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A NEW APPROACH TO THE STUDY OF HARRIS TYPE MARKOV OPERATORS

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Harris operators are generalizations of Markov matrices. It is our purpose to present an elementary discussion of the theory of Harris operators. In Chapter 1 we introduce most of the results about Markov operators to be used later. In Chapter 2 we study Orey's Lemma. And in the rest of the paper we use Orey's Lemma to give elementary proofs of Harris' Theorem, Ornstein-Metivier-Brunel Theorem, Doeblin's theorem, and Pointwise Convergence of uP^n.

1. Introduction. We shall use the definitions and notation of [**3**] and [**4**].

Recall that *if* λ *is* σ *finite measure on* (X, Σ), *then a Markov operator,* P, *is a linear operator on* $L_\infty(X, \Sigma, \lambda)$ *satisfying*

$$P1 \leq 1; \quad f \geq 0 \Rightarrow Pf \geq 0; \quad f_n \downarrow 0 \Rightarrow Pf_n \to 0$$

All inequalities, here and elsewhere are in the a.e. sense. Denote $\langle u, f\rangle = \int uf d\lambda; u \in L_1$ and $f \in L_\infty$. The dual operator acts on L_1 by $\langle uP, f\rangle = \langle u, Pf\rangle; u \in L_1$ and $f \in L_\infty$.

We may extend P, by monotone continuity, so that Pf and uP are defined for *all* non-negative measurable functions [**3**, Chapter I].

THEOREM 1.1. *Let* P *be conservative and ergodic. Then:*

(1) $P1 = 1$.

(2) $f \geq 0, Pf \leq f \Rightarrow f = \text{Const}$.

(3) $f \geq 0,\ f \neq 0 \Rightarrow \Sigma P^n f \equiv \infty$.

(4) $u \geq 0,\ u \neq 0 \Rightarrow \Sigma uP^n \equiv \infty$.

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(5) *There is at most one function, up to a multiplicative constant, such that*

$$0 \leq u(x) < \infty, \quad uP = u.$$

If $u \neq 0$, *then* $u(x) > 0$.

Elementary proofs for (1)-(4) are given in [**4**, Chapter II] and for (5) in [**3**; Chapter VI, Theorem A].

An integral kernel is an operator of the form

$$Kf(x) = \int k(x,y)f(y)\lambda(dy)$$

where $k \geq 0$ is $\Sigma \times \Sigma$ measurable and $K1 \leq 1$.

We shall use "The Harris Decomposition" [**3**; Chapter V]: $P^n = Q_n + R_n, Q_n \geq 0, R_n \geq 0$ *and* Q_n *is the largest integral kernel bounded by* P^n.

DEFINITION. P is a Harris operator provided:

(a) P is conservative and ergodic.

(b) $Q_j \neq 0$ for some integer j.

2. Orey's Lemma. Let h, w be non-negative and non-trivial functions.

Denote the integral kernel of $h(x)w(y)$ by $h \otimes w$, thus:

$$(h \otimes w)f = \langle w, f\rangle h.$$
$$u(h \otimes w) = \langle u, h\rangle w.$$

Note also that

$$P(h \otimes w) = (Ph) \otimes w$$
$$(h \otimes w)P = h \otimes (wP).$$

Orey proved the following theorem [**13**, Theorem 2.1].

THEOREM 2.1. *Let* P *be Harris. If* Σ *is separable, then* $P^r \geq h \otimes w$ *for some integer* r *and non-negative non-trivial functions* h *and* w.

CONJECTURE. *Separability of Σ is not necessary.*

We shall prove two versions of Orey's Lemma where Σ is not assumed to be separable.

LEMMA 2.2. *Let P be Harris, then*

$$\sum_{n=1}^{\infty} q_n(x,y) = \infty \text{ a.e. } \lambda^2.$$

PROOF. By [**3**; Chapter V, Equation (5.5)],

$$Q_{j+n} \geq P^n Q_j.$$

Hence

$$\sum_{n=1}^{\infty} Q_{j+n}1 \geq \sum_{n=1}^{\infty} P^n(Q_j 1) \equiv \infty.$$

Choose Y with $\lambda(Y) = 0$ such that, if $x \notin Y$, then $q_m(x,\cdot) \neq 0$ for some m. Now

$$q_{n+m}(x,y) \geq [q_m(x,\cdot)P^n](y)$$

by [**4**, p. 298].

Thus, if $x \notin Y$, then

$$\sum_{n=1}^{\infty} q_n(x,y) \geq \sum_{n=1}^{\infty} q_{n+m}(x,y) \geq \sum_{n=1}^{\infty} [q_m(x,\cdot)P^n](y) = \infty$$

for almost all y, by Theorem 1.1. □

LEMMA 2.3. *Let $s_n(x,y) \geq 0$ be $\Sigma \times \Sigma$ measurable. If $s_n \uparrow s < 0$ a.e. λ^2, then there exists an integer n, a positive constant ε, and two sets f and G, of positive measure such that*

$$\int s_n(x,z)s_n(z,y)\lambda(dz) \geq \varepsilon 1_F(x) 1_G(y).$$

PROOF. Let $\lambda_1 \sim \lambda$ with $\lambda_1(X) = 1$. Put:

$$\varphi_n(x) = \lambda_1(\{z : s_n(x,z) \geq 1/n\}),$$
$$\psi_n(y) = \lambda_1(\{z : s_n(z,y) \geq 1/n\}).$$

Then $0 \leq \phi_n(x), \psi_n(y) \leq 1$. Also,

$$\int \phi_n(x)\lambda_1(dx) \to 1 \text{ and } \int \psi_n(y)\lambda_1(dy) \to 1.$$

Thus $\phi_n(x) \uparrow 1, \psi_n(y) \uparrow 1$ a.e. λ_1, hence a.e. λ.

Given $\delta > 1/2$ find n such that, if

$$F = \{x : \varphi_n(x) \geq \delta\} \text{ and } G = \{y : \psi_n(y) \geq \delta\},$$

then $\lambda(F) > 0, \lambda(G) > 0$.

Then we may find $\varepsilon > 0$ with

$$\int s_n(x,z)s_n(z,y)\lambda(dz) \geq \varepsilon$$

provided that, for $x \in F, y \in G$,

$$\lambda_1(\{z : s_n(x,z) < 1/n\}) < 1 - \delta$$
$$\lambda_1(\{z : s_n(z,y) < 1/n\}) < 1 - \delta.$$

Hence

$$\lambda_1(\{z : s_n(x,z) \geq 1/n\} \cap \{z : s_n(z,y) \geq 1/n\})$$
$$\geq 1 - (2 - 2\delta) = 2\delta - 1.$$

Thus

$$\lambda(\{z : s_n(x,z) \geq 1/n\} \cap \{z : s_n(z,y) \geq 1/n\}) = \varepsilon' > 0.$$

Put $\varepsilon = \varepsilon'/n^2$. □

The above argument was used in [**12**].

THEOREM 2.4. *Let P be Harris. There exists an integer N and two non-negative non-trivial functions h, w, such that*

$$1/N \sum_{k=1}^{N} P^k \geq h \odot w.$$

PROOF. Let $s_n = \sum_{j=1}^{n} q_j$. By Lemma 2.3,

$$\begin{aligned}\varepsilon 1_F(x) 1_G(y) &\leq \int \Big(\sum_{j=1}^{n} q_j(x,z) \Big) \Big(\sum_{i=1}^{n} q_i(z,y) \Big) \lambda(dz) \\ &\leq \sum_{i,j=1}^{n} q_{i+j}(x,y) \leq n \sum_{k=1}^{2n} q_k.\end{aligned}$$

Put $N = 2n$, $h = \frac{\varepsilon}{2n^2} 1_F$, $w = 1_G$. □

REMARK. We used $\sum_1^\infty q_n > 0$. One may prove Theorem 2.4. for nonconservative operators.

In Chapter 6 we shall need a third version of Orey's Lemma:

THEOREM 2.5. *Let P satisfy:*

$$P1 = 1;\ \lambda(A) > 0 \Rightarrow P1_A \geq \alpha(A) > 0,$$

where $\alpha(A)$ is a constant. Then

$$P^5 \geq 1 \odot w$$

where $w \geq 0$ and $w \neq 0$.

PROOF. Let $\lambda_1 \sim \lambda$ and $\lambda_1(X) = 1$.

(a). There exists an $\varepsilon > 0$ such that

$$\lambda_1(A) \geq 1 - \varepsilon \Rightarrow P1_A \geq \varepsilon.$$

Otherwise, find sets A_n with

$$\lambda_1(A_n) \geq 1 - 1/2^n, \ \lambda_1(\{x : P1_{A_n}(x) < 1/2^n\}) \neq 0.$$

Put $A = \cap_{n=2}^{\infty} A_n$. Then $\lambda_1(A) \geq 1/2$ and hence $\lambda(A) > 0$. Also $\lambda_1(\{x : P1_A(x) < 1/2^n\}) \neq 0$, thus

$$\lambda(\{x : P1_A(x) < 1/2^n\}) \neq 0,$$

a contradiction. (This argument was used in [**7**]).

(b). Let $K_0 f = \int f d\lambda_1$. Then

$$\begin{aligned}(P \wedge K_0)1 &\geq \varepsilon \\ (P \wedge K_0)1 &= \inf\{P1_A + \lambda_1(A')\}.\end{aligned}$$

If $\lambda_1(A) \geq 1-\varepsilon$, then $P1_A \geq \varepsilon$ by (a). If $\lambda_1(A) < 1-\varepsilon$, then $\lambda_1(A') \geq \varepsilon$.

(c) $Q_1 1 \geq \varepsilon$. NY [**3**, Chapter V] $P \wedge K_0$ is an integral kernel, hence $P \wedge K_0 \leq Q_1$.

(d) $q_2(x,y) > 0$ a.e. λ^2. $q_2(x,y) \geq [q_1(x,\cdot)P](y)$ by [**4**, p. 28]. It suffices to prove that if $0 \leq u \in L_1$ and $u \neq 0$, then $uP(x) > 0$ a.e.: Given A with $\lambda(A) > 0$, then

$$\langle uP, 1_A\rangle = \langle u.P1_A\rangle \geq \alpha(A)\langle u, 1\rangle \neq 0.$$

(e). There exist two non negative non trivial functions h and w' with $q_4(x,y) \geq h(x)w'(y)$: Apply Lemma 2.3 to $s_n = q_2$.

(f) $P^5 \geq 1 \otimes w$ where $w \geq 0,\ w \neq 0 : P^5 \geq PQ_4 \geq (Ph) \otimes w' \geq 1 \otimes w$ where $w = \alpha(h)w'$. □

In all the three versions of Orey's Lemma we have

ASSUMPTION 1. *There exist $a_1, a_2, \ldots, a_N$ with*

$$a_n \geq 0,\ a_N \neq 0,\ \sum_1^N a_n = 1$$

such that

$$S = \sum_{n=1}^{N} a_n P^k = h \otimes w + T$$

where $T \geq 0$*, the functions* h, w *are nonnegative and nontrivial..*

Moreover S *is conservative and ergodic.*

We need to prove only the last statement. S is conservative by [**4**, Theorem 2.7]. For ergodicity:

(1). If $S = 1/N \sum_{n=1}^{N} P^n$, then, whenever $S1_A = 1_A$, $P1_A(x) = 0$ for all $x \in A'$. Thus $P1_A \leq 1_A$; hence, since P is conservative and ergodic, A is trivial.

(2). If $P1 = 1$ and $\lambda(A) > 0 \Rightarrow P1_A \geq \alpha(A) > 0$, then $S = P^5, P^5 1 = 1$ and

$$\lambda(A) > 0 \Rightarrow P^5 1_A \geq \alpha(A) > 0.$$

(3). $S = P^r \geq h \otimes w$. For any k,

$$P^{r+k} \geq (P^k h) \otimes w.$$

It suffices to show that P^j is ergodic for some $j \geq r$. By [**6**] and [**4**, Theorem 3.5], there exists a fixed integer d such that

$$\sum_i (P^j) \subset \sum_i (P^d)$$

for every integer j. Choose $j \geq r$ with $(j, d) = 1$. Let $nj + md = 1$. If $n \leq 0$, then, whenever $A \in \sum_i (P^j)$, we have

$$1_A = P^{md} 1_A = P\, P^{-nj} 1_A = P1_A.$$

Thus A is trivial. □

3. Existence of an invariant measure.

LEMMA 3.1. *Let Assumption* 1 *hold. Then* $T^n 1 \downarrow 0$.

PROOF. Let $T^n 1 \downarrow g$. Then

$$0 \le g \le 1, \quad Tg = g.$$

Thus $Sg \ge g$, therefore by Theorem 1.1. $g = \text{Const}$. Hence $\langle w, g\rangle = 0$ or $g = 0$. □

LEMMA 3.2. *Let Assumption* 1 *hold. Then*

$$\sum_0^\infty T^n h = 1/\langle w, 1\rangle.$$

PROOF.

$$\langle w, 1\rangle \sum_{n=0}^N T^n h = \sum_{n=0}^N T^n (h \otimes w) 1 = \sum_{n=0}^N T^n (S - T) 1$$
$$= 1 - T^{N+1} 1 \uparrow 1.$$

□

NOTATION. $v = \sum_{n=0}^\infty wT^n$.

COROLLARY. $\langle v, h\rangle = 1$.

$$\langle v, h\rangle = \left\langle \sum_0^\infty wT^n, h\right\rangle = \left\langle w, \sum_0^\infty T^n h\right\rangle.$$

Let us show that $v < \infty$. In fact a stronger result is valid, i.e., $0 \le u \in L_1 \Rightarrow \sum_0^\infty uT^n < \infty$: $\langle \sum_0^N uT^n, 1 - T^k 1\rangle \le k$. Hence

$$\sum_0^\infty uT^n < \infty \text{ on } \cup_k \{x : T^k 1(x) < 1\} = X.$$

THEOREM 3.3. *Let Assumption* 1 *hold. Then*

$$vS = v, \; 0 < v(x) < \infty.$$

PROOF. $vS = vT + v(h \otimes w) = \sum_1^\infty wT^n + \langle v, h\rangle w = v$ by Corollary to Lemma 3.2. Finally, $v(x) > 0$ by Theorem 1.1. □

THEOREM 3.4. (HARRIS' THEOREM). *If Assumption* 1 *holds, then*

$$vP = v.$$

PROOF. $vP^N \leq a_N^{-1} vS = a_N^{-1} vM\infty$. If $0 < f \in L_\infty$ satisfies $\langle vP^N, f\rangle < \infty$, then

$$\langle vP^i, P^{N-i} f\rangle < \infty, \quad 0 \leq i \leq N.$$

Now $(P^{N-i}f)(x) > 0$:

$$\lim_{n\to\infty} (P^{N-i}(nf))(x) \geq P^{N-i}1(x) = 1.$$

Hence $vP^i(x) < \infty$, for $0 \leq i \leq N$. Thus $v_1 = \sum_{n=1}^N a_n v(I + \cdots + P^{n-1}) < \infty$ and

$$0 = v - vS = v_1(I - P).$$

Finally, $v_1 = v_1 P$ implies $v_1 = v_1 S$ and v_1 is a multiple of v, by Theorem 1.1. Thus $v = vP$. □

4. The Ornstein-Metivier-Brunel Theorem.

THEOREM 4.1. *Let Assumption* 1 *hold. If*

$$\sum_0^\infty T^n|f| \in L_\infty \text{ and } \langle v, f\rangle = 0$$

then

$$f \in \text{Range}\,(I - P),$$

hence

$$\left\|\sum_{n=0}^N P^n f\right\| \leq \text{Const}.$$

PROOF. Let us check that $\langle v, |f|\rangle < \infty$:

$$\langle v, |f|\rangle = \Big\langle \sum_0^\infty wT^n, |f|\Big\rangle = \Big\langle w, \sum_0^\infty T^n|f|\Big\rangle < \infty.$$

Put $g = \sum_0^\infty T^n f$. Then $g \in L_\infty$ and

$$\begin{aligned}(I-S)g &= (I-T)g - \Big\langle w, \sum T^n f\Big\rangle h = (I-T)g - \langle v, f\rangle h \\ &= (I-T)g = \lim_{N\to\infty}(f - T^{N+1}f) = f.\end{aligned}$$

This last step was by Lemma 3.1. Now

$$f = (I-S)g = (I-P)\sum_{n=1}^N a_n(I + \cdots + P^{n-1})g = (I-P)g_1.$$

Finally, if $f = (I-P)g_1$, then

$$\Big\|\sum_0^N P^n f\Big\| \le 2\|g_1\|.$$

□

REMARKS. Put

$$\Omega = \{e : e \ge 0 \text{ and } \Sigma T^n e \in L_\infty\}.$$

By Lemma 3.2, $h \in \Omega$. If $e \in \Omega$, then $Se = Te + \langle w, e\rangle h \in \Omega$. Thus if $0 \le e \le \Sigma S^j h$, then $e \in \Omega$. If $e \in \Omega$ and $A = \{x : e(x) \ge \delta > 0\}$, then $1_A \le \delta^{-1}e$ so $1_A \in \Omega$.

Therefore, there exists a sequence of sets A, such that

$$A_k \uparrow X, 1_{A_k} \in \Omega.$$

Let $1_A \in \Omega$. If support $f \subset A$ and $\langle v, f\rangle = 0$, then $f = (I-P)g_1$ where

$$\begin{aligned}|g_1| &= \Big|\sum_{n=1}^N a_n(I + \cdots + P^{n-1})\sum_{j=0}^\infty T^j f\Big| \\ &\le \|f\|\Big(\sum_{n=1}^N a_n(I + \cdots + P^{n-1})\sum_{j=0}^\infty T^j 1_A\Big) \\ &\le \text{Const}\,.\|f\|.\end{aligned}$$

The constant depends on the set A but not on the function f. Thus

$$\left|\left|\sum_0^N P^n f\right|\right| \leq 2||g_1|| \leq 2\text{Const}\,.||f||$$

where the constant depends on A alone.

If we write $f = f_1 - f_2$ where support $f_1 \subset A$, support $f_2 \subset A$ and $\langle v, f_1\rangle = \langle v, f_2\rangle$, then

$$\left|\left|\sum_0^N P^n f_1 - \sum_0^N P^n f_2\right|\right| \leq 2\text{Const}\,.(||f_1|| + ||f_2||).$$

This leads to "Ration Limit Theorems".

Let us conclude this Chapter with a dual result.

THEOREM 4.2. *Let Assumption* 1 *hold. If*

$$\sum_0^\infty |u|T^n \in L_1, \quad \langle u, 1\rangle = 0,$$

then

$$u \in \text{Range}\,(I - P).$$

Hence

$$\left|\left|\sum_0^N uP^n\right|\right|_1 \leq \text{Const}\,.$$

PROOF. Put $s = \sum_0^\infty uT^n$. Then, by assumption, $s \in L_1$. Now $s(I - S) = s(I - T) - \langle s, h\rangle w$. But

$$\langle s, g\rangle = \left\langle \sum_0^\infty uT^n, h\right\rangle = \left\langle u, \sum_0^\infty T^n h\right\rangle = 0$$

by Lemma 3.2. Moreover,

$$s(I - T) = \lim_{N\to\infty} (u - uT^{N+1}) = u.$$

Finally,

$$s(I-S) = \Big(\sum_{1}^{n} a_n s(I + \cdots + P^{n-1})\Big)(I-P) = s_1(I-P).$$

□

REMARK. Let

$$\Omega_1 = \{y : y \geq 0 \text{ and } \sum_{0}^{\infty} yT^n \in L_1\}.$$

We do not know if $\Omega_1 \neq \{0\}$ unless $v \in L_1$ (in which case $w \in \Omega_1$ and Ω_1 is invariant under S).

5. Doeblin's Theorem.

THEOREM 5.1. *Let P_1 be a Markov operator satisfying*

$$P_1 1 = 1,\ \lambda(A) > 0 \Rightarrow P_1 1_a \geq \alpha(A) > 0.$$

Then P_1^n converges in the operator norm.

PROOF. By Theorem 2.5.,

$$P_1^5 = 1 \otimes w + T,\ T \geq 0.$$

By Theorem 3.4., if $v = \sum wT^n$ then $vP = v$. Note that

$$T1 = 1 - \langle w, 1\rangle < 1.$$
$$||T^n|| \leq (1 - \langle w, 1\rangle)^n.$$

Recall that, by Corollary to Lemma 3.2., $\langle v, 1\rangle = 1$. Put

$$Ef = \langle v, f\rangle.$$

Then $||E|| \leq 1, E^2 = E = EP = PE$. Now

$$P^{5n} = T^n + 1 \otimes w + 1 \otimes (wT) + \cdots + 1 \otimes (wT^{n-1}).$$

We know this for $n = 1$l let us prove, by induction,

$$P^5 P^{5n} = P^5 t^n + 1 \otimes w + \cdots + 1 \otimes (wT^{n-1})$$
$$= T^{n+1} + 1 \otimes (wT^n) + 1 \otimes w + \cdots + 1 \otimes (wT^{n-1}).$$

If $0 \le f \le 1$, then

$$|P^{5n} f - Ef| = \left| T^n f + \sum_{j=0}^{n-1} \langle wT^j, f\rangle - \sum_{j=0}^{\infty} \langle wT^j, f\rangle \right|$$
$$\le ||T||^n + \sum_{j=n}^{\infty} \langle wT^j, 1\rangle \le ||T||^n \left(1 + ||w||_1 \sum_0^{\infty} ||T||^n\right)$$
$$= 2(1 - \langle w, 1\rangle)^n.$$

Thus $||P^{5n} - E|| \le 4(1 - \langle w, 1\rangle)^n \to 0$ as $n \to \infty$. Finally

$$||P^k p^{5n} - E|| = ||P^k P^{5n} - P^k E|| \le ||P^{5n} - E|| \to 0.$$

□

LEMMA 5.2. *Let P satisfy*

$$P1 = 1, \quad \lambda(A) > 0 \Rightarrow \sum_{n=1}^{N(A)} P^n 1_A \ge \alpha(A) > 0.$$

Then there exists a function v with $0 < v \in L_1, vP = v$ and

$$\text{Range}\,(I - P) = \{f : \langle v, f\rangle = 0\}.$$

PROOF. Put $P_1 = \sum_{n=1}^{\infty} 1/2^n P^n$. The operator P_1 satisfies the assumptions of Theorem 5.1.

If $v = vP_1$, then

$$0 = \left(\sum_1 1/2^n v(I + \cdots + P^{n-1})\right)(I - P) = v_1(I - P).$$

But $v_1 = v_1 P$ implies $v_1 = v_1 P_1$ and, by Theorem 1.1., v_1 is a multiple of $v : v = vP$. Put

$$Ef = \langle v, f\rangle; L = \{f : \langle v, f\rangle = 0 = \{f : Ef = 0\}.$$

Then $P_1E = EP_1 = E$, so $P_1L \subset L$.

By Theorem 5.1.,

$$||P_1^j/L|| \leq ||P_1^j(I-E)|| = ||P_1^j - E|| \rightarrow \text{ as } j \rightarrow \infty.$$

If $||P_1^j/L|| \leq 1$, then the restriction of $I - P_1^j$ to L is invertible. Thus $L \subset \text{Range}\,(I - P_1^j)$. Now

$$I - P_1^j = (I - P_1)(I + P_1 + \cdots + P_1^{j-1})$$

or $\text{Range}\,(I - P_1^j) \subset \text{Range}\,(I - P_1)$. Finally

$$(I - P_1) = (I - P)\sum_{n=1}^{\infty} 1/2^n(I + \cdots + P^{n-1})$$

or $\text{Range}\,(I - P) \subset \text{Range}\,(I - P)$. Thus $L \subset \text{Range}\,(I - P)$.

Conversely, if $g = (I - P)f$, then

$$\langle v, g\rangle = \langle v, f\rangle - \langle vP, f\rangle = 0.$$

□

The next Theorem is Horowitz's version of Doeblin's Theorem, see [**9**]. A similar result is proved in [**15**].

THEOREM 5.3. *Let* $P1 = 1$. *The following conditions are equivalent:*

(1) $\lambda(A) > 0 \Rightarrow \sum_0^{N(A)} P^n 1_A \geq \alpha(A) > 0$.

(2) *There exists* $v, 0 < v < L_1$, *and*

$$\text{Range}\,(I - P) = \{f : \langle v, f\rangle = 0\}.$$

(3) *There exists* $v, 0 < v \in L_1$, *and if* $Ef = \langle v, f\rangle$, *then*

$$\left|\left|1/N \sum_0^{N-1} P^n - E\right|\right| \rightarrow 0.$$

(4) *There exists* $v, 0 < v \in L_1$, *and*

$$\left\|1/N \sum_0^{N-1} P^n f - \langle v, f\rangle\right\| \to 0$$

for every $f \in L_\infty$.

(5) *There exists* $v, 0 < v \in L_1$, *and*

$$\text{Closure Range}\,(I - P) = \{f : \langle v, f\rangle = 0\}.$$

PROOF. (1)⇒(2). Lemma 5.2.

(2)⇒(3). By the Closed Graph Theorem there exists a constant C such that

$$\langle v, f\rangle = 0 \Rightarrow f = (I - P)g \text{ and } \|g\| \le C\|f\|.$$

Thus

$$\begin{aligned}\left\|1/N \sum_0^{N-1} P^n f - \langle v, f\rangle\right\| &= \left\|1/N \sum_0^{N-1} P^n (f - \langle v, f\rangle)\right\| \\ &= \left\|1/N \sum_0^{N-1} P^n (I - P)g\right\| \le 2\|g\|/N \\ &\le 2C/N\|f - \langle v, f\rangle\| \le 4C/N\|f\|.\end{aligned}$$

(3)⇒(4). Obvious.

(4)⇒(1). Take $f = P1_A$. Then

$$1/N \sum_1^N P^n 1_A - \langle v, 1_A\rangle \ge -1/2\langle v, 1_A\rangle$$

if N is large enough. Thus

$$\sum_1^N P^n 1_A \ge N/2\langle v, 1_A\rangle = \alpha(A) > 0.$$

Now (2)⇒(5) is clear.

(5)⇒(4). Put $P_N = \frac{1}{N}\sum_0^{N-1} P^n$. Now

$$P_N f = P_N(f - \langle v, f\rangle) + \langle v, f\rangle \to \langle v, f\rangle$$

since $P_N g \to 0$ when $g \in$ Closure Range $(I - P)$. □

6. Pointwise convergence. Let P be a conservative and ergodic operator with a σ finite invariant measure μ:

$$d\mu = v d\lambda, \ \ vP = v.$$

By [**3**, Chapter VII],

$$\int |Pf| d\mu \le \int |f| d\mu,$$
$$\int |Pf|^2 d\mu \le \int |f|^2 d\mu.$$

Thus P is a contraction on $L_2(\mu)$.

Given $u \in L_1(\lambda)$, then $u = u_0 v$ where $u_0 \in L_1(\mu)$. Define

$$(u_0 v)P^* = v \cdot P u_0 (uP^* = v \cdot P(u/v)).$$

If $0 \le u \in L_1(\lambda)$, then $0 \le u_0 \in L_1(\mu)$ and

$$uP^* \ge 0, \ \ \int uP^* d\lambda = \int P u_0 d u_o d\mu \le \int_0 u_0 d\mu = \int u d\lambda.$$

If $u \in L_1(\lambda)$, put $u = u^+ - u^-$. Then

$$\int |uP^*| d\lambda \le \int u^+ P^* d\lambda + \int u^- P^* d\lambda \le \int (u^+ + u^-) d\lambda = \int |u| d\lambda :$$

P^* is the dual of a Markov operator.

Let us compute $P^* f$:

$$\langle u_0 v, P^* f\rangle = \langle (u_0 v)P^*, f\rangle = \int P u_0 \cdot v f d\lambda$$
$$= \langle u_0 v, 1/v[(vf)P]\rangle.$$

THEOREM 6.1. *Let P be a conservative and ergodic operator with a σ finite invariant measure $\mu (d\mu = cd\lambda)$. Define*

$$P^* f = 1/v[(vf)P].$$

Then P^* is a Markov operator and

$$(u_0 v)P^* = v \cdot Pu_0, \quad u_0 \in L_1(\mu).$$

The operator P^* is conservative and ergodic, too, and $vP^* = v$. Now $P^{**} = P$ and P, P^* are adjoint operators on $L_2(\mu)$.

Finally

$$P^r \geq h \otimes w \Rightarrow P^{*r} \geq (w/v) \otimes (vh).$$

PROOF. Let $0 \leq f \in L_\infty$ be such that $0 \neq f \in L_1(\lambda)$. Then

$$\infty = \sum_0^\infty (vf)P^n = v \sum_0^\infty P^{*n} f.$$

Thus P^* is conservative and ergodic, too. Now

$$vP^* = vP1 = v,$$

$$P^{**} f = 1/v[(fv)P^*] = Pf.$$

If $f, g \in L_1(\mu) \cap L_\infty(\lambda)$, then

$$\int Pf \cdot g d\mu = \langle vg, Pf \rangle = \langle 1/v[(vg)P], vf \rangle = \int f \cdot P^* g d\mu.$$

Finally, if $P^r \geq h \otimes w$, then, for every $f \geq 0$, we have

$$\begin{aligned} P^{*r} f &= 1/v[(fv)P^r] \geq 1/v[(fv)h \otimes w] \\ &= \left(\int fvh d\lambda \right) w/v = ((w/v) \otimes (vh))f. \end{aligned}$$

Let us quote Theorem A and B of [**3**, Chapter VIII]. Define

$$\sum\nolimits_k = \left\{ A : \int (p^n 1_A)^2 d\mu = \int (P^{*n} 1_A)^2 d\mu = \mu(A) < \infty \text{ for all } n \right\}.$$

Then

(1) $\sum_K$ is a field. If $A_n \in \sum_K$ and $A_n \uparrow A$ where $\mu(A) < \infty$, then $A \in \sum_K$.

(2) If $A \in \sum_K$, then $P1_A$ and P^*1_A are characteristic functions of sets in $\sum_K$.

(3) If K is the subspace of $L_2(\mu)$ generated by $\sum_K$, then K is invariant under P and P^*, and if $f \in K$, then

$$P^*Pf = PP^*f = f.$$

(4) If $\int_A f d\mu = 0$ for every set $A \in \sum_K$, then

$$(vP^n f, g) \to 0,\ \langle vP^{*n} f, g\rangle \to 0$$

for every $g \in L_2(\mu)$.

Let us use the main result of [**6**]:

(5) If P is Harris, then either $\sum_K = \{\emptyset\}$ or $\sum_K$ contains an atom.

Let $\sum_K \neq \{\emptyset\}$ and let A_0 be an atom of $\sum_K$. Put $1_{A_j} = P^j 1_{A_0}$. A_j is again an atom of $\sum_K$. We can not have $A_0 \cap A_j = \emptyset$ for all $j \geq 1$ since $\sum P^j 1_{A_0} \equiv \infty$. Let d be the first integer such that $A_d = A_0$.

If $0 \leq i < j \leq d-1$, then

$$A_i \cap A_j = \emptyset (A_0 \cap A_{j-i} = \emptyset).$$

Finally, the set $\cup_{i=0}^{d-1} A_i$ is invariant under P, so must be X. Hence

$$\mu(X) = d\mu(A_0) < \infty.$$

Conversely, if $\mu(X) < \infty$, then $X \subset \sum_K$ contains an atom. □

Let us summarize.

THEOREM 6.2. *Let P be a Harris operator with an invariant measure $\mu(d\mu = vd\lambda)$.*

(1) *If $\mu(X) = \infty$, then $\sum_K = \{\emptyset\}$; hence*

$$\langle (u_0 v)P^n, f\rangle \to 0 \text{ as } n \to \infty,$$
$$\langle vP^n u_0, f\rangle \to 0 \text{ as } n \to \infty$$

whenever $u_0 \in L_1(\mu), f \in L_2(\mu) \cap L_\infty(\lambda)$.

(2) *If* $\mu(X) < \infty$, *then* $\sum_K = \{A_0, A_1, \dots, A_{d-1}\}$ *so that the sets* A_i *are disjoint.*

$$P1_{A_i} = 1_{A_{i+1}} \ (A_d = A_0).$$
$$(v1_{A_i})P = v1_{A_{i-1}} \ (A_{-1} = A_{d-1}).$$

If $u_0 \in L_1(\mu)$, *and* $f \in L_\infty(\lambda)$ *and*

$$\alpha_i = \mu(A_i)^{-1} \int_{A_i} u_0 d\mu,$$

then

$$\left\langle \left(\left(u_0 - \sum_{i=0}^{d-1} \alpha_i 1_{A_i} \right) v \right) P^n, f \right\rangle \to 0,$$
$$\left\langle vP^n \left(u_0 - \sum_{i=0}^{d-1} \alpha_i 1_{A_i} \right), f \right\rangle \to 0.$$

PROOF. (1). If $\mu(X) = \infty$, then $\sum_K = \{\emptyset\}$. Thus

$$\langle (u_0 v) P^n, f \rangle \to 0,$$
$$\langle vP^n u_0, f \rangle \to 0$$

whenever $u_0 \in L_2(\mu)$ and $f \in L_2(\mu)$.

Fix $f \in L_2(\mu) \cap L_\infty(\lambda)$. Then by continuity, we may take $u_0 \in L_1(\mu)$.

(2). We showed that, if $\mu(X) < \infty$, then

$$X = \cup_{i=0}^{d-1} A_i, \ A_i \cap A_j = \emptyset, \ 0 \le i < j < d,$$
$$P1_{A_i} = 1_{A_{i+1}} \ (A_d = A_0).$$

Now

$$(v1_{A_i})P = (v1_{A_i})P^{**} = v \cdot P^* 1_{A_i} = v1_{A_{i-1}}$$

since $P^* 1_{A_i} = P^* P 1_{A_{i-1}} = 1_{A_{i-1}}$. By the choice of α_i,

$$u_0 - \sum_0^{d-1} \alpha_i 1_A \text{ is orthogonal to } \sum_K .$$

By (4),

$$\left\langle \left(\left(u_0 - \sum_{i=0}^{d-1} \alpha_i 1_{A_i} \right) v \right) P^n, f \right\rangle \to 0,$$

$$\left\langle v P^n \left(u_0 - \sum_{n=0}^{d-1} \alpha_i 1_{A_i} \right), f \right\rangle \to 0$$

if $u_0 \in L_2(\mu)$ and $f \in L_2(\mu)$.

Fix $f \in L_\infty(\lambda)$. Then, by continuity, we may take $u_0 \in L_1(\mu)$ in the above equations. □

In the rest of this paper we elaborate on results of Horowitz [**10**].

ASSUMPTION 2. *Let P be a conservative and ergodic Markov operator such that there exists an integer r with*

$$P^r \geq h \otimes w,$$

where h, w are non negative and non-trivial.

Recall Theorem 3.4.: There exists v with $0 < v(x) < \infty$ and $vP = v$.

NOTE. If P is Harris and $\sum$ is separable, then Assumption 2 follows from Orey's Lemma (Theorem 2.1.).

Now

$$P^r = h \otimes w + T, \; T \geq 0.$$

Let us show that

$$P^{rn} = T^n + (P^{r(n-1)} h) \otimes w + \cdots + h \otimes (w T^{n-1}).$$

Let us prove by induction:

$$\begin{aligned} P^{r(n+1)} &= P^r T^n + (P^{rn} h) \otimes w + \cdots + (P^r h) \otimes (w T^{n-1}) \\ &= T^{n+1} + h \otimes (w T^n) + (P^{rn} h) \otimes w + \cdots + (P^r h) \otimes (w T^{n-1}). \end{aligned}$$

LEMMA 6.3. *Let Assumption* 2 *hold. If* $u_0 \in L_1(\mu)$, *then*
(1) $\langle (u_0 v)P^n, h\rangle \to 0 \Rightarrow (u_0 v)P^n \to 0$.
(2) $\langle P^n u_0, w\rangle \to 0 \Rightarrow P^n u_0 \to 0$.

PROOF. (1). Let $u = u_0 v$, where $u_0 \in L_1(\mu)$.

$$(uP^k)P^{rn} \le |uP^k|T^n + v \max_{1\le i\le k} |\langle uP^{r(n-i)+k}, h\rangle| + ||u||_1\ ||h||_\infty \sum_{i=k} wT^i.$$

$$|uP^k|T^n \to 0 \text{ (as } n \to \infty) \text{ since } \sum_{n=0}^{\infty} |uP^k|T^n < \infty.$$

$$\sum_{i=k}^{\infty} wT^i \to 0(\text{as } k \to \infty)0 \text{ since } \sum_{i=0}^{\infty} wT^i < \infty.$$

$$\langle uP^n, h\rangle \to 0 \text{ (as } n \to \infty) \text{ by assumption.}$$

(2). $P^{*r} \ge (w/v) \otimes (vh)$; hence, by (1),

$$\langle (u_0 v)P^{*n}, w/v\rangle \to 0 \Rightarrow (u_0 v)P^{*n} \to 0,$$

or

$$\langle P^n u_0, w\rangle \to 0 \Rightarrow P^n u_0 \to 0.$$

□

THEOREM 6.4. *Let Assumption* 2 *hold. Let* $u_0 \in L_1(\mu)$. *Then:*
(1) *If* $\mu(X) = \infty$, *then*

$$(u_0 v)P^n \to 0,$$
$$P^n u_0 \to 0.$$

(2) *If* $\mu(X) < \infty$ *let* $X = \cup_{i=0}^{d-1} A_i$ *as in Part* (2) *of Theorem* 6.2. *Put*

$$\alpha_i = \mu(A_i)^{-1} \int_{A_i} u_0 d\mu.$$

Then

$$\Big(\Big(u_0 - \sum_{i=0}^{d-1} \alpha_i 1_{A_i}\Big)v\Big)P^n \to 0,$$

$$P^n\Big(u_0 - \sum_{i=0}^{d-1} \alpha_i 1_{A_i}\Big) \to 0.$$

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