodesic then no three segments in the path have a nonempty intersection. Hence, each segment in a standard geodesic is a part of a representative of at most one cusp, or in other words it is never the case that (σ_j, σ_{j+1}) and $(\sigma_{j+1}, \sigma_{j+2})$ are both representatives of cusps.

Standard geodesic paths are not unique, but the lack of uniqueness as a set of points is all due to the lack of uniqueness in the representation of cusps as two segments. We make this more precise with the following definition.

Definition 5. Let γ be a standard geodesic from x to y. Define $GS_{(x,y)}$, the *geodesic spine* of γ, to be the union of all segments of γ which do not, together with an adjacent segment, give a representative of a cusp, together with all of the cusp points of γ.

Although $GS_{(x,y)}$ is defined as a set, it has a natural linear order inherited from the geodesic γ. This order on a particular $GS_{(x,y)}$ will not everywhere be compatible with the orientation of T in the sense that the initial point of an oriented segment in $GS_{(x,y)}$ may come after the final vertex of that segment in this order. An example of this can be seen by considering the geodesic spine between two cusp points separated by more than one oriented segment in the implicit line in Figure 2. Because of this natural linear order, we will sometimes abuse language and call the geodesic spine a path, even though it has gaps between cusp points. When a pair of segments representing a cusp $[p_1, p_2]^c$ is on γ, it is only p_1 and p_2 which are on $GS_{(x,y)}$. However, we will again sometimes abuse language and say that $[p_1, p_2]^c$ is on $GS_{(x,y)}$ to stress the fact that p_1 and p_2 are cusp points.

As the notation suggests, the geodesic spine of γ depends only on the endpoints of the geodesic. It is independent of the particular choice of the standard geodesic γ. The uniqueness of the geodesic spine of a standard geodesic between x and y will follow from the following theorem.

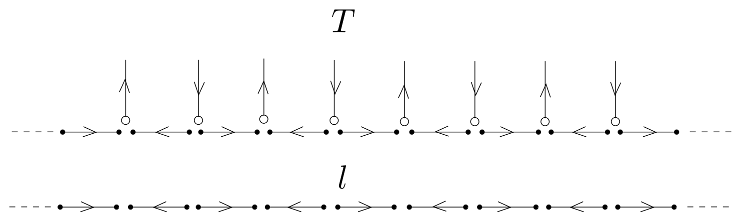

FIGURE 2. An oriented order tree T and implicit line l.

Theorem 2 (see [**15**, Theorem 3.6]). *The geodesic spine of a standard geodesic from x to y is the intersection of all paths from x to y.*

The standard notions of trivial actions (ones with global fixed points) and minimal actions (those with no proper invariant subtree) must be modified slightly to suit the order tree situation.

Definition 6. If $g \in G$ where the group G acts on an order tree T, we say that g has a generalized fixed point $x \in T$ if gx is not separated from x (every segment containing x intersects every segment containing gx).

An action of G on T is nontrivial if there is no point $x \in T$ which is a generalized fixed point for every element of G. Another type of action which is trivial in spirit is an action with a unique fixed end. In the case of an **R**-tree, an end can be defined as an equivalence class of rays, where a ray is an embedding of $[0, \infty)$. We adapt the definitions slightly to the case of a (possibly not Hausdorff) order tree.

Definition 7. A *ray* in an order tree T is a subset ρ of T that can be written as an infinite increasing union of geodesic spines,

$$\rho = \bigcup_{i=1}^{\infty} GS_{(x,x_i)},$$

with $GS_{(x,x_i)} \subset GS_{(x,x_j)}$ if $i < j$. The ray ρ is said to be *infinite* if the sequence $\{x_i\}$ does not converge in T.

Definition 8. If the ray ρ is not infinite, we call a limit of the sequence $\{x_i\}$ an *endpoint* of ρ. If T is not Hausdorff, a finite ray may have multiple endpoints, but all endpoints of the same ray are nonseparable from each other.

Lemma 2. *If ρ is a finite ray starting at the point y of an order tree T and a is an endpoint of ρ, then $\rho = GS_{(a,y)} - \{a\}$.*

Proof. Suppose that ρ has the expression, $\rho = \cup_{i=1}^{\infty} GS_{(y,y_i)}$. First suppose that $\alpha \in GS_{(a,y)}$ and $\alpha \neq a$. If $\alpha = y$ then $\alpha \in \rho$. If $\alpha \neq y$ then α separates y and a. All but finitely many y_i lie in the neighborhood of a consisting of the connected component of $T - \{\alpha\}$ that contains a. So there is a j such that α does not separate y_j from a, which implies that $\alpha \notin GS_{(a,y_j)}$. Hence, $\alpha \in GS_{(y,y_j)}$, which implies that $\alpha \in \rho$. Thus, ρ contains the entire set, $GS_{(a,y)} - \{a\}$.

For the other inclusion, suppose that $y_i \notin GS_{(a,y)}$. Then a and y lie in the same component of $T - \{y_i\}$. But for $j > i$, y_i separates y from y_j, so y_j does not lie in the connected component of $T - \{y_i\}$ that contains a. Since this is true of all y_j with $j > i$, the sequence $\{y_i\}$ cannot have a as a limit point, contradicting the fact that a is an endpoint of ρ. So y_i belongs to $GS_{(a,y)}$ for each i, and ρ is a subset of $GS_{(a,y)}$. Now, $a \notin \rho$ for otherwise a would have to lie in $GS_{(y,y_i)}$ for some i. In this case either $a = y_i$, which is nonsense, or only finitely many y_j would lie in the connected component of $T - \{y_i\}$ containing a, namely y_j with $j < i$. Therefore, $\rho \subset GS_{(a,y)} - \{a\}$. $\square$

There is an equivalence relation on infinite rays in T given by $\rho_1 \approx \rho_2$ if the intersection $\rho_1 \cap \rho_2$ contains an infinite ray. Note that, since geodesic spines are unique, whenever $\rho_1 \cap \rho_2$ contains an infinite ray, the two rays ρ_1 and ρ_2 eventually coincide. Additionally, if $\rho_1 \not\approx \rho_2$, then $\rho_1 \cap \rho_2$ is contained in a finite geodesic spine, and the two rays ρ_1 and ρ_2 eventually separate.

Definition 9. An *end* of the order tree T is an equivalence class of infinite rays in T.

Now, G acts on the set of infinite rays and for any $g \in G$, $\rho_1 \approx \rho_2$ if and only if $g\rho_1 \approx g\rho_2$. Therefore, G acts on the set of ends of T.

And finally, the standard notions of invariant subtrees and minimal actions must be adapted to this situation.

Definition 10. If T contains a G-invariant subset T', with the property that for any two points x and y of T', we have $GS_{(x,y)} \subseteq T'$, we call T' an invariant *implicit subtree.* Of course, an invariant subtree is one special case of an invariant implicit subtree. We call T' an invariant *implicit line* if T' admits a total ordering (without a greatest or a least element) with the following property: Choose any $x, y \in T'$ and let $[x, y]$ denote the interval of T' determined by the total ordering. Then $GS_{(x,y)} = [x, y]$ and furthermore, the natural ordering on $GS_{(x,y)}$ agrees with the total order on $[x, y]$.

Figure 2 shows an example of an order tree T and an implicit line l in T.

Definition 11. Let a group G act on an order tree T. The action is *minimal* if T contains no proper invariant implicit subtree.

Analogous to the case for Hausdorff trees, we have

Lemma 3. *If G acts on the order tree T fixing two distinct ends, then G fixes an implicit line l, the ends of which are fixed by G.*

Proof. Suppose that G fixes the ends ε_1 and ε_2. Choose rays $\rho_1 = \cup_{i=1}^{\infty} GS_{(a,a_i)}$ and $\rho_2 = \cup_{i=1}^{\infty} GS_{(b,b_i)}$ representing ε_1 and ε_2, respectively. Now, ρ_1 and ρ_2 may coincide for a time, but since $\varepsilon_2 \neq \varepsilon_2$ the rays ρ_1 and ρ_2 eventually separate. Choose points a_k and b_l after the point of separation. We define,

$$l := \left(\bigcup_{i=k}^{\infty} GS_{(a_k,a_i)} \right) \cup \left(GS_{(a_k,b_l)} \right) \cup \left(\bigcup_{i=l}^{\infty} GS_{(b_l,b_i)} \right).$$

Now, l is an implicit line whose ends are fixed by G because one represents ε_1 and the other represents ε_2.

To see that l is fixed by G, let $p \in l$ and $g \in G$. Then p separates l into two infinite rays,

$$\alpha_1 = \bigcup_{i=1}^{\infty} GS_{(p,p_i)}$$
$$\alpha_2 = \bigcup_{i=1}^{\infty} GS_{(p,q_i)}$$

with α_i representing ε_i. Since G fixes ε_1 and ε_2, $g\alpha_1$ eventually coincides with α_1 and $g\alpha_2$ eventually coincides with α_2. Choose $p_m \in g\alpha_1 \cap \alpha_1$ and $q_n \in g\alpha_2 \cap \alpha_2$. Let σ_1 be the initial segment $GS_{(p,p_m)}$ and σ_2 the initial segment $GS_{(p,q_n)}$. Then $\sigma_1 \cap \sigma_2 = \{p\}$, so that $g\sigma_1 \cap g\sigma_2 = \{gp\}$. Since $g\sigma_i$ is an initial segment of $g\alpha_i$, the initial segments $GS_{(gp,p_m)}$ and $GS_{(gp,q_n)}$ intersect only in the one point set $\{gp\}$. Therefore, $GS_{(p_m,gp)} \cup GS_{(gp,q_n)}$ is the geodesic spine $GS_{(p_m,q_n)}$. But $GS_{(p_m,q_n)}$ is contained in l. So

$$gp \in GS_{(p_m,gp)} \cup GS_{(gp,q_n)} = GS_{(p_m,q_n)} \subset l,$$

showing that if p is in l and g is in G, then gp is in l. Therefore, G fixes the implicit line l. $\square$

3. Partial orders. The points of an oriented simply connected 1-manifold T are often considered to be partially ordered in a natural way, by declaring two points to be comparable if and only if they lie in a submanifold which is homeomorphic to $\mathbf{R}$ by orientation preserving homeomorphism, and the smaller of a comparable pair of points is determined by the orientation on the manifold. This partial order is obviously preserved by all orientation preserving homeomorphisms of T. There is an extension to this partial order, which is detailed in [**14**, Definition 4.4]. This extension is also naturally inspired by the orientation on the manifold, and it is also preserved by all orientation preserving homeomorphisms of T. In fact, it is a maximal such extension. We review this order here. If $x \in T$, let I_x be an open set in T containing x which is homeomorphic (as an oriented manifold) to $\mathbf{R}$. There is a total order on the points of I_x which is induced by the homeomorphism. Let I_x^+ be the set of elements of $I_x - \{x\}$ which are greater than x, and let $I_x^- = I_x - (I_x^+ \cup \{x\})$. Now, $T - \{x\}$ has exactly two connected components, and we let x^+ be the component containing I_x^+ and let x^- be the component containing I_x^-. A partial order on the points of T is given by $x \leq y$ if and only if $y^+ \subseteq x^+$.

Notation. For an arbitrary poset S and for $x, y \in S$, if x and y are not comparable, write $x \sim y$. If so, and they have a common upper bound, write $x \sim_u y$, and if they have a common lower bound write $x \sim_l y$. In a general partial order, a pair of elements could have either no common bounds, just a common upper bound, just a common lower bound, or both types of common bounds.

We observe that the poset of the points on a simply connected 1-manifold satisfies the following properties, which a general poset may or may not satisfy:

Definition 12. A poset S is *strongly connected* if for any pair of noncomparable elements of the poset S, say $x \sim y$, either $x \sim_u y$ or $x \sim_l y$.

We remark that this is, as the name suggests, stronger than the standard definition of a connected poset. A poset is generally considered to be connected if, given any pair of elements $x, y \in S$, there is a finite sequence $x = a_0, a_1, \dots, a_n = y$ in S such that a_i is comparable to a_{i+1} for all $0 \le i < n$ (see [**12**]). Using this language, a totally ordered set is a connected poset in which a sequence can always be chosen to have $n = 1$, and a strongly connected poset is a connected poset where a sequence can always be chosen with $n \le 2$.

Definition 13. A poset S is *acyclic* if for any three elements $x, y, z \in S$, if $x \sim_u y$ and $x \sim_l z$, then $z > y$.

Definition 14. A partially ordered set will be called *simply connected* if it is both strongly connected and acyclic.

Note that, in particular, the partial order on the points of an oriented simply connected 1-manifold is a simply connected partial order.

Remark. If two elements in a simply connected partially ordered set S are not comparable, then acyclicity implies that it is not possible for the pair to have both common upper and lower bounds, and strong connectivity implies that the pair must have one type of common bound. Hence, for a pair of elements x and y in S with $x \neq y$, exactly one of the four possible relationships hold; namely, one of $x < y$, $x > y$, $x \sim_l y$, and $x \sim_u y$ holds. We note that this condition is weaker than simple connectivity. Consider the example of a poset with six elements, $\{a, b, c, x, y, z\}$, with the relationships $y < b < x$, $z < c < x$, $y < a$, and $z < a$. Here, each noncomparable pair has a common upper bound or a common lower bound, but not both. However, this poset is not acyclic, for $a \sim_l b$, $b \sim_u c$, and $a \sim_l c$, and hence it is not simply connected.

In order to characterize partial orders which groups acting (minimally and without fixing a unique end) on simply connected 1-manifolds must admit, simply connected partial orders are exactly the right thing to consider. However, when considering actions on more general order trees, it turns out that simply connected partial orders are too restrictive. For a complete characterization, the notion of a formal extension is necessary.

Definition 15. Let S be an acyclic partially ordered set. A *formal extension* of the partial order is the set S along with the partial order, with an additional structure as follows. Each pair of elements, $x, y \in S$ with $x \sim y$ which has neither a common upper or lower bound is formally assigned exactly one of the two types $\sim_u$ or $\sim_l$, so that for each pair $x \neq y$, exactly one of the four possible formal relationships holds: $x < y$, $x > y$, $x \sim_l y$ or $x \sim_u y$. The formal extension is said to be a *simply connected extension* if the resulting set of formal relationships, which by construction is strongly connected, also formally satisfies acyclicity.

The necessity for this definition is seen in the following example:

Example 1. Consider the infinite dihedral group $G = \langle s, t \mid tst^{-1} = s^{-1}, t^2 = e\rangle$. This group does not admit a left invariant simply-connected partial order. However, it does have a partial order with a simply connected formal extension. This order comes naturally from an action of the group on the real line, where the real line is viewed as an order tree T with maximal positively oriented segments $\sigma_i = [i, i+1]$, $i \in \mathbf{Z}$ with $o(\sigma_i) = i$ and $f(\sigma_i) = i+1$ if i is even, and $o(\sigma_i) = i+1$ and

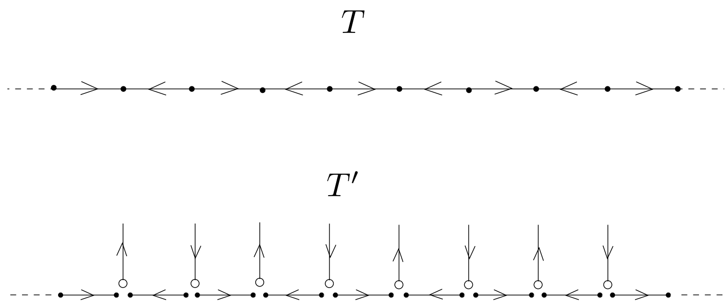

FIGURE 3. An oriented order tree structure on $\mathbf{R}$ and associated 1-manifold.

$f(\sigma_i) = i$ if i is odd, as shown in Figure 3. Positive translations shift T two units to the right, and reflections reflect about the integer points. Note that the action of the dihedral group on T is minimal, while the action on the associated non Hausdorff 1-manifold T' stabilizes any submanifold consisting of the horizontal geodesic spine together with all the translates of any open proper subinterval at the bottom of a single vertical edge. Therefore, the action of the dihedral group on T' is not minimal, nor does it contain a minimal G-invariant submanifold. The group is embedded naturally in T' by identifying it with the orbit of a point with trivial stabilizer, and the simply connected partial order on T' induces a partial order on this orbit, and hence the group, which is acyclic but is not even connected, let alone strongly connected. However, the embedding in T' gives a recipe for a formal extension of the order structure which is simply connected.

Note that the partial order on any set in which nothing is comparable to anything else always has a trivial simply connected extension defined simply by declaring all pairs of elements to be the same incomparability type. Hence we define:

Definition 16. A simply connected extension of a partial order is *trivial* if at least one pair of elements is non-comparable, and all such non-comparable pairs are of a single type.

Remark. A partial order which is already simply connected has a simply connected extension–the one in which no additional relationships are assigned. Such a partial order may or may not be trivial in the above sense. From now on, when we say that a simply connected partial order is trivial, we mean in the sense of Definition 16. In addition, note that there are certainly groups which cannot admit nontrivial extensions; for example, a torsion group. More interesting examples from the point of view of actions on trees appear in Section 6.

We record next a lemma about these partial orders.

Lemma 4. *Let S be a set with a simply connected extension of a partial order and let $x, y, z \in S$. Then,*

- *if $x \sim_l y$ and $y < z$, then $x \sim_l z$.*
- *If $x \sim_u y$ and $y > z$ then $x \sim_u z$.*

Proof. First suppose that $x \sim_l y$ and $y < z$. Then x and y are not comparable and have no common upper bound. Recall that x and z always satisfy exactly one of the four relationships. We show that $x \sim_l z$ by eliminating the other three possibilities. First, $x < z$ implies that z is a common upper bound for x and y, which is impossible. Second, $x > z$ implies that $x > y$, which is not true. Finally, if $x \sim_u z$, then by acyclicity it must be true that $y > z$, which is once again a contradiction. Hence, it must be that $x \sim_l z$. The proof of the second statement is similar. □

We will be interested in the special case where the partially ordered set is a group and the order is left-invariant. In that setting, if $g \sim h$ in the original partial order, and the pair have neither an upper nor a lower common bound, then no pair of translates $fg \sim fh$ will either. We will be interested only in extensions which are left-invariant.

Definition 17. Let G be a group with a left-invariant acyclic partial order. We say that the partial order has a left-invariant simply connected extension if it has a simply connected extension as a poset which is left-invariant in the sense that $g \sim_u h$ implies that $fg \sim_u fh$ for every $f \in G$, and similarly for $\sim_l$.

For use in later sections, we investigate the properties of an acyclic partial order on a set S, which we assume has a simply connected extension. In what follows, we are always referring to the full set of comparability relationships in a chosen simply connected extension of the given partial order, even in the special case that the partial order is already simply connected and does not need to be extended.

First we define the notion of betweenness inspired by the geodesic spines in **R**-order trees.

Definition 18 (Betweenness). Given two points a and c with $c \neq a$, we say that b is between a and c if any of the following three conditions holds:

(1) In the case that $a < c$, either $a < b < c$, or else both $a \sim_u b$ and $b \sim_l c$.

(2) In the case that $a \sim_l c$, either both $a \sim_l b$ and $b < c$, or else both $c \sim_l b$ and $b < a$.

(3) In the case that $a \sim_u c$, either both $a \sim_u b$ and $b > c$, or else both $c \sim_u b$ and $b > a$.

Definition 19. For any $a, b \in G$ with $a \neq b$, we define the nonempty set $B_{a,b}$ by

$$B_{a,b} = \{a, b\} \cup \{x \mid x \text{ is between } a \text{ and } b\}.$$

We call a set of the form $B_{a,b}$ a *between-set*.

The following lemma, which can easily be proven by careful case-by-case analysis, records some basic properties of between-sets.

Lemma 5. *Let S be a partially ordered set with a simply connected extension, and let $a, b, c, d \in S$. Then,*

(1) *for any three elements, a, b, c we have $B_{a,b} \subset B_{a,c} \cup B_{c,b}$.*

(2) *$c \in B_{a,b}$ if and only if $B_{a,b} = B_{a,c} \cup B_{c,b}$.*

(3) *If $c \in B_{a,b}$, then $B_{a,c} \cap B_{c,b} = \{c\}$.*

(4) *If $b \in B_{a,c}$ and $c \in B_{b,d}$, then $b, c \in B_{a,d}$.*

From this lemma, it is easy to see that each of these sets $B_{a,b}$ comes with a natural total order. The only ambiguity is in declaring whether a is the least or greatest element. As in the case of the total order on geodesic spines in an order tree, this partial order will not always agree with the original partial order on S.

Definition 20. We define a total order, $\preceq$, on the set $B_{a,b}$ as follows. If x and y belong to $B_{a,b}$, then $x \preceq y$ if $x \in B_{a,y}$ and $x \succeq y$ if $x \in B_{y,b}$. (Of course, here a is considered the least element and b the greatest).

Corollary 1. *The order on $B_{a,b}$ is a total order.*

Definition 21. A *path* from x to y in G is an ordered *n-tuple* of the form $(B_{s_1,t_1}, \dots, B_{s_n,t_n})$ with $s_1 = x$, $t_n = y$, and $t_i = s_{i+i}$ for $1 \leq i \leq n-1$.

Corollary 2. *$B_{x,y}$ is the intersection of all paths from x to y.*

Note that if the set S is the set of points on an oriented simply connected 1-manifold with the order described earlier, then $B_{a,b} = GS_{(a,b)}$. Since these geodesic spines can always be expressed as finite unions of oriented segments in order trees, our extensions of partial orders will require a finiteness condition on the paths $B_{a,b}$. To articulate this, we define a relation on each set $B_{a,b}$.

Definition 22. We define an equivalence relation $O_{a,b}$ on $B_{a,b}$ by declaring that, for x and y in $B_{a,b}$, we have $xO_{a,b}y$ if and only if $B_{x,y}$ is totally ordered in the original order on the set S.

Lemma 6. *The relation $O_{a,b}$ is an equivalence relation on $B_{a,b}$.*

Proof. The relation $O_{a,b}$ is clearly reflexive and symmetric. Transitivity follows from the properties of the extension of the order on S. □

It is now clear that the equivalence classes in a set $B_{a,b}$ are themselves totally ordered in the original order, and that if x and y are in the same equivalence class, then all points between x and y are also in that equivalence class.

Definition 23. We say that the simply connected extension of a partial order on a set S is *rectifiable* if, for every two elements a and b in S, the between-set $B_{a,b}$ is a finite union of equivalence classes under the relation $O_{a,b}$.

Partial orders with rectifiable simply connected extensions are naturally inspired by the partial orders on simply connected 1-manifolds, and they satisfy algebraic properties commonly found in the theory of partially ordered groups. We now discuss a few of these. First, if a group has such a partial order, then so does any subgroup (though the order restricted to the subgroup may be trivial). Furthermore, these orders also pass to appropriate quotient groups. It is a standard result in the theory of partially ordered groups that, if G has a left invariant partial order, and H is a normal, convex subgroup, then a partial order is naturally induced on the quotient group G/H (see [**12**, subsection 1.6.3]). In the case of a simply connected partial order, or more generally a simply connected extension, the order on the quotient group is of the same special type as long as we require a stronger version of convexity for the subgroup, obtained by replacing the notion of totally ordered sets by sets of the form $B_{a,b}$.

Definition 24. Let G be a group with a left-invariant partial order with left-invariant simply connected extension. Then a subgroup H is called *completely convex* if for any pair of elements $h_1, h_2 \in H$, we have $B_{h_1,h_2} \subset H$.

We have:

Theorem 3. *Let G be a group with a left-invariant partial order with left-invariant simply connected extension. Let H be a completely convex normal subgroup. Then G/H also admits a left-invariant partial order with left-invariant simply connected extension. Furthermore, if the extension is rectifiable, then the extension induced on G/H is also rectifiable.*

Proof. First we define four possible relationships between cosets in G/H. Given two cosets $g_1H \neq g_2H$, define

(1) $g_1H < g_2H$ if there exists an element $h \in H$ such that $g_1 < g_2h$.

(2) $g_1H > g_2H$ if there exists an element $h \in H$ such that $g_1 > g_2h$.

(3) $g_1H \sim_u g_2H$ if there exists an element $h \in H$ such that $g_1 \sim_u g_2h$.

(4) $g_1H \sim_l g_2H$ if there exists an element $h \in H$ such that $g_1 \sim_l g_2h$.

Since H is normal in G and the partial order is left invariant, if a pair of cosets satisfies one of the above conditions for a particular choice of coset representatives, then they will satisfy that condition with any choice of coset representatives. Also, for any pair of distinct cosets g_1H and g_2H, exactly one of the relationships, $g_1 < g_2$, $g_2 < g_1$, $g_1 \sim_u g_2$ or $g_1 \sim_l g_2$ holds since we have a full extension of a partial order in G. So each pair of distinct cosets will be related in at least one of the above ways. In addition, it is clear that these relationships are invariant under left multiplication by elements of G.

We now prove that the relationships defined above satisfy the following property:

Property 1. *For $g_1, g_2, g_3 \in G$,*

(1) *if $g_1H < g_2H$ and $g_2H < g_3H$, then $g_1H < g_3H$.*

(2) *If $g_1H \sim_u g_2H$ and $g_2H \sim_l g_3H$, then $g_3H < g_2H$.*

(3) *If $g_1H \sim_u g_2H$ and $g_3H < g_2H$, then $g_1H \sim_u g_3H$.*

(4) *If $g_1H \sim_l g_2H$ and $g_2H < g_3H$ then $g_1H \sim_l g_3H$.*

Proof. The proofs of the four statements are nearly identical to each other. In each case, the normality of H gives the desired relationship as long as $g_1H \neq g_3H$, and the complete convexity of H implies that the two cosets cannot in fact be the same. We provide the explicit argument for the first statement. If $g_1H < g_2H$ and $g_2H < g_3H$, then $g_1 < g_2h$ and $g_2 < g_3h'$ for some $h, h' \in H$. By the normality of H, $g_2h < g_3h''$ for some $h'' \in H$. Therefore, $g_1 < g_3h''$, which implies that $g_1H < g_3H$, the desired conclusion, as long as $g_1H \neq g_3H$. But $g_3H = g_1H$ implies that $g_1^{-1}g_3 \in H$. Notice that, translating the above relationships by g_3^{-1}, we see that $g_3^{-1}g_1 < g_3^{-1}g_2h < h''$. But this implies that $g_3^{-1}g_2h \in B_{g_3^{-1}g_1,h''}$, which in turn is contained in H by the complete convexity of H. Hence, $g_3^{-1}g_2 \in H$, or equivalently $g_3H = g_2H$, which is ruled out by the assumption that $g_2H < g_3H$. □

We now prove that a given pair of distinct cosets can satisfy at most one of the four possible relationships. By symmetry, there are only four possibilities which must be eliminated for the pair $g_1H \neq g_2H$.

(1) Suppose that $g_1H < g_2H$ and $g_2H < g_1H$. Then by the first part of Property 1, $g_1H < g_1H$, which is impossible.

(2) Suppose that $g_1H \sim_u g_2H$ and $g_2H \sim_l g_1H$. Then by the second part of Property 1, $g_1H < g_1H$.

(3) Suppose $g_1H \sim_u g_2H$ and $g_1H < g_2H$. Then by the third part of Property 1, $g_1H \sim_u g_1H$, which is impossible.

(4) Suppose $g_1H \sim_l g_2H$ and $g_2H < g_1H$. Then by the fourth part of Property 1, $g_1H \sim_l g_1H$.

At this point, we see that we have defined a partial order and a formal set of relationships $\sim_u$ and $\sim_l$ which satisfy all the properties of a simply connected extension. Finally, to ensure that this structure is really an extension of the partial order, we must check that the formal relationships $\sim_u$ and $\sim_l$ actually agree with any existing relationships based on common bounds. Suppose that g_1H and g_2H are two distinct cosets which are not comparable according to the above definition of comparability, but do have a common upper (respectively lower) bound. We show that in the above formal assignments, it is indeed the case that $g_1H \sim_u g_2H$ (respectively $g_1H \sim_l g_2H$). We argue the case of an upper bound. Suppose that g_3H is this common upper bound. Then $g_1 \leq g_3h$ and $g_2 \leq g_3h'$ for some $h, h' \in H$. Hence, by normality of H, $g_2h'' \leq g_3h$ for some $h'' \in H$, so g_1 and g_2h'' share an upper bound in G. Now if g_1 and g_2h'' were comparable in G, then g_1H and g_2H would be comparable in G/H. Hence g_1 and g_2h'' are not comparable, and $g_1 \sim_u g_2h''$ in G. This implies that $g_1H \sim_u g_2H$ in G/H. The argument for lower bounds is the same.

To complete the proof of Theorem 3, we now show that if the original extension is rectifiable, then so is the induced extension of the order on G/H. Suppose towards a contradiction that the extended order on G/H is not rectifiable. Since the order is left-invariant, we may assume that, for some $g \in G$, there are infinitely many equivalence classes in $B_{gH,H}$ under the relation $O_{gH,H}$. It follows that we may choose a sequence of cosets $\{g'_0H = gH, g'_1H, g'_2H, g'_3H, \dots\}$ such that the following hold:

$$g_i'H \neq H, \quad g_i'H \in B_{g_{i-1}'H,H}, \quad g_i'H \in B_{g_{i-1}'H,g_{i+1}'H},$$

and $g_i'H$ is not comparable to $g_{i-1}'H$ for all $i \geq 1$. From this we will inductively construct a sequence $\{g = g_0, g_1, g_2, \dots\}$ and $h \in H$ satisfying

$$g_iH = g_i'H, \quad g_i \in B_{g_{i-1},h}, \quad g_i \in B_{g_{i-1},g_{i+1}},$$

and such that g_{i-1} is not comparable to g_i for any $i \geq 1$. This is impossible, because all of the $\{g_i\}$ lie in $B_{g,h}$, yet they all are in distinct equivalence classes under $O_{g,h}$, violating the fact that the extension of the order in G is rectifiable. Before constructing the sequence, we state an easily verified lemma:

Lemma 7. *Suppose that $gH \in B_{fH,kH}$ where $g, f, k \in G$. Then there are elements h and h' in H such that $gh' \in B_{f,kh}$.*

To construct the desired sequence, first note that $g_1'H \in B_{gH,H}$. By Lemma 7, $g_1'h_1$ belongs to $B_{g,h}$ for some $h, h_1 \in H$. Set $g_1 = g_1'h_1$. Now, $g_1 \in B_{g_0,h}$ and g_1 is not comparable to g_0 as desired. For the inductive step, suppose that we have already constructed $\{g_1, g_2, \dots, g_n\}$ with the desired properties. Since $g_{n+1}'H$ belongs to $B_{g_nH,H}$, Lemma 7 yields $h_{n+1}, h' \in H$ so that $g_{n+1}'h_{n+1} \in B_{g_n,h'}$. Again, let $g_{n+1} = g_{n+1}'h_{n+1}$. We have already that $g_{n+1}H = g_{n+1}'H$, and also that $g_{n+1} \in B_{g_n,h'}$.

Now, by the complete convexity of H, we have that $B_{h,h'}$ is contained in H, which implies that $g_{n+1} \notin B_{h,h'}$ since $g_{n+1} \notin H$. Hence it must be that $g_{n+1} \in B_{g_n,h}$ as desired, for if not, $g_{n+1} \notin B_{h,h'} \cup B_{g_n,h}$, which contradicts the fact that

$$g_{n+1} \in B_{g_n,h'} \subset B_{h,h'} \cup B_{g_n,h}.$$

In addition, it is clear that $g_n \in B_{g_{n+1},g_{n-1}}$ and g_{n+1} and g_n are noncomparable. So we have inductively constructed the sequence needed to complete the proof. $\square$

Next, in the theory of left-invariant partially ordered groups, the positive cone is an important tool, and there is an analogous structure for left-invariant simply connected extensions. In a left-invariant partially ordered group G, the *positive cone* is the subset $\mathcal{P} = \{g \in G \mid e < g\}$. The positive cone satisfies the two conditions

$$\mathcal{P} \cap \mathcal{P}^{-1} = \varnothing \tag{3.1}$$

$$\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}, \tag{3.2}$$

where $\mathcal{P}^{-1} = \{x^{-1} \mid x \in \mathcal{P}\}$. The importance of the positive cone comes from the fact that any subset $\mathcal{P} \subset G$ satisfying conditions 3.1 and 3.2 defines a left-invariant partial order on G by the equation $g < h$ if and only if $g^{-1}h \in \mathcal{P}$. The partial order defined by $\mathcal{P}$ is a total order if and only if $G = \mathcal{P} \cup \mathcal{P}^{-1} \cup \{e\}$. Many other properties of partially ordered groups may be defined and studied via $\mathcal{P}$. For example, a partially ordered group is said to be *directed up* if any two elements of the group share an upper bound, which is equivalent to the condition

$$G = \mathcal{P} \cup \mathcal{P}^{-1} \cup \left(\mathcal{P} \cdot \mathcal{P}^{-1}\right).$$

Similarly, partially ordered groups with left-invariant simply connected extensions may be classified in terms of the positive elements and the elements sharing lower or upper bounds with the identity.

More precisely, if G has a simply connected extension of a partial order, then we make the definitions:

$$\mathcal{P} = \{g \mid e < g\}, \quad \mathcal{U} = \{g \mid e \sim_u g\}, \quad \mathcal{L} = \{g \mid e \sim_l g\}.$$

For each of these sets, we define:

$$\begin{aligned} \mathcal{P}^{-1} &= \{g \in G \mid g^{-1} \in \mathcal{P}\}, \\ \mathcal{U}^{-1} &= \{g \in G \mid g^{-1} \in \mathcal{U}\}, \\ \mathcal{L}^{-1} &= \{g \in G \mid g^{-1} \in \mathcal{L}\}. \end{aligned}$$

Various properties of these sets follow from the formal properties of the partial order. For instance, if $h \in \mathcal{L}$, then $h \sim_l e$. Since the order is left-invariant, it follows that $e \sim_l h^{-1}$, and hence $h \in \mathcal{L}^{-1}$. Therefore, we have $\mathcal{L} = \mathcal{L}^{-1}$. Also, if $g \in \mathcal{U}$ and $h \in \mathcal{L}$, then since $h \sim_l e$ and the order is left-invariant, it follows that $gh \sim_l g$. Now acyclicity implies that $gh \in \mathcal{P}$, and therefore $\mathcal{U} \cdot \mathcal{L} \subset \mathcal{P}$. As in the case of a general partial order, these sets characterize the order here as follows:

Proposition 2. *The group G has a left-invariant partial order with simply connected extension if and only if there exist subsets $\mathcal{P}, \mathcal{L}$ and $\mathcal{U}$ of G such that*:

(1) $\mathcal{P} \cap \mathcal{P}^{-1} = \varnothing$, $\mathcal{L}^{-1} = \mathcal{L}, \mathcal{U}^{-1} = \mathcal{U}$.

(2) $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}$.

(3) $\mathcal{L} \cdot \mathcal{P} \subset \mathcal{L}$.

(4) $\mathcal{P} \cdot \mathcal{U} \subset \mathcal{U}$.

(5) $\mathcal{U} \cdot \mathcal{L} \subset \mathcal{P}$.

(6) *G is the disjoint union $G = \mathcal{P} \sqcup \mathcal{P}^{-1} \sqcup \mathcal{U} \sqcup \mathcal{L} \sqcup \{e\}$.*

We remark that the disjointness of the union of item (6) would follow from items (1)–(5) if we added the assumption that neither $\mathcal{U}$ nor $\mathcal{L}$ contains the identity. The proof of this proposition is similar to the proof for ordinary left-invariant orders. If G has a simply connected extension of a partial order, one checks that the remaining properties also follow from the properties of the partial order. Conversely, if such sets $\mathcal{P}, \mathcal{L}$ and $\mathcal{U}$ exist, then we define:

$$g < h \text{ if } g^{-1}h \in \mathcal{P}, \quad g \sim_u h \text{ if } g^{-1}h \in \mathcal{U}, \quad g \sim_l h \text{ if } g^{-1}h \in \mathcal{L}.$$

Condition (5) is essentially acyclicity and (6) is strong connectivity.

Just as in the cases of other kinds of partial orders, many properties of simply connected extensions may be stated in terms of these sets. For example, the partial order on G given by P is simply connected and needs no extension whenever G is equal to the union,

$$G = \mathcal{P} \cup \mathcal{P}^{-1} \cup (\mathcal{P} \cdot \mathcal{P}^{-1}) \cup (\mathcal{P}^{-1} \cdot \mathcal{P}).$$

4. Groups with partial orders act on order trees. In this section we consider groups with partial orders which have left-invariant rectifiable simply connected extensions. We will prove the following theorem, which proves that (1) implies (2) in both main theorems.

Theorem 4. *If G is a countable group which admits a partial order with a nontrivial left-invariant rectifiable simply connected extension, then G acts faithfully, minimally and without fixing a unique end by* *orientation preserving homeomorphisms on an oriented order tree T. Moreover, T can be chosen to have a point x with trivial stabilizer so that identifying G with the orbit Gx in the associated 1-manifold induces the original partial order on G. In the case that the partial order was simply connected (needing no extension), the induced action on the simply connected 1-manifold will also be minimal.*

We will prove this theorem by constructing an oriented order tree T on which the group acts by orientation preserving homeomorphisms. In the construction, a subset of T will be labeled by group elements, and distinct elements of the group will label distinct points of T, so the action will be faithful. We model our construction on the proof of Theorem 6.8 of [**11**], in which points on the real line are labeled by the elements of G in order to construct the desired action. In the construction, we express G as a union of sets of the form B_{x_i,y_i}, and T will be a quotient space of the union of compact subintervals of the real line, one called I_i for each set B_{x_i,y_i}. In order to specify how to glue the intervals together to form a tree as well as to obtain the G-action, we will label points on each interval I_i by the group elements in B_{x_i,y_i}. Since each set B_{x_i,y_i} is the union of equivalence classes under the relation O_{x_i,y_i}, each such equivalence class will label points on a subinterval of I_i. Since each equivalence class has a total order induced by the partial order on G, these orders will be used to orient the labeled subintervals of I_n. The only difficulty here is in the case that an equivalence class consists of a single element. To avoid this, we simply blow up G to a larger set G^+ by adding formal elements g_+ and g_- for each element g of G. We extend the order and the G-action in a natural way to the set G^+, so that former singleton equivalence classes under O_{x_i,y_i} of the form $\{g\}$ will now have two additional formal elements, allowing us to orient the corresponding labeled subinterval.

Definition 25. Let G be a partially ordered group. Define the induced poset G^+ with a G action as follows:

(1) As a set, $G^+ = \{g, g_-, g_+ | g \in G\}$.

(2) (G action on G^+.) For $g, h \in G$, define an action of G on G^+ via

$$g(h) = gh, \quad g(h_-) = (gh)_-, \quad g(h_+) = (gh)_+.$$

(3) (Order on G^+.) For each $x \in G$, $x_- < x < x_+$ and if $x < y$ then $x_+ < y_-$.

It is not hard to see that Definition 25 gives G^+ a left-invariant partial order. If the partial order on G has a simply connected extension, the induced partial order on G^+ can be extended following the manner in which the order on G is extended. To define this extension, let $g_\varepsilon, h_\delta \in G^+$ where ε and δ are $+$, $-$ or absent. Note that if $g_\varepsilon \sim h_\delta$, then $g \sim h$ in G and, furthermore, if g_ε and h_δ share an upper or lower bound in G^+ if and only if g and h share one in G.

Definition 26. For h_δ and g_ε as above, define a formal extension of the order on G^+ by declaring $g_\varepsilon \sim_u h_\delta$ if $g \sim_u h$ in the extension of the order on G and $g_\varepsilon \sim_l h_\delta$ if $g \sim_l h$ in the extension of the order on G.

Lemma 8. *If G is a group with a rectifiable simply connected extension of a partial order, then the natural order on G^+ also is a rectifiable simply connected extension of the corresponding order on G^+. In addition, the left G-action preserves the extended partial order, and for any x and y in G with $x \neq y$, no $O_{x,y}$ class consists of a single point.*

Proof. The only change in structure is that any time b appeared in a set $B_{x,y} \subset G$, the triple $b_- < b < b_+$ appears in the corresponding set $B_{x,y} \subset G^+$. So in particular, the number of equivalence classes under $O_{x,y}$ does not change, and there are no equivalence classes consisting of only one point, except possibly at the endpoints if x or y belongs to $G^+ - G$. As previously mentioned, the G-action clearly preserves the order. $\square$

We now record an easily verifiable lemma which holds in the setting of the previous lemma.

Lemma 9. *If G is a group with a partial order with left-invariant simply connected extension, and G^+ is the poset with extension described above, and $a, b \in G$, then*

$$\begin{aligned} a < b &\iff \{a_+, b_-\} \subset B_{a_-,b_+} \\ a \sim_u b &\iff \{a_+, b_+\} \subset B_{a_-,b_-} \\ a \sim_l b &\iff \{a_-, b_-\} \subset B_{a_+,b_+}. \end{aligned}$$

Definition 27. Let R be the relation on the augmented group G^+ defined by: xRy if $B_{x,y} = \{x, y\}$, where $x, y \in G^+ - G$, and xRx if $x \in G$.

Lemma 10. *R is an equivalence relation, and the equivalence classes are all of order one or two.*

Proof. The relation R is clearly reflexive and symmetric. Furthermore, the only nontrivial case to check for transitivity is that $g_s R h_t$ and $h_t R f_r$ where $g, h, f \in G$ and $s, t, r \in \{+, -\}$ and $g_s \neq h_t, h_t \neq f_r$. Now,

$$B_{g_s,f_r} \subseteq B_{g_s,h_t} \cup B_{h_t,f_r} = \{g_s, h_t, f_r\},$$

and we claim that the only possibility is for $g_s = f_r$. This then implies that $g_s R f_r$, showing that R is transitive. Moreover, this implies that equivalence classes containing elements of $G^+ - G$ have at most two elements, while for $g \in G$, the equivalence class of g is always the singleton set $\{g\}$. To verify the claim that $g_s = f_r$, note first that if $g \neq f$, then by Lemma 9, B_{g_s,f_r} contains at least four elements of G^+, which is impossible as we saw above that it can contain at most three elements. Hence, it must be that $f = g$, but if also $r \neq s$, then $B_{g_s,f_r} = B_{g_s,g_r} = \{g_s, g, g_r\}$, which is also impossible since we saw above that it contained no elements of G. $\square$

We now construct an oriented order tree on which the group acts. We follow the ideas in [**9**, Section 3], in which order trees are described as countable increasing unions of segment which intersect in a very restricted way. We begin by constructing a decomposition of the group into a countable union of subsets, which will guide the construction of the tree.

Since G is countable, G can be expressed as a countable union of sets of the form $B_{a,b}$, where a and b are elements of G. We claim the union can be chosen to be of a certain type.

Lemma 11. *Let G be a countable group with a left-invariant partial order with rectifiable simply connected extension. Then, for some* *choices of $x_i, y_i \in G$, we have*

$$G = \bigcup_{i=1}^{\infty} B_i, \text{ where } B_i = B_{x_i,y_i},$$

and, in addition, if we set $G_n = \cup_{i=1}^{n} B_i$, then

(1) $G_n \cap B_{n+1} \neq \varnothing$.

(2) *For every a and b in G_n, we have $B_{a,b} \subseteq G_n$.*

(3) *$G_n \cap B_{n+1}$ contains x_{n+1} and has one of the two following forms:*

(a) $G_n \cap B_{n+1} = \{x_{n+1}\}$.

(b) *$G_n \cap B_{n+1}$ is half open and totally ordered in G_n.*

Proof. Choose an arbitrary element of G, and call it x_1. Now arbitrarily enumerate the elements of $G - \{x_1\}$ as $\{y_i\}_{i=1}^{\infty}$, and let $B_i = B_{x_1,y_i}$. Then G is certainly the union of the sets B_i as desired, and property (1) is satisfied. Moreover, property (2) follows from the properties of the sets of the form $B_{a,b}$ developed in Section 3. We claim that, by changing some of the sets in the union, we can ensure that property (3) holds as well. Notice that if y_{n+1} is in G_n, then $G_n = G_{n+1}$ since the entire set B_{n+1} is contained within G_n. So after eliminating such indices we can re-index so that $G_n \cap B_{n+1}$ contains x_1 and has one of the following forms:

(1) $G_n \cap B_{n+1} = \{x_1\}$.

(2) $G_n \cap B_{n+1}$ is half open, i.e., has more than one element, but is not of the form (3).

(3) $G_n \cap B_{n+1} = B_{x_1,z_{n+1}}$ for some $z_{n+1} \in G_n$.

If the intersection is of the first form, it already satisfies condition (a). Now if the intersection is of the third form, just replace $B_{n+1} = B_{x_1,y_{n+1}}$ by $B_{z_{n+1},y_{n+1}}$, which reduces the form of the intersection to condition (a). If the intersection is of the second type, recall that the equivalence classes of B_{n+1} under the equivalence relation $O_{x_1,y_{n+1}}$ are linearly ordered. If we consider the smallest class to be the one containing x_1 and the largest class to be the one containing y_{n+1}, then choose any element from the largest equivalence class which contains elements from G_n, say z_{n+1}. If we then replace B_{n+1} by $B_{z_{n+1},y_{n+1}}$ as in case (3), the intersection is totally ordered in G_n since it is completely contained in a single equivalence class along $B_{z_{n+1},y_{n+1}}$. Hence, it satisfies condition (b). Furthermore, these replacements did not interfere with conditions (1) or (2) of the lemma. $\square$

We will now use this expression of G as the union of the sets B_i to construct a tree T with a labeling by $\nu : G^+ \to T$ as follows. T will be the quotient space of a disjoint union $\cup_{i=0}^{\infty} I_i$ modulo a set of identifications $\{R_i\}_{i=1}^{\infty}$, where each I_i is a compact subinterval of the real line, and the identification R_i identifies one endpoint of I_i with one point in the disjoint union $\cup_{k=0}^{i-1} I_k$. The set T_n will be defined as the quotient space $\cup_{i=0}^{n} I_i/\{R_1, \dots , R_n\}$, so that $T_{n+1} = (T_n \cup I_{n+1})/R_{n+1}$. Then, for any n, $T = (T_n \cup \cup_{i=n+1}^{\infty} I_i)/\{R_i\}_{i=n+1}^{\infty}$. Thus, to define T, it suffices to construct T_0 and then inductively construct T_{n+1} from T_n.

To define the labeling function ν, we let $G_n^+ = \{g_+, g, g_- | g \in G_n\}$, and we will label a subset of points in I_n by $G_n^+ - G_{n-1}^+$. Apart from a technical detail, this will be the temporary labeling $\widetilde{\nu}_n$ defined below. Since G^+ is the disjoint union of all these subsets, we let ν be the labeling of T by G^+ obtained by defining ν restricted to each disjoint subset $G_n^+ - G_{n-1}^+$ to be the appropriate labeling map above, and we define ν_n to be the labeling of T_n obtained similarly. Finally, each tree T_n will naturally be homeomorphic to a subtree of both T_{n+i} and T, and we will sometimes abuse notation and refer to all of these copies by the same name, T_n.

The inductive construction of T. Let G be a group with a left-invariant rectifiable simply connected extension of a partial order. Express the group as a union of the form:

$$G = \bigcup_{i=1}^{\infty} B_i, \text{ where } B_i = B_{x_i, y_i}$$

as in Lemma 11.

Base step. Let $T_0 = I_0$ be a copy of the unit interval, and define

$$\nu_0 : \{(x_1)_-, x_1, (x_1)_+\} \longrightarrow T_0$$

by

$$\nu_0((x_1)_-) = 0, \quad \nu_0(x_1) = 0.5, \quad \nu_0((x_1)_+) = 1.$$

Inductive step. Suppose T_n has been constructed, along with $\nu_n : G_n^+ \to T_n$. We must label a closed segment of the real line, I_{n+1}, by the elements of $G_{n+1}^+ - G_n^+$ and then specify the identification R_{n+1} to construct T_{n+1}. The new labeling $\nu_{n+1} : G_{n+1}^+ \to T_{n+1}$ will be defined to be the map ν_n restricted to G_n^+ and the labeling specified below of I_{n+1} restricted to $G_{n+1}^+ - G_n^+$. The details will depend upon the form of $G_n \cap B_{n+1}$.

Case 1. $B_{n+1} \cap G_n = \{x_{n+1}\}$. To simplify notation, suppress subscripts and abbreviate x_{n+1} by x and y_{n+1} by y. Now suppose B_{n+1} is made up of k equivalence classes under $O_{x,y}$; each is totally ordered with respect to the partial order on G, and the set of equivalence classes has a natural linear order along $B_{x,y}$. We index these equivalence classes by i where $1 \leq i \leq k$, where we take the first class, or $i = 1$, to be the one containing x. We construct a labeling of a copy of $[0, k] \subset \mathbf{R}$. For each index i, augment the equivalence class by adding, for each g in the class, g_- and g_+. First, identify pairs of elements which are equivalent $\bmod R$. The resulting set is still totally ordered. Now define $(\nu_{n+1})_i$ from this totally ordered set of labels to $[i-1, i]$ as in the classical construction in the proof of Theorem 6.8 of [**11**], which lays a totally ordered set down on the real line. If this modified equivalence class happens to have least (or greatest) elements, label $i-1$ (or i, respectively) by these elements. If not, the endpoints i and $i+1$ receive no labels, but will be limit points of labeled points. Notice that since no augmented equivalence class has only one element, there is never confusion about whether to label $i-1$ or i by a given extremal element of the modified equivalence class. Since $B_{n+1} \cap G_n = \{x\}$, if we take $\widetilde{\nu}_{n+1}$ to be the union of all these maps, the domain of $\widetilde{\nu}_{n+1}$ intersects the domain of ν_n only at the triple x_-, x, x_+. Exactly one of these augmented elements x_-, x_+ is between x and y, for ease of discussion suppose it is x_-. Then, on the interval $[0, 1]$, $0 = \widetilde{\nu}_{n+1}(x_+) < \widetilde{\nu}_{n+1}(x) < \widetilde{\nu}_{n+1}(x_-) \leq 1$, and furthermore, the open segments $(0, \widetilde{\nu}_{n+1}(x))$ and $(\widetilde{\nu}_{n+1}(x), \widetilde{\nu}_{n+1}(x_-))$ contain no images under $\widetilde{\nu}_{n+1}$. We remove the segment $[0, \widetilde{\nu}_{n+1}(x_-))$ from $[0, k]$, leaving just $[\widetilde{\nu}_{n+1}(x_-), k]$. This segment, $[\widetilde{\nu}_{n+1}(x_-), k]$, is the segment I_{n+1}. It is possible that $\widetilde{\nu}_{n+1}(x_-) = 1$ in the case that there were no other group elements besides x in the equivalence class of x along B_{n+1}. Otherwise, $\widetilde{\nu}_{n+1}(x_-) < 1$.

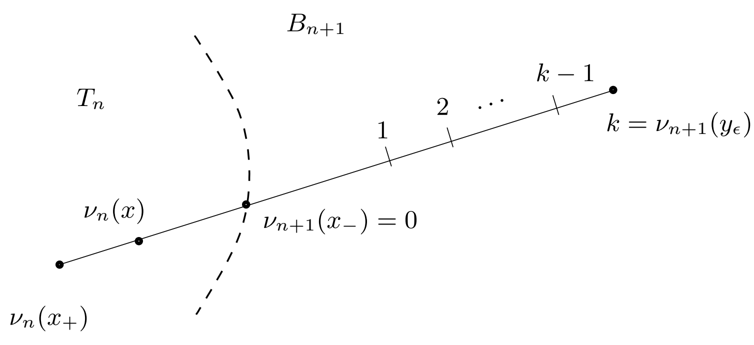

FIGURE 4. T_{n+1} after the gluing of B_{n+1}, Case 1.

To define the gluing relation R_{n+1}, identify the point $\widetilde{\nu}_{n+1}(x_-)$ to the point $\nu_n(x_-)$ in T_n to form T_{n+1}. That is, we define T_{n+1} by,

$$T_{n+1} = (T_n \cup [\widetilde{\nu}_{n+1}(x_-), k]) \,/\, (\widetilde{\nu}_{n+1}(x_-) = \nu_n(x_-)),$$

and we define $\nu_{n+1} : G^+_{n+1} \to T_{n+1}$ as just ν_n restricted to G^+_n and $\widetilde{\nu}_{n+1}$ restricted to $G^+_{n+1} - G^+_n$, as in Figure 4. Notice that the domains of $\widetilde{\nu}_{n+1}$ and ν_n intersect in precisely the element x_-, and the relation R_{n+1} identifies $\widetilde{\nu}_{n+1}(x_-)$ with $\nu_n(x_-)$.

Case 2. $B_{n+1} \cap G_n$ is half open and totally ordered in G. Now, $B_{n+1} \cap G_n$ is a totally ordered set of group elements, and x is either the greatest or the least of them. For ease of discussion, assume it is the least, so that there is no greatest element. Choose a countable, increasing subsequence $\{a_i\} \subset B_{n+1} \cap G_n$ which is not bounded above (with respect to the total order) by any element in $B_{n+1} \cap G_n$. These group elements label points in T_n, and since T_n is a finite union of compact intervals, there is some limit point l of the sequence $\{\nu_n(a_i)\}$ in T_n. Now consider the set $B_{n+1} - (G_n \cap B_{n+1})$, which intersects only finitely many (say k) of the equivalence classes in B_{n+1}. Label a copy of $[0, k]$ by elements in $G^+_{n+1} - G^+_n$ as in Case 1, with either $\widetilde{\nu}_{n+1}(y_-) = k$ or $\widetilde{\nu}_{n+1}(y_+) = k$. In this case, the segment $[0, k]$ is I_{n+1}. If an element $g \in B_{n+1} - (G_n \cap B_{n+1})$ closest to x exists, then let $\widetilde{\nu}_{n+1}(g_s) = 0$, $s \in \{+, -\}$, where $g_s \in B_{x,g}$ in the augmented group. If no such

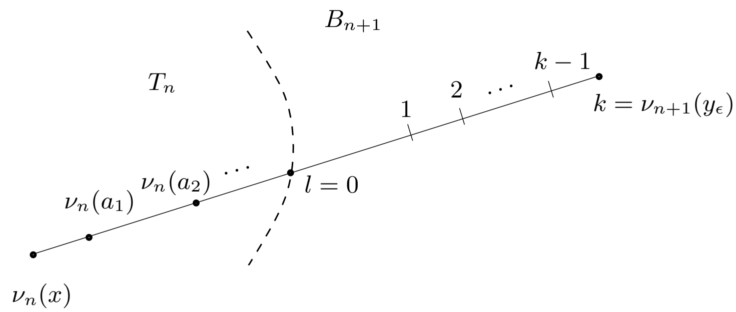

FIGURE 5. T_{n+1} after the gluing of B_{n+1}, Case 2.

g exists, 0 remains unlabeled. For every other h in that equivalence class, points in $(0,1]$ are labeled by h_-, h, h_+ as usual. Finally, define the gluing relation R_{n+1} by identifying the point $0 \in I_{n+1}$ to the point $l \in T_n$. That is, we define T_{n+1} by

$$T_{n+1} = \frac{(T_n \cup I_{n+1})}{l = 0},$$

and we define $\nu_{n+1} : G^+_{n+1} \to T_{n+1}$ to agree with ν_n when restricted to G^+_n and $\tilde{\nu}_{n+1}$ restricted to $G^+_{n+1} - G^+_n$, as in Figure 5.

One shows inductively that T_n has the following properties:

Lemma 12. *In the construction above, for each n we have*:

(1) *T_n is a tree.*

(2) *The connected components of the complement of the closure of $\nu_n(G^+_n)$ are precisely the intervals of the form $(\nu_n(g), \nu_n(g_-))$ or $(\nu_n(g), \nu_n(g_+))$, for some $g \in G_n$.*

(3) *For all $a, b \in G_n$, the unique interval $[\nu_n(a), \nu_n(b)]$ in T_n contains points, labeled in the natural order by the elements of the set $B_{a,b} \subset G^+_n$. Furthermore, no other elements of G can label these points, and if an element $c_t \in (G^+_n - G_n) \cap B_{a,b}$ labels a point along $[\nu_n(a), \nu_n(b)]$ both of the following conditions are satisfied*:

(a) *Either* $c_t R d_s$ *for* $d \in B_{a,c} \cap B_{a,b} \subset G_n^+$, *or* $\nu_n(c_t)$ *is a limit point of points labeled by elements of* $B_{a,c} \cap B_{a,b}$.

(b) *Either* $c_t R f_s$ *for* $f \in B_{b,c} \cap B_{a,b} \subset G_n^+$, *or* $\nu_n(c_t)$ *is a limit point of points labeled by elements of* $B_{b,c} \cap B_{a,b}$.

(4) *For all* $x, y \in G_n^+$, $\nu_n(x) = \nu_n(y)$ *if and only if* xRy.

(5) *If a point* $x \in T_n$ *is not an element of* $\nu_n(G_n^+)$, *then either* $x \in (\nu_n(h), \nu_n(h_s))$ *for some* $h \in G_n$ *and* $s \in \{+, -\}$, *or else for all* $z \in T_n$ *with* $z \neq x$, *the point* x *is a limit point of points in* $[x, z) \cap \nu_n(G_n^+)$.

Figure 6 is an illustration of one of the possibilities described in condition (3). In this case, $c_t = c_+$, and for both parts (a) and (b) of condition (3), the condition not involving limit points holds, with $d_s = d_+$ in part (a) and $f_s = f_-$ in part (b).

Note that the first property shows that the space T is simply connected. The fourth will ensure that the labeling by G^+, though not injective, induces an injective map from G^+/R to T, so that in particular distinct group elements label distinct points of T. The second and third properties will ensure that the natural action of G on the set of labeled points extends to an orientation preserving action on the space T.

We now prove the following proposition, which proves all assertions of Theorem 4 except for the last two sentences, and directly establishes that implication (1) implies (2) of Main theorem 2:

Proposition 3. *The space* T *constructed above has the following properties*:

(1) T *has the structure of an oriented order tree.*

(2) *The group* G *acts faithfully on* T *by orientation preserving homeomorphisms.*

(3) *The action is minimal.*

(4) G *does not fix a unique end of* T.

Proof. We prove the four properties of T listed above:

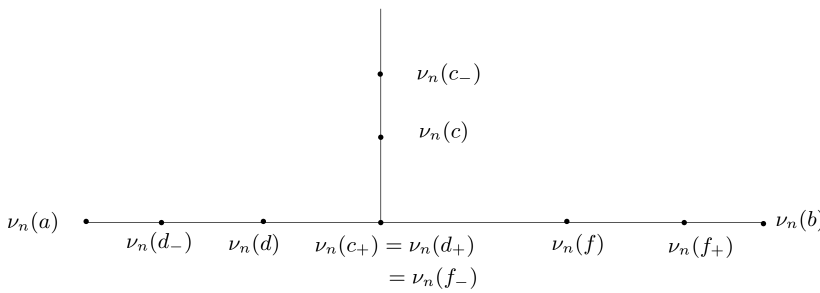

FIGURE 6. One possibility for multiple labelings in Property 4.

Property (1). To see that T has the structure of an oriented order tree, consider the set of the images of the closed segments $[j, j+1] \subset I_i$ for all integers $i, j \geq 0$. Recall that in Case 1 of the construction, the interval $[0, k]$ was shortened by removing an open subinterval and leaving $I_i = [\widetilde{\nu}_i(x_-), k]$. Therefore, the initial subinterval of I_i that we wish to consider in Case 1 is $[\widetilde{\nu}_i(x_-), 1]$, rather than $[0, 1]$. For the rest of the proof, we will abuse notation by referring to the left endpoint of any such interval, $[\widetilde{\nu}_i(x_-), 1]$, by 0. Denote by σ the image in T of one of the above closed intervals $[j, j+1]$. By construction, each of these has at least one interior point labeled by a group element g. If g_- labels a point in $[j, \nu(g))$, let $i(\sigma)$ be the endpoint j, and let $f(\sigma)$ be $j+1$. On the other hand, if g_- labels a point in the image of $(\nu(g), j+1]$, let $i(\sigma)$ be $j+1$ and let $f(\sigma)$ be j. Note that this definition is independent of the choice of g, since the group elements which are labels in $[j, j+1]$ are totally ordered. Let S^+ be the set of all such σ, together with their closed subsegments and unions of any pairs $\{\sigma_1, \sigma_2\}$ where $\sigma_1 \cap \sigma_2 = f(\sigma_1) = i(\sigma_2)$ (with the obvious assignments of initial and final endpoints). This set of positively ordered segments gives T the structure of an oriented order tree.

Property (2). To establish the action we will use the labeling $\nu : G^+ \to T$. Note that, for any g and h in G, if $\nu(g) = \nu(h)$ then $g = h$, and that the set $\nu(G^+)$ spans T. We term a *branch point* any point in the tree which was the image of an integer point or an endpoint of I_n for some n. Branch points, by construction, are always either labeled by elements of $G^+ - G$ or are limit points of labeled points. Note that this set of branch points includes all points where the tree genuinely branches, in addition to any point where the orientation changes. Hence, if a connected subset of T contains no branch points, it is homeomorphic by order preserving homeomorphism to an interval of the real line.

To see that the group acts on T, let L be the subset of T which is labeled by elements of G^+. Consider the complement of the closure of L. Since the complement of the closure of L contains no branch points, any connected component of this complement is an open interval of the form (p, q). We claim that (p, q) has the form $(\nu(f), \nu(f_s))$ for some f in G and $s \in \{+, -\}$. For suppose that $x \in (p, q)$. Then $x \in (\nu_n(f), \nu_n(f_s))$ for some $f \in G_n$ and $s \in \{+, -\}$. But note that, in T_{n+i}, points in the segment $(\nu_{n+i}(f), \nu_{n+i}(f_s))$ can never be labeled, for by Property 3 of the construction, they could only be labeled by elements in $G^+ - G_n^+$ if they were limit points of labeled points, or if they were already labeled. So since $(\nu(f), \nu(f_s))$ is contained in the complement of L, and both $\nu(f)$ and $\nu(f_s)$ are in L, it follows that $(\nu(f), \nu(f_s))$ is a connected component of the closure of L which happens to contain x. Therefore, the intervals (p, q) and $(\nu(f), \nu(f_s))$ are the same. Hence, since $\nu(f)$ and $\nu(f_s)$ belong to L, we have $(p, q) = (\nu(f), \nu(f_s))$.

Now since G acts on G^+ respecting the relation R, G clearly acts on L, and we would like to extend this action continuously to the closure of L. Let x be a point in this closure but not in L. We may have many segments of the form $[a, x]$, any two of which intersect only at x, each containing an increasing (in the total order on $[a, x]$ with x being greatest) sequence of labeled points converging to x. Suppose $[a, x]$ and $[b, x]$ are any two such segments, and let $\{\nu(a_i)\}$ and $\{\nu(b_i)\}$ be the two sequences of points. Note that for some n, $[a, x] \cup [x, b]$ is contained in T_n. Now let g be an arbitrary group element, and we worry about ambiguity in defining gx. The worry is that after transforming by the group element g, the two sets of points $\{\nu(ga_i)\}$ and $\{\nu(gb_i)\}$ along $[\nu(ga), \nu(gb)]$ will have different limit points, say x_a and x_b.

We claim there can be no labeled points in the open segment (x_a, x_b). If there were, then that label (say h) would lie in the subset B_{ga_i, gb_j} of G^+ for every i and j. But applying g^{-1} we see that $g^{-1}h \in B_{a_i, b_j}$ for every i and j. Therefore, $g^{-1}h$ must label a point along $[a, b]$ between the points $\nu(a_i)$ and $\nu(b_j)$ for every i and j. However, x

is the only such point, and it is unlabeled by hypothesis. So x_a and x_b must be endpoints of some connected component of the complement of the closure of L; hence, they are actually labeled points. Since they are different points, the labels are distinct, and not even equivalent modulo R. But then, applying g^{-1} again, we produce distinct labels not equivalent modulo R which are between a_i and b_j for any i and j; hence, they label distinct points in $[\nu(a_i), \nu(b_j)]$ for every i and j. But this is nonsense, as there is only one point between $[\nu(a_i), \nu(b_j)]$ for every i and j. So the group action may be extended continuously to the closure of L. Since the complement of the closure is a disjoint union of open intervals, extend the action to all of T by extending linearly across these intervals. The action clearly preserves the orientation since $g(h_-) = (gh)_-$ and $g(h_+) = (gh)_+$ for any elements g and h of G. Since the labeling ν restricted to G is injective, the action is faithful.

Property (3). The fact that the action is minimal follows from the nontriviality of the extension of the partial order. First, suppose to the contrary that T' is a proper invariant subtree of T. We claim that T' can contain no points in $\nu(G^+)$. There are three important G-invariant subsets of $\nu(G^+)$, namely,

$$S = \{\nu(g) \mid g \in G\},$$
$$S_+ = \{\nu(g^+) \mid g \in G\},$$

and

$$S_- = \{\nu(g^-) \mid g \in G\}.$$

If T' contains all three subsets, then T' contains the span of $\nu(G^+)$, which is all of T. So since T' is a proper subtree, it is missing at least one subset. If T' contains neither S_+ nor S_-, then it cannot contain S either, since $[\nu(g), \nu(h)]$ always contains some element of either S_- or S_+. But if T' does not contain S_+ and does contain S_-, then it follows that $g \sim_l h$ for all g and h in G. Similarly, if T' does not contain S_- but does contain S_+, then $g \sim_u h$ for all g, h in G, both of which contradict the nontriviality of the partial order. We claim that, additionally, T' cannot contain any points in the complement of the closure of L either, for if $x \in T'$ is such a point, then $x \in (\nu(h_s), \nu(h))$

for some $h \in G$, $s \in \{+,-\}$, which implies that $gx \in (\nu((gh)_s), \nu(gh))$ for every element g in G. Now $[x, gx]$ is contained in T', and since $(\nu(h_s), \nu(h))$ and $(\nu((gh)_s), \nu(gh))$ contain no branch points, if $x \neq gx$, at least one of the points $\nu(h_s)$, $\nu(h)$, $\nu((gh)_s)$, $\nu(gh)$ must be in T'. This is impossible as we just showed that T' contains no labeled points. Therefore, it must be that $x = gx$, which implies that $gh = h$, so $g = e$. This is impossible, because the argument holds for every element in G.

Therefore, T' can contain only limit points of the set of labeled points which are themselves unlabeled. If x is such a point, we claim that $gx = x$ for every $g \in G$. For, if not, consider $[x, gx] \in T'$. Then by Property 5 of Lemma 12, $[x, gx)$ must contain labeled points, another contradiction. So T' must be just the single point x. We claim that this forces the extension of the partial order to be trivial. For, choose any group element $g \in G$, and consider the segment $[\nu(g), x]$. If $[\nu(g), x]$ is not an oriented segment of T, because there are at most finitely many switches in orientation along it, we may re-choose g so that $[\nu(g), x]$ is oriented. But, given any other $h \in G$, since $g(h^{-1})$ fixes x and preserves orientation, $[\nu(h), x]$ must also be an oriented segment, and $[\nu(h), x]$ and $[\nu(g), x]$ are either both oriented towards or away from x. Hence, all segments of the form $[\nu(h), x]$ must be either oriented towards x or oriented away. But then, given elements f and h of G, since the interval $[\nu(f), \nu(h)]$ is contained in $[\nu(f), x] \cup [\nu(h), x]$, and since labels along these segments correspond to between-sets in G, either f and h are comparable, or $f \sim_u h$ (in the case $x > \nu(g)$ for every $g \in G$) or $f \sim_l h$ (in the case that $x < \nu(g)$ for all $g \in G$). Note that, since x is a limit point of labeled points, by construction there must be two elements g and h in G such that $x \in [\nu(g), \nu(h)]$, so that $g \sim h$, and hence the extension is trivial.

Property (4). Next we must show that there is no unique fixed end. Suppose to the contrary that the action has a unique fixed end e. For an arbitrary element g, choose a ray ρ from $\nu(g)$ representing e. If $\nu(g_-) \in \rho$, we say that $\nu(g)$ points towards the end e represented by ρ, and if $\nu(g_+) \in \rho$ we say $\nu(g)$ points away from e. Since the group acts transitively on itself but fixes the end e, for any group element h, $\nu(h)$ will point the same way with respect to e as $\nu(g)$. If both $\nu(g)$ and $\nu(h)$ point towards e, either g and h are comparable, or $g \sim_l h$; and if both $\nu(g)$ and $\nu(h)$ point away from e, they are either comparable, or $g \sim_u h$. So all noncomparable pairs must be of the same type. Notice that there is at least one such pair, for if not, G is totally ordered, and then $T = \mathbf{R}$, and e is not the unique fixed end. So the extension is in fact trivial if there is a unique fixed end. □

Remark. The proof of Proposition 3 established all of the conclusions of Theorem 4 except for the last two claims, so we verify them here.

First we check that there is a point with trivial stabilizer, and that there is a natural order induced on that orbit which recovers the order on G. Let $x = \nu(e)$, and denote the orbit of x in T by $\mathcal{O}$. Since $g\nu(h) = \nu(gh)$, we have $\mathcal{O} = \{\nu(g) \mid g \in G\}$, and since $\nu(h) = \nu(g)$ if and only if $g = h$, the stabilizer of any element of $\mathcal{O}$ is trivial. As described in Proposition 1, one can blow T up to an oriented non Hausdorff 1-manifold T' on which the group acts. Since branch points of T were never labeled by group elements, there is a labeling of T' by G, also denoted by ν, that retains the properties that each group element labels a distinct point and that each point of $\mathcal{O}$ has a trivial stabilizer. The orientation on T' induces an extension of a simply connected partial order on $\mathcal{O}$, as described at the beginning of Section 3, and this naturally induces an order on G by identifying $\nu(g)$ with g. Since the original order on G was characterized in Lemma 9 in terms of between-sets in G^+, and since property (3) of Lemma 12 insures that the labels on shortest paths in T' between elements of $\nu(G^+)$ are essentially the same as the corresponding between-sets in G^+, the induced order on G is the order with which we began.

Finally, we check that in the case that the partial order is simply connected to begin with, the action on T' is minimal. If there is a branch point in T which is a sink or a source, then there is a pair of noncomparable elements in G which do not have a common lower or upper bound in G. Therefore, if the original order was simply connected, there will be no branch points in T which are sinks or sources. Thus, no rays need to be added in the process of blowing up to a 1-manifold. This implies that the resulting action is still minimal. This establishes that implication (1) implies (2) of Main theorem 1 as well.

5. Groups acting on simply connected 1-manifolds have simply connected partial orders. The goal of this section is to prove the converse to Theorem 4, thus proving that implication (2) implies (1) in Main theorem 1, stated in the introduction, for groups acting on simply connected 1-manifolds. Namely, we prove:

Theorem 5. *If a countable group G acts minimally, and without fixing a unique end, on an oriented, simply connected 1-manifold T by orientation preserving homeomorphisms and there is some point in the manifold with trivial stabilizer, then G admits a nontrivial left-invariant rectifiable simply connected partial order.*

Definition 28. For an oriented non Hausdorff simply connected 1-manifold T, we say that *x points at y* if $y \in x^-$, and *x points away from y* if $y \in x^+$.

Throughout the rest of this section, whenever we have a group G acting on an order tree, T, we will denote the orbit under G of the element of $x \in T$ by $\mathcal{O}(x)$.

Lemma 13. *Let T be an oriented non Hausdorff simply connected 1-manifold, and suppose that G acts minimally on T. If the elements x and y in $\mathcal{O}(v)$ are not comparable and share a lower bound in T, then x and y share a lower bound in $\mathcal{O}(v)$, and if x and y share an upper bound in T, then x and y share an upper bound in $\mathcal{O}(v)$.*

Proof. We consider only the case of lower bounds; the case for upper bounds is similar. First we claim:

Claim. *Suppose that points x and y in $\mathcal{O}(v)$ are not comparable and share a lower bound in T, but share no lower bound in $\mathcal{O}(v)$. Then there is a point a of $T - \mathcal{O}(v)$ such that every point of $\mathcal{O}(v)$ points at a.*

If the claim holds, then consider the element $a \in T$. Since every element of $\mathcal{O}(v)$ points at a, every element of $\mathcal{O}(v)$ points at every element of $\mathcal{O}(a)$. Let A be the set

$$A = \bigcup_{\alpha,\beta \in \mathcal{O}(a)} GS_{(\alpha,\beta)}.$$

Now, A is a G-invariant implicit subtree, but no element of $\mathcal{O}(v)$ can be in A, since it would have to point at both ends of some geodesic spine $GS_{(\alpha,\beta)}$. This contradicts the fact that the action is minimal, and therefore it must be the case that x and y in fact do share a lower bound in $\mathcal{O}(v)$.

To prove the claim, suppose that x and y are noncomparable and share a lower bound in T but share no lower bound in $\mathcal{O}(v)$. Recall that the geodesic spine $GS_{(x,y)}$ contains finitely many cusp pairs. If an interior segment of $GS_{(x,y)}$ contains a point z of $\mathcal{O}(v)$ that is noncomparable either to x or to y, then z shares a lower bound in T with this element, and is less than the other one. Suppose that z shares a lower bound with y and is less than x. Any lower bound of y and z is also a lower bound of x and z. Therefore, by possibly replacing x and y with other elements of $\mathcal{O}(v)$ on the segment of $GS_{(x,y)}$, we may assume that no interior segment of $GS_{(x,y)}$ contains points of $\mathcal{O}(v)$.

Let t be the first point in a cusp pair along $GS_{(x,y)}$ on the way from x to y. Choose an open neighborhood N' of t homeomorphic to $\mathbf{R}$. Let N be the open segment of N' not contained in $GS_{(x,y)}$. Then N is directed away from both x and y, so N can contain no points of $\mathcal{O}(v)$. Finally, since no interior segment of $GS_{(x,y)}$ intersects $\mathcal{O}(v)$, every point of $GS_{(x,y)} \cap \mathcal{O}(v)$ points at every point of N.

Let $a \in N$. We will show that everything in $\mathcal{O}(v)$ points at a. Since we already have everything in $\mathcal{O}(v) \cap GS_{(x,y)}$ pointing at N, let $z \in \mathcal{O}(v) - GS_{(x,y)}$. Since z does not lie in N or $GS_{(x,y)}$, the three elements x, y and a all lie in the same component of $T - \{z\}$. There are two cases to consider.

Case 1. $z \in x^- \cap y^-$. This case is illustrated in Figure 7. Since x and y share no lower bound in $\mathcal{O}(v)$, $x, y \in z^-$, for otherwise we would have $x > z$ and $y > z$. Therefore, a must also lie in z^-, which means that z points at a.

Case 2. $z \notin x^- \cap y^-$. This case is illustrated in Figure 8. We consider the case that $z \in x^+$; the case $z \in y^+$ is handled similarly. We wish to show that z points at x and hence at a. Choose $g \in G$ with $w := gx$ in the component of $T - a$ that is contained in $x^- \cap y^-$. By Case 1, $a \in w^-$. And, since $z \in x^+$, we have $gz \in w^+$. Therefore, a and gz lie in opposite components of $T - w$, so a and w lie in the same component of $T - (gz)$.

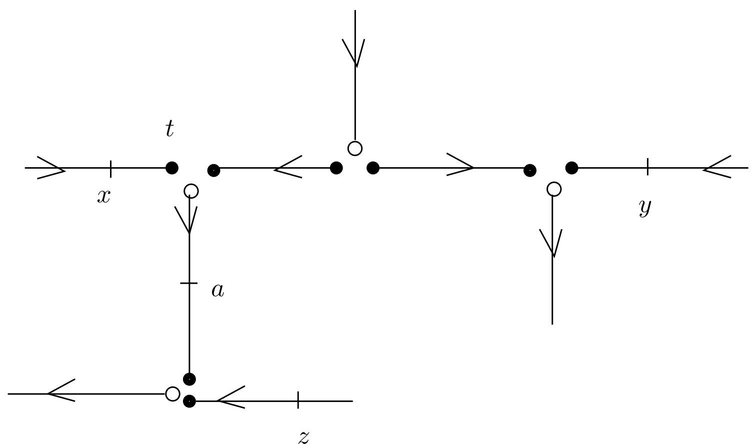

FIGURE 7. Case 1, $z \in x^- \cap y^-$.

Now, $w \in x^- \cap y^-$ so $w^+ \subset x^- \cap y^-$, for otherwise we would have $w^- \subset x^- \cap y^-$ forcing w to be a lower bound for x and y. Therefore,

$$gz \in w^+ \subset x^- \cap y^-,$$

and by Case 1, $a \in (gz)^-$. Since a and w lie in the same component of $T - (gz)$, we have $w \in (gz)^-$. In other words, $gx \in (gz)^-$ so that $x \in z^-$. Since $z \in x^+$ (by the assumption for Case 2), we have $x^- \subset z^-$. Since $a \in x^-$, we have $a \in z^-$ showing that z points at a. Therefore, every element of $\mathcal{O}(v)$ points at a, and the claim is proved. □

We now prove Theorem 5.

Proof of Theorem 5. Since T is an oriented simply connected 1-manifold, T is a simply connected poset. Choose $x \in T$ such that $\mathrm{Stab}\,(x)$ is trivial. By Lemma 13 the subposet $\mathcal{O}(x)$ is strongly connected. Since $\mathcal{O}(x)$ is a subposet of the acyclic poset T, $\mathcal{O}(x)$ is itself acyclic. Therefore, the partial order of $\mathcal{O}(x)$ is simply connected. Also, since the poset T is rectifiable, so is $\mathcal{O}(x)$. Therefore, the left-invariant partial order given to G by identifying it with $\mathcal{O}(x)$ is rectifiable and

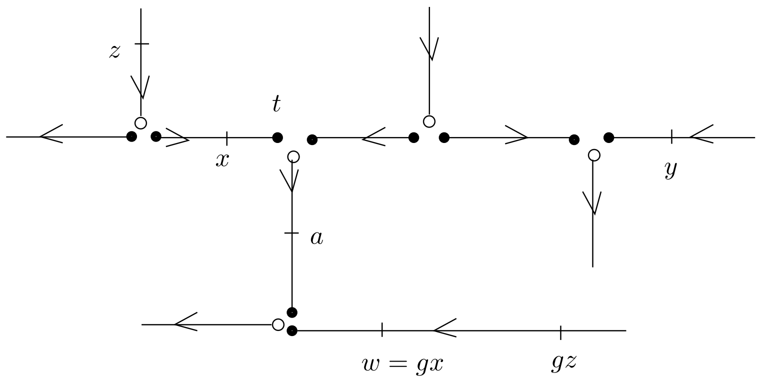

FIGURE 8. Case 2, $z \notin x^- \cap y^-$.

simply connected. The rest of the proof is devoted to proving that the simply connected partial order of $\mathcal{O}(x)$ (and therefore of G) is nontrivial.

We will show that the assumption that the order is trivial leads to a contradiction. This is done by showing that, under this assumption, T is homeomorphic to $\mathbf{R}$. Since G acts faithfully on T, this shows that the order on $\mathcal{O}(x)$ is total, and hence nontrivial.

Assume towards a contradiction that the order on $\mathcal{O}(x)$ is trivial. Since a total order is not trivial, $\mathcal{O}(x)$ must contain noncomparable pairs of elements. Moreover, all noncomparable pairs must be of the same type; either all such pairs satisfy $\sim_l$ or all satisfy $\sim_u$. We consider only the case in which each two noncomparable elements $x, y \in \mathcal{O}(x)$ satisfy $x \sim_l y$, because the case for $x \sim_u y$ is similar.

Our first step towards proving that T is a line is to show that there exists a pair of comparable elements in $\mathcal{O}(x)$. Suppose not. Then, $y \sim_l z$ for any $y, z \in \mathcal{O}(x)$. Therefore, for any elements y and z in $\mathcal{O}(x)$, we have $\mathcal{O}(x) \cap B_{y,z} = \{y, z\}$. But $GS_{(y,z)} \cap \mathcal{O}(x) = B_{y,z}$, so no element of $\mathcal{O}(x)$ ever separates two other elements of $\mathcal{O}(x)$. Therefore, the set

$$I := \Bigg(\bigcup_{y,z \in \mathcal{O}(x)} GS_{(y,z)} \Bigg) - \mathcal{O}(x)$$

is a G-invariant implicit subtree of T. Since the G acts minimally on T, $I = \varnothing$, which implies that $GS_{(y,z)} = \{y, z\}$ for any $y, z \in \mathcal{O}(x)$. Therefore, no two points of $\mathcal{O}(x)$ are separable, so that $\mathcal{O}(x)$ itself is a proper G-invariant implicit subtree of T, again contradicting the minimality of the action. Thus, there must be at least two comparable elements of $\mathcal{O}(x)$.

Now, for $y \in \mathcal{O}(x)$, we consider the set

$$L_y := \{w \in \mathcal{O}(x) \mid w \leq y\},$$

which by the above paragraph and the transitivity of the action of G on $\mathcal{O}(x)$ must contain at least one element other than y. Again, by transitivity, L_y cannot have a minimal element. Therefore, L_y must be infinite. Additionally, each pair of elements in $L_y - \{y\}$ has y as an upper bound, so by the assumption that no pair of noncomparable elements in $\mathcal{O}(x)$ share an upper bound, L_y must be totally ordered. Therefore, the set

$$\rho_y := \bigcup_{w \in L_y} GS_{(y,w)} \tag{5.1}$$

can be written as an infinite increasing union of geodesic spines, so it is a ray.

We claim that, for any elements s and t in $\mathcal{O}(x)$, the intersection $\rho_s \cap \rho_t$ contains ρ_w for some $w \in \mathcal{O}(x)$. To prove this, there are three cases to consider: first $s < t$, second $t < s$, and third $s \sim t$. In the first case, $\rho_s \subset \rho_s \cap \rho_t$ and in the second case, $\rho_t \subset \rho_s \cap \rho_t$. In the third case, $s \sim_l t$ since we assumed that $x \sim_l y$ for every noncomparable pair $x \sim y$ in $\mathcal{O}(x)$. Since $\mathcal{O}(x)$ is strongly connected, s and t share a lower bound in $\mathcal{O}(x)$, say w. In this case, $\rho_w \subset \rho_s \cap \rho_t$, proving the claim.

Let R be the set of rays of the form given in equation (5.1). We now prove that every ray in R is infinite. Note that, for any $g \in G$, we have $gL_y = L_{g \cdot y}$, so $g\rho_y = \rho_{g \cdot y}$, and G transitively permutes the set R. Therefore, either all rays in R are infinite or all are finite. Suppose that they are all finite. Since every two rays in R eventually overlap, every two rays in R have exactly the same set of endpoints. Let E be the set of endpoints of any (hence every) ray in R. Then G permutes the components of $T - E$. Only one component contains points of $\mathcal{O}(x)$, and G must fix that component, which is therefore an invariant implicit subtree. Since G acts minimally on T, this is impossible, so the rays of R must be infinite.

Since all rays in R are infinite and any two eventually overlap, they all define the same end ε, which is fixed by G. Since we assumed G not to fix a unique end of T, G must fix another end δ of T. By Lemma 3, G fixes the implicit line defined by ε and δ which, by the minimality of the action, must be the entire manifold. Therefore, $T \approx \mathbf{R}$, and G acts faithfully on $\mathbf{R}$. Thus, the order on $\mathcal{O}(x)$ is a total order, and we have reached our desired contradiction. □

6. Left-orderable stabilizers. The following proposition shows that we cannot completely drop the condition in Theorem 5 that some point have a trivial stabilizer.

Proposition 4. *There exists a countable group which acts faithfully and minimally, without fixing a unique end, on an oriented order tree, yet does not admit a partial order with left-invariant nontrivial rectifiable simply connected extension.*

Proof. Let $G_1 = \langle g_1 \rangle$ and $G_2 = \langle g_2 \rangle$ be nonisomorphic finite cyclic groups each containing a proper subgroup isomorphic to the group H. Form the free product of G_1 and G_2 amalgamated along H; $G = G_1 *_H G_2$. Standard Bass-Serre theory yields a simplicial tree T with a G action. The quotient of T by G is a graph with two vertices, v_1 and v_2, and one edge, e. If we orient e arbitrarily and lift the orientation to a G-equivariant orientation of T, then T has the structure of an oriented order tree and the action of G is by orientation-preserving homeomorphisms. Moreover, the action is faithful, nontrivial and fixes no unique end. However, G cannot admit a partial order with nontrivial rectifiable simply connected extension, which can be seen as follows.

Suppose that G does admit a partial order with nontrivial rectifiable simply connected extension. Theorem 4 then applies to G, so we may let T' be the tree constructed in that theorem. Recall from [**14**] that, if an element $g \in G$ has no fixed point in T', then g acts as translation along some axis, so no nonzero power of g has a fixed point. Since g_1 and g_2 are torsion elements in G, we may choose fixed points x and y

of g_1 and g_2, respectively. The points x and y are fixed by all of G_1 and G_2, respectively, because g_i generates G_i. Since H is contained in the intersection, $G_1 \cap G_2$, it must stabilize the geodesic spine $GS_{(x,y)}$. We will show that the action on the interior of this geodesic spine is faithful. Since the interior of $GS_{(x,y)}$ is homeomorphic to $\mathbf{R}$, this will show that H has a faithful action on $\mathbf{R}$. Such an action is impossible since it would induce on H a total order, as in [**11**, Theorem 6.8], but a finite group cannot admit a total order. Therefore, G cannot admit a partial order with nontrivial rectifiable simply connected extension.

To show that the action of H on the interior of $GS_{(x,y)}$ is faithful, it will suffice to show that the interior of $GS_{(x,y)}$ contains a point labeled by an element of G, since the stabilizer of such a point is trivial. Note first that since both endpoints have nontrivial stabilizers, neither x nor y can be labeled by an element of G, so if one is labeled, the label must belong to $G^+ - G$. On the other hand let,

$$A := \bigcup_{g \in G} gGS_{(x,y)}.$$

Now, any g has a decomposition $g = h_1k_1h_2k_2 \cdots h_mk_m$ with $h_i \in G_1$ and $k_i \in G_2$, which produces a path

$$h_1GS_{(x,y)},\ h_1k_1GS_{(x,y)},\ h_1k_1h_2GS_{(x,y)}, \dots ,\ h_1k_1 \cdots h_mk_mGS_{(x,y)}$$

from $GS_{(x,y)}$ to $gGS_{(x,y)}$. Therefore, A is a connected G-invariant subset of T', so it is an invariant subtree. Since the action is minimal, A must be all of T'. But, T' contains points labeled by elements in G, and this set of points is invariant under the action of G. Since neither x nor y is labeled by an element of G, none of their translates are. Hence, G-labeled points must lie in the translates of the interior of $GS_{(x,y)}$, and so a G-labeled point must lie in the interior of $GS_{(x,y)}$ itself. □

Note that, in the example above, the groups G_1 and G_2 stabilize the points $\widetilde{v}_1$ and $\widetilde{v}_2$, which are lifts of the vertices v_1 and v_2 to T. In fact, every point of T has a nontrivial stabilizer. This illustrates the fact that a nonleft orderable stabilizer constitutes a major obstruction to defining an extended simply connected partial order on G by using its action on T.

Although we are unable to entirely omit restrictions on the stabilizer of the point x we choose, we can replace it by the condition that there exists a point in T with left-orderable stabilizer, proving that (3) implies (1) in Main theorem 1, completing the proof of the classification theorem for groups acting on simply connected 1-manifolds.

Theorem 6. *If a countable group G acts minimally and without fixing a unique end, on an oriented, simply connected 1-manifold T by orientation preserving homeomorphisms, and there is some point $x \in T$ where* $\mathrm{Stab}\,(x)$ *is left-orderable, then G admits a nontrivial left-invariant rectifiable simply connected partial order.*

Proof. By the same reasoning in Theorem 5, we see that the left cosets of $\mathrm{Stab}\,(x)$ admit a rectifiable simply connected partial order. This extends to an order on the group G as follows. Choose $g_1 \neq g_2 \in G$. If $g_1^{-1}g_2 \notin H$, then $g_1H \neq g_2H$, and we assign $g_1 < g_2$ if $g_1H < g_2H$ in the partial order on the cosets, and similarly $g_2 < g_1$ if $g_2H < g_1H$. Note that, if such a pair has not been assigned to be comparable, then the cosets are also not comparable, so they have either an upper or a lower bound, which in turn provides g_1 and g_2 with common upper or lower bounds. On the other hand, if $g_1^{-1}g_2 \in H$, then either $e < g_1^{-1}g_2$ or $e > g_1^{-1}g_2$. In the first case we set $g_1 < g_2$ and in the second case we set $g_2 < g_1$. It is easy to see that the result is a left invariant rectifiable simply connected partial order. It cannot be trivial, since already at the level of cosets the order was not trivial. $\square$

7. Oriented order trees. We have seen that groups that act minimally on oriented simply connected 1-manifolds have simply connected partial orders. The goal of this section is to show that although groups acting on more general order trees do not always admit such partial orders, they do always admit partial orders with simply connected rectifiable extensions. The group theoretic motivation for this distinction can be seen in Example 1 in Section 3. This example shows an action of the infinite dihedral group on an order tree that gives rise to an acyclic (but not strongly connected) partial order with rectifiable simply connected extension. Moreover, this action induces an action on a simply connected 1-manifold which we have shown is not minimal, and has no minimal invariant submanifold, corresponding to the fact that the infinite dihedral group cannot admit a full simply connected partial order.

In the case of 3-manifold groups, this distinction has a natural topological interpretation. Namely, if a 3-manifold contains a minimal, codimension-1 foliation, then the space of leaves is a simply connected 1-manifold T, and the induced action of $\pi_1(M)$ on T is a minimal action [**15**, 8.3]. However, if the foliation is not minimal, then the action will not necessarily be minimal.

As we did for groups acting on 1-manifolds, we first consider an action with a trivial stabilizer in Theorem 7, proving that (2) implies (1) in Main theorem 2 and then extend to actions with a left orderable stabilizer in Theorem 8, proving that (3) implies (1) in Main theorem 2.

Theorem 7. *If a countable group G acts minimally and without fixing a unique end on an oriented order tree T by orientation preserving homeomorphisms, and there is some point with trivial stabilizer, then G admits a partial order with nontrivial left-invariant rectifiable simply connected extension.*

First, we have two lemmas necessary for the proof of Theorem 7. Recall the map, $\varphi : T' \to T$ mentioned after the proof of Proposition 1 and the notions of incoming and outgoing rays at a branch point of an order tree.

Lemma 14. *Let T be an oriented order tree, let T' be the associated 1-manifold, and let $\widehat{T}$ be the core of T'. If $x \in \widehat{T}$ and X_1 is a component of $T' - \{x\}$, then $\varphi(X_1) \neq T$.*

Proof. Since T' is a simply connected 1-manifold, the point x disconnects T' into two components X_1 and X_2. Let $y = \varphi(x)$. We first claim that $X_2 \not\subset \varphi^{-1}(y)$. There are two possibilities for y. Either y is a branch point or a regular point of T. If y is regular, then $\varphi^{-1}(y)$ is the single point x, so $X_2 \not\subset \varphi^{-1}(y)$. If y is a branch point, then the preimage of y depends on the in-degree $n_0(y)$ and the out-degree $n_f(y)$. We consider the case $n_0(y) = 0$ and $n_f(y) \geq 2$, that is, y is a sink. The other cases are similar. As in [**14**, Section 5], $n_f(y) = |R(y, f)| \geq 2$, where $R(y, f)$ is the set of incoming rays at y, where such a ray is an equivalence class of segments σ with $f(\sigma) = y$ and where $\sigma_1 \approx \sigma_2$ if and only if $\{y\} \subsetneq \sigma_1 \cap \sigma_2$. Now, $\varphi^{-1}(y)$ consists of an entire open ray

σ_y (the distinguished ray) and a set of points $\{x_{r_\sigma}\}$, where r_σ ranges over all of the rays in $R(y, f)$. Since $n_f(y) \geq 2$, there are at least two of these points, one $x = x_{r_\sigma}$ and one $x_{r_\tau} \neq x$, where σ and τ are segments representing r_σ and r_τ, respectively. Then $\sigma - \{x\}$ and τ lie in different connected components of $T' - \{x\}$, so X_2 contains either $\sigma - \{x\}$ or τ. But both $\sigma - \{x\}$ and τ contain points not in $\varphi^{-1}(y)$, so in the case that $n_0(y) = 0$ and $n_f(y) \geq 2$, we have $X_2 \not\subset \varphi^{-1}(y)$. The other possibilities for $n_0(y)$ and $n_f(y)$ are similar.

Now suppose towards a contradiction that $\varphi(X_1) = T$, and let $\alpha \in X_2$. Then there exists $\beta \in X_1$ such that $\varphi(\beta) = \varphi(\alpha)$. Since $\varphi^{-1}(\varphi(\alpha))$ is an implicit subtree of T' and, since α and β belong to $\varphi^{-1}(\varphi(\alpha))$, we have $GS_{(\alpha,\beta)} \subset \varphi^{-1}(\varphi(\alpha))$. Since $x \in GS_{(\alpha,\beta)}$, we have $\varphi(\alpha) = \varphi(x) = y$. This is true for any point of X_2, so $X_2 \subset \varphi^{-1}(y)$, contradicting the previous claim. □

Lemma 15. *Suppose that G acts minimally on the oriented order tree T, and let T' be the associated 1-manifold with the G-action. Every nonempty invariant implicit subtree of T' contains the core $\widehat{T}$.*

Proof. If I is an invariant implicit subtree that does not contain $\widehat{T}$, choose $x \in \widehat{T} - I$. Then I is contained in one component X_1 of $T' - \{x\}$. Since $\varphi(X_1) \neq T$, $\varphi(I) \neq T$. But, $\varphi(I)$ is an invariant subtree of T, contradicting minimality of the action of G on T. □

We now are in a position to prove Theorem 7.

Proof of Theorem 7. Choose a point $\alpha \in T$ with trivial stabilizer. Blow T up in the usual way to an oriented 1-manifold T' on which G acts. In the process, some points of T may be split apart, and new rays and open intervals may be added. The points of T' form a simply connected partially ordered set. Since $\varphi(\widehat{T}) = T$, we may choose a point $x \in \widehat{T}$ such that $\varphi(x) = \alpha$ and x has trivial stabilizer. As in Theorem 5, we define a left-invariant partial order on G by identifying it with $\mathcal{O}(x)$. It may be the case that some noncomparable pairs in $\mathcal{O}(x)$ have no common bounds in $\mathcal{O}(x)$, but they can be assigned the type $\sim_u$ or $\sim_l$ according to how they relate in T'. The resulting extension will clearly satisfy both Definitions 12 and 13, so it is a simply connected extension. Since the poset T' is rectifiable, the partial order of $\mathcal{O}(x)$ is as well. The rest of the proof is devoted to showing that this extension is nontrivial.

We follow the proof of Theorem 3, using Lemma 15 instead of minimality to show that if the order is trivial, there must be two comparable elements of $\mathcal{O}(x)$. Again, we then consider the set R of rays of the form of equation (5.1) and note that G acts transitively on R, so either all rays are infinite or all are finite.

The fact that, unlike in Theorem 5, G need not act minimally on T', will complicate the rest of the proof. Suppose that all the rays are finite, and let E denote the set of endpoints of any (hence every) ray in R. Now, E is G-invariant, and if no two points of E are separable from each other, E itself is an implicit subtree of T'. In this case, E certainly cannot contain all of $\widehat{T}$, contradicting Lemma 15. If there are two separable points in E, then as in Theorem 5, the set

$$I := \left(\bigcup_{a,b\in E} GS_{(a,b)} \right) - \mathcal{O}(x)$$

is a G-invariant implicit subtree of T'. Since $\widehat{T}$ contains all of $\mathcal{O}(x)$, the set I does not contain $\widehat{T}$, again contradicting Lemma 15. Therefore, all rays of R must be infinite and, as in Theorem 5, they all define the same end ε', which is fixed by G.

We now use ε' to find an invariant implicit subtree or a fixed end of T. First note that if ω is a geodesic spine in T', then $\varphi(\omega)$ is a single point or a geodesic spine in T. Since φ maps points of $\mathcal{O}(x)$ to distinct points in T, $\varphi(\rho_y)$ is a ray for any $\rho_y \in R$. Since any two rays in R eventually overlap, the same is true of any two rays of the form $\varphi(\rho_y)$. Moreover, G transitively permutes the rays $\varphi(\rho_y)$. So, either all are infinite or all are finite. If they are finite, they all have the same set of endpoints, say E_1, which is a proper G-invariant implicit subtree of T. Therefore, all rays $\varphi(\rho_y)$ are infinite, and they all define the same end ε, which is fixed by G.

Since G was assumed not to fix a unique end of T, there must be another fixed end δ. By Lemma 3, G fixes the implicit line l defined by ε and δ. Since G does act minimally on T, l must be the entire tree, T. Therefore, $T \approx \mathbf{R}$, and we have reached our desired contradiction. $\square$

As in the case of a minimal action on a simply connected 1-manifold, the assumption on the stabilizer of x can be weakened. The proof is essentially the same as for Theorem 6.

Theorem 8. *If a countable group G acts minimally and without fixing a unique end on an oriented order tree T by orientation preserving homeomorphisms, and there is some point $x \in T$ where* $\mathrm{Stab}\,(x)$ *is left-orderable, then G admits a partial order with left-invariant nontrivial rectifiable simply connected extension.*

Remark. We have restricted ourselves to $\mathbf{R}$-order trees, but with some technical improvements Theorems 7 and 8 can be extended to arbitrary order trees. This is achieved by replacing "simply connected oriented 1-manifold" by "oriented order tree without branching" in Theorems 5 and 6 and defining the geodesic spine $GS_{(x,y)}$ to be the intersection of all paths from x to y. The technical improvements involve dealing with the facts that, in an arbitrary order tree, a finite ray need not have any endpoint and a geodesic spine may not decompose into a disjoint union of segments. Indeed, the definition of finite ray would be changed to include any ray that is contained in some finite geodesic spine.

Acknowledgments. We would like to thank Rachel Roberts for the many helpful conversations and useful advice and suggestions she gave throughout the writing of this paper. We would also like to thank Indira Chatterji for sparking our interest in this class of partially ordered groups. And finally, we are extremely grateful to the anonymous referee, whose careful and helpful comments greatly improved the exposition of this paper.

REFERENCES

1. R. Alperin and H. Bass, *Length functions of group actions on Λ-trees*, Annals Math. Stud., Princeton University Press **111** (1987), 265–378.

2. T. Barbot, *Actions de groupes sur les 1-variétés non séparées et feuilletages de codimension un*, Ann. Fac. Sci. Toulouse Math. **7** (1998), 559–597 (in French).

3. B. Bowditch, *Treelike structures arising from continua and convergence groups*, Mem. Amer. Math. Soc. **662** (1999), 1–86.

4. B. Bowditch and J. Crisp, *Archimedean actions on median pretrees*, Math. Proc. Cambridge Philos. Soc. **130** (2001), 383–400.

5. S. Boyer, D. Rolfsen and B. Wiest, *Orderable* 3*-manifold Groups*, Ann. Inst. Fourier (Grenoble) **55** (2005), 243–288.

6. M. Culler and J.W. Morgan, *Group actions on* $\mathbf{R}$*-trees*, Proc. London Math. Soc. **55** (1987), 571–604.

7. P. Dehornoy, *Braid groups and left distributive operations*, Trans. Amer. Math. Soc. **345** (1994), 115–150.

8. D. Gabai, *Eight problems in the geometric theory of foliations and laminations on 3-manifolds*, Studies Adv. Math. **2** (1997), 1–33.

9. D. Gabai and W. Kazez, *Order trees and essential laminations of the plane*, Math. Research Lett. **4** (1997), 603–616.

10. D. Gabai and U. Oertel, *Essential laminations in* 3*-manifolds*, Ann. Math. **130** (1989), 41–73.

11. E. Ghys, *Groups acting on the circle*, Ens. Math. **47** (2001), 329–407.

12. V. Kopytov and N. Medvedev, *Right-ordered groups*, Siberian School of Algebra and Logic, Consultants Bureau, Plenum, 1996.

13. F. Paulin, *Actions de groupes sur les arbres*, Sém. Bourbaki 1995/96, Astérisque **241** (1997), 97–137.

14. R. Roberts, J. Shareshian and M. Stein, *Infinitely many hyperbolic* 3*-manifolds which contain no reebless foliation*, J. Amer. Math. Soc. **16** (2003), 639–679.

15. R. Roberts and M. Stein, *Group actions on order trees*, Topology Appl. **115** (2001), 175–201.

16. P. Shalen, *Dendrology of groups*: *An introduction*, in *Essays in group theory*, S.M. Gersten, ed., M.S.R.I. Pub. **8**, Springer-Verlag, 1987.

17. H. Short and B. Wiest, *Orderings of mapping class groups after Thurston*, Enseign. Math. **46** (2000), 279–312.

Department of Mathematics, Statistics and Computer Science, UW–Stout, Menomonie, WI 54751
Email address: horakm@uwstout.edu

Department of Mathematics, Trinity College, Hartford, CT 06106
Email address: mstein@trincoll.edu