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EVERYWHERE WILD CELLS AND SPHERES

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1. **Introduction.** Wild cells of all possible codimensions in E^n, $n \geqq 3$, were first constructed by W. A. Blankinship in [7]. Morton Brown [9] made a nice application of a result of J. J. Andrews and M. L. Curtis [3] to construct wild cells and spheres of all possible codimensions in E^n, $n \geqq 3$. Although Brown did not go into detail on his suggested method for producing wild codimension two spheres, such spheres had previously been given by J. C. Cantrell and C. H. Edwards in [11]. The work of Cantrell and Edwards was carried further by Ralph Tindell [19] to obtain some interesting wild embeddings in codimension two. Everywhere wild arcs in E^3 have been constructed by R. H. Bing [6] and W. R. Alford [1]. Also, everywhere wild arcs have been constructed in E^n, $n \geqq 4$, [9].

In this paper we establish the following result.

THEOREM. *In E^n, $n \geqq 3$, there are cellular, everywhere wild cells and everywhere wild spheres of all codimensions.*

Thus, it is immedate that there are also closed, everywhere wild strings and half-strings of all codimensions. In §6 we obtain noncellular everywhere wild cells. Finally, in §7, everywhere wild arcs are exhibited in E^n which pierce a locally flat $(n-1)$-sphere at a single point. The author expresses gratitude to C. E. Burgess and T. M. Price for helpful suggestions regarding this paper. For applications of this paper see [12], [17] and [18].

2. **Definitions.** Euclidean n-space is denoted by E^n, $E_+{}^n = E^{n-1} \times [0, \infty) \subset E^n$, $I^n = \{(x_1, x_2, \cdots, x_n) \mid -1 \leqq x_i \leqq 1, \quad i = 1, 2, \cdots, n\} \subset E^n$, and S^n is the boundary of I^{n+1}. An *n-string, n-half-string, n-cell, n-sphere* is a set which is homeomorphic to E^n, $E_+{}^n$, I^n, S^n, respectively. A topological k-manifold M in E^n is said to be *locally flat* at a point $x \in M$ if there is a neighborhood U of x in E^n such that the pair $(U, U \cap M)$ is homeomorphic to (E^n, E^k) if $x \in \operatorname{Int} M$ or to $(E^n, E_+{}^k)$ if $x \in \operatorname{Bd} M$. M is said to be *locally tame* at $x \in M$ if there is a neighborhood U of x in E^n and a homeomorphism of U onto E^n which carries $U \cap M$ onto a subpolyhedron of E^n. M

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is *wild* at $x \in M$ if it is not locally tame at x, and M is *everywhere wild* if it is wild at every point.

Let $A \subset X \subset Y$ be topological spaces. The set $Y - X$ is *projectively 1-connected* at A if each neighborhood U of A contains a neighborhood V of A such that each loop in $V - X$ is null-homotopic in $U - X$. In the case that A is a point, we say that $Y - X$ is 1-LC at A. (Projective 1-connectivity of $Y - X$ at X is the same as the cellularity criterion of [**16**] for X.) Let $X \subset Y$ be topological spaces. Then, $Y - X$ is said to be 1-SS (1-short shrink) at $x \in X$ if for every neighborhood U of x there is a neighborhood $V \subset U$ of x such that every loop in $V - X$ which is null-homotopic in $Y - X$ is also null-homotopic in $U - X$. We say that $Y - X$ is 1-LA (1-locally abelian) at $x \in X$ if for any sequence $\{V_i\}$ of closed neighborhoods of x such that $V_1 \supset V_2 \supset \cdots$ and $\bigcap_{i=1}^{\infty} V_i = x$, there is an integer N for which $\pi_1(V_N - X)$ injects into $\pi_1(V_1 - X)$ as an abelian subgroup.

3. **Construction of everywhere wild cells and spheres in codimension two.** A form of the first lemma appears in [**12**]; however, we include it here for completeness.

LEMMA 1. (a) *Let* $\Sigma^{n-2} \subset S^n(E^n)$ *be an* $(n-2)$*-sphere which is locally flat at a point* x. *Then,* $S^n - \Sigma^{n-2}(E^n - \Sigma^{n-2})$ is 1-SS *at* x.

(b) *Let* $X^{n-2} \subset E^n$ *be a closed* $(n-2)$*-string which is locally flat at a point* x. *Then* $E^n - X^{n-2}$ *is* 1-SS *at* x.

PROOF. We will establish only the case where $\Sigma^{n-2} \subset S^n$ is an $(n-2)$-sphere which is locally flat at x, because the proofs of the other cases are similar. Let U be any neighborhood of x and let $V \subset U$ be a flattening cell neighborhood for Σ^{n-2} at x, i.e., $(V, V \cap \Sigma^{n-2}) \approx (I^n, I^{n-2})$. Let l be a loop in $V - \Sigma$ which is null-homotopic in $S^n - \Sigma$. By pushing radially away from x we see that l is homotopic in $V - \Sigma$ to a loop l' in Bd $V - \Sigma$ which is null-homotopic in $S^n - (\text{Int } V \cup \Sigma)$. The proof will be complete if we can show that l' is null-homotopic in Bd $V - \Sigma$. Since we know that l' is null-homotopic in $S^n - (\text{Int } V \cup \Sigma)$, it will suffice to show that the injection

$$\pi_1(\text{Bd } V - \Sigma) \to \pi_1(S^n - (\text{Int } V \cup \Sigma))$$

is a monomorphism. In order to do this consider the following Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to H_2(S^n - (\Sigma - \text{Int } V)) &\to H_1(\text{Bd } V - \Sigma) \\ &\to H_1(S^n - (\text{Int } V \cup \Sigma)) \oplus H_1(V - (\text{Bd } V \cap \Sigma)) \\ &\to H_1(S^n - (\Sigma - \text{Int } V)) \to \cdots. \end{aligned}$$

By using Alexander duality this sequence becomes

$$\cdots \to 0 \to Z \to Z \oplus 0 \to 0 \to \cdots.$$

Hence, the inclusion of $\mathrm{Bd}\, V - \Sigma$ into $S^n - (\mathrm{Int}\, V \cup \Sigma)$ induces an isomorphism on first homology. But now any loop l in $\mathrm{Bd}\, V - \Sigma$ which is null-homotopic in $S^n - (\mathrm{Int}\, V \cup \Sigma)$ is also null-homologous in $S^n - (\mathrm{Int}\, V \cup \Sigma)$, consequently null-homologous in $\mathrm{Bd}\, V - \Sigma$. Since $\pi_1(\mathrm{Bd}\, V - \Sigma)$ is abelian, it follows that l is null-homotopic in $\mathrm{Bd}\, V - \Sigma$ and so the injection

$$\pi_1(\mathrm{Bd}\, V - \Sigma) \to \pi_1(S^n - (\mathrm{Int}\, V \cup \Sigma))$$

is a monomorphism as desired.

Lemma 2. *Let $A \subset X \subset Y$ be topological spaces such that $Y - X$ is not 1-SS at any point of A. Suppose $W \subset Z \subset E^1$, where W is open in E^1, and suppose $R \subset Y \times (E^1 - W)$. Then, $(Y \times E^1) - ((X \times Z) \cup R)$ is not 1-SS at any point of $A \times W$.*

Proof. Suppose that $(Y \times E^1) - ((X \times Z) \cup R)$ is 1-SS at some point $p = (a, t)$ of $A \times W$. Let U be any neighborhood of a in Y. Then, $U \times W$ is a neighborhood of p in $Y \times E^1$. Therefore, there is a neighborhood $V_* \subset U \times W$ of p such that every loop in $V_* - ((X \times Z) \cup R)$ which is null-homotopic in $(Y \times E^1) - ((X \times Z) \cup R)$ is null-homotopic in $(U \times W) - ((X \times Z) \cup R)$. Choose a neighborhood V of a in Y and an $\epsilon > 0$ such that

$$p = (a, t) \in V \times (t - \epsilon, t + \epsilon) \subset V_*.$$

By our construction, it is easy to see that every loop in $V - X$ which is null-homotopic in $Y - X$ is also null-homotopic in $U - X$ and so it follows that $Y - X$ is 1-SS at a which is a contradiction.

Although a manifold $M \subset E^n$ of codimension two may be locally tame at a point $x \in M$ and yet fail to be locally flat at x, the following lemma is easily established.

Lemma 3. *If a manifold $M \subset E^n$ fails to be locally flat at every point, then M is everywhere wild.*

Example 1. *For $n \geqq 3$, there is an $(n-2)$-cell $F^{n-2} \subset E^n$ which lies on the boundary of an $(n-1)$-cell D^{n-1} such that $D^{n-1} - F^{n-2}$ is locally flat and such that $E^n - \mathrm{Bd}\, D^{n-1}$ fails to be 1-SS at every point of $\mathrm{Int}\, F^{n-2}$. (Hence, by Lemma 1, F^{n-2} fails to be locally flat at every point and so is everywhere wild by Lemma 3.) Furthermore, F^{n-2} is cellular in E^n.*

W. R. Alford in [1] constructs a wild 2-sphere S in E^3 whose set of wild points is an arc F^1. Let D^2 be a 2-cell in S which has F^1 on its boundary. Then, $D^2 - F^1$ is locally flat. It follows from the construction of S that $E^3 - S$ is not 1-SS at any point of Int F^1. In particular, there is a neighborhood U of each $x \in \text{Int } F^1$ such that for any neighborhood $V \subset U$ of x there is a loop in $V - S$ which is null-homotopic in $E^3 - S$, but not in $U - S$. For a fixed $x \in \text{Int } F^1$, let $B \subset U$ be a flattening open disk neighborhood of Bd D^2 in S at x, i.e., $(B, B \cap \text{Bd } D^2) \approx (E^2, E^1)$. Now let $U' \subset U$ be a neighborhood of x in E^3 such that $U' \cap S \subset B$ and suppose that $E^3 - \text{Bd } D^2$ is 1-SS at x. Then, there is a neighborhood $V \subset U'$ such that each loop in $V - \text{Bd } D^2$ which is null-homotopic in $E^3 - \text{Bd } D^2$ is also null-homotopic in $U' - \text{Bd } D^2$. Let $l : \text{Bd } I^2 \rightarrow V - S$ be a loop which is null-homotopic in $E^3 - S$, but not in $U - S$. By the above assumption, there is an extension $f: I^2 \rightarrow U' - \text{Bd } D^2$ of l. Clearly, there are two closed disks D_+ and D_- in $B - \text{Bd } D^2$ such that $f(I^2) \cap S \subset D_+ \cup D_-$. Let G denote the complementary domain of S in E^3 which contains $l(\text{Bd } I^2)$. Let X denote the component of $f^{-1}(G)$ which contains Bd I^2 and consider the components of $I^2 - X$. Let A_+ be the union of all of those components having frontiers whose images are contained in D_+ and let A_- be the union of all of those components having frontiers whose images are contained in D_-. (By unicoherence (see Theorem 5.19 on p. 60 of [**20**]), those frontiers are connected and so their images are contained in either D_+ or D_-.) Then, by Tietze's extension theorem $f | A_+ \cap f^{-1}(S)$ can be extended to a map $f_+ : A_+ \rightarrow D_+$ and $f | A_- \cap f^{-1}(S)$ can be extended to a map $f_- : A_- \rightarrow D_-$. Redefine f to be f_+ on A_+ and f_- on A_-. By using a collar of D_+ and D_- in Cl $G \cap U$ (which exist since D_+ and D_- are locally flat), we can "pull in" f to obtain $f_* : I^2 \rightarrow U - S$ and so l would be null-homotopic in $U - S$ which is a contradiction. Hence, $E^3 - \text{Bd } D^2$ is not 1-SS at x. Consequently, F^1 is not locally flat at x by Lemma 1 and so F^1 is everywhere wild by Lemma 3. (In [**14**, §3], Gillman shows that F^1 does not pierce any disk which also shows it everywhere wild.) It is easy to see that F^1 satisfies the cellularity criterion of [**16**] and is thus cellular. (Gillman observes that F^1 is cellular in [**14**, §3], and Alford does the same in [**2**].)

Inductively, assume that there is an $(n-2)$-cell $F^{n-2} \subset E^n$ which lies on the boundary of an $(n-1)$-cell D^{n-1} such that $D^{n-1} - F^{n-2}$ is locally flat and such that $E^n - \text{Bd } D^{n-1}$ fails to be 1-SS at every point of Int F^{n-2}. Let $F^{n-1} = F^{n-2} \times [-1, 1]$, and $D^n = D^{n-1} \times [-1, 1]$. Then, $F^{n-1} \subset \text{Bd } D^n \subset E^{n+1} = E^n \times E^1$. Since $D^{n-1} - F^{n-2}$ is locally flat in E^n, it follows that $D^n - F^{n-1}$ is locally flat in E^{n+1}. Since

$$\text{Bd}\, D^n = (\text{Bd}\, D^{n-1} \times [-1, 1]) \cup (D^{n-1} \times \{-1, 1\}),$$

Lemma 2 implies that $E^{n+1} - \text{Bd}\, D^n$ is not 1-SS at any point of Int F^{n-1}. It is a consequence of the next lemma that F^{n-1} is cellular in E^{n+1}. The proof of this lemma is straightforward.

LEMMA 4. *If $A \times E^1 \approx E^{n+1}$ and $X \subset A$ is cellular in E^{n+1}, then $X \times [-1, 1]$ is also cellular in E^{n+1}.*

EXAMPLE 2. *For $n \geqq 3$, there is a closed $(n-2)$-string X^{n-2} in E^n such that $E^n - X^{n-2}$ fails to be 1-SS at every point of X^{n-2}. (Hence, by Lemma 1 and Lemma 3, X^{n-2} is everywhere wild.) Furthermore, X^{n-2} is the boundary of a closed $(n-1)$-half-string Y^{n-1} such that $Y^{n-1} - X^{n-2}$ is locally flat in E^n.*

Do the construction of [1] a countable number of times on $E^2 \subset E^3$ so as to make each interval $[n, n+1]$, n an integer, the wild arc. (In carrying out the construction on $[n, n+1]$, run the hooks from n toward $n+1$.) The resulting 1-string X^1 obviously lies on a closed two-string W^2 such that $E^3 - W^2$ fails to be 1-SS at every point of X^1. By an argument similar to that following Example 1, we see that $E^3 - X^1$ fails to be 1-SS at every point of X^1. It is clear that X^1 bounds a closed 2-half-string Y^2 such that $Y^2 - X^1$ is locally flat in E^3. The fact that $X^{n-1} = X^{n-2} \times E^1$ and $Y^n = Y^{n-1} \times E^1$ satisfy Example 2 in dimension n follows from Lemma 2.

EXAMPLE 3. *For $n \geqq 3$, there is an everywhere wild $(n-2)$-sphere Σ^{n-2} in E^n. Furthermore, Σ^{n-2} lies on the boundary of an $(n-1)$-cell D^{n-1} such that $D^{n-1} - \Sigma^{n-2}$ is locally flat in E^n.*

To get Example 3 simply one-point compactify the triple (X^{n-2}, Y^{n-1}, E^n) of Example 2 to obtain $(\Sigma^{n-2}, D^{n-2}, S^n)$ and then remove a point not on D^{n-2}.

4. A theorem for generating everywhere wild cells and spheres in codimension two.

LEMMA 5. *If $M^{n-2} \subset E^n$ is an $(n-2)$-manifold which is locally flat at $x \in$ Int M, then M is 1-LA at x.*

PROOF. (This lemma is a generalization of the italicized proposition on p. 988 of [4].) Let $B_\epsilon^k = \{x \in E^k \mid \|x\| < \epsilon\}$. Let $\{V_i\}$ be the sequence of neighborhoods given in the definition of 1-LA. Since M is locally flat at x, we may assume that x is the origin, that B_1^{n-2} is a neighborhood of x in M and that $B_1^n \cap M = B_1^{n-2}$. Choose ϵ such that $0 < \epsilon < 1$ and $B_\epsilon^n \subset V_1$. Next choose an integer N such that $V_N \subset B_\epsilon^n$ and pick a point in $V_N - M$ to serve as the base point for

$\pi_1(V_N - M)$, $\pi_1(B_\epsilon^n - M)$ and $\pi_1(V_1 - M)$. Since the injection of $\pi_1(V_N - M)$ into $\pi_1(V_1 - M)$ is compounded of the injection of $\pi_1(V_N - M)$ into the infinite cyclic group $\pi_1(B_\epsilon^N - M)$ and the injection of $\pi_1(B_\epsilon^N - M)$ into $\pi_1(V_1 - M)$, the conclusion follows.

LEMMA 6. *Let $A \subset X \subset Y$ be topological spaces such that $Y - X$ fails to be 1-LA at each point of A. Let $W \subset Z \subset E^1$, where W is open in E^1. Then, $(Y \times E^1) - (X \times Z)$ is not 1-LA at any point of $A \times W$.*

The proof of Lemma 6 is similar to that of Lemma 2 and so will not be included.

THEOREM 1. (a) *For $n \geqq 3$, suppose that $D^{n-2} \subset E^n$ is an $(n-2)$-cell such that $E^n - D^{n-2}$ fails to be 1-LA at every interior point of D^{n-2}. Then, $D^{n-1} = D^{n-2} \times [-1, 1] \subset E^n \times E^1 = E^{n+1}$ is an $(n-1)$-cell such that $E^{n+1} - D^{n-1}$ fails to be 1-LA at every interior point of D^{n-1}. (Hence, by Lemmas 5 and 3, D^{n-1} is everywhere wild in E^{n+1}.)*

(b) *For $n \geqq 3$, suppose that $X^{n-2} \subset E^n$ is a closed $(n-2)$-string such that $E^n - X^{n-2}$ fails to be 1-LA at every point of X^{n-2}. Then, $X^{n-1} = X^{n-2} \times E^1 \subset E^n \times E^1 = E^{n+1}$ is a closed $(n-1)$-string such that $E^{n+1} - X^{n-1}$ fails to be 1-LA at every point of X^{n-1}. (By Lemmas 5 and 3, X^{n-1} is everywhere wild in E^{n+1}.) Thus, there is an everywhere wild $(n-2)$-sphere in E^n for $n \geqq 3$.*

Theorem 1 follows directly from Lemma 6.

By using the techniques developed in [4] and [8], one ought to be able to show that any subarc of "Bing's sling" [6] and the wild arc of [1] satisfy the hypothesis of part (a) of Theorem 1. It would then follow that the 1-string obtained by one-point compactifying E^3 and then removing a point of "Bing's sling" and the 1-string obtained by doing Alford's construction down E^1 on $E^2 \subset E^3$ would satisfy the hypothesis of part (b) of Theorem 1.

5. Construction of everywhere wild cells and spheres in codimensions other than two.

EXAMPLE 4. *For $n \geqq 3$, there is an arc α_n in E^n such that $E^n - \alpha_n$ is not projectively 1-connected at α_n.*

The referee points out that it can be shown by the technique of proof of Theorem 1 of [16] that any arc α_n for which $\pi_1(E^n - \alpha_n) \neq 0$, $n \geqq 3$, is an example, and such arcs are explicit in [7]. (Furthermore, it follows from [16] that if $n \neq 4$, $E^n - \alpha_n$ is projectively 1-connected at α_n if and only if α_n is cellular and if $n = 4$, cellularity implies projective 1-connectivity while projective 1-connectivity implies that $\pi_1(E^n - \alpha_n) = 0$.)

We will now give a short explicit construction based on some work of [9]. If α_3 is the arc of Example 1.1 of [4], then $S_3 - \alpha_3$ is neither 1-connected nor projectively 1-connected at α_3. Suppose that $\alpha_n \subset S^n$ is an arc such that $S^n - \alpha_n$ is neither 1-connected nor projectively 1-connected at α_n, and let $a_{n+1} = S(\alpha_n) \subset S(S^n/\alpha_n)$. ($S$ denotes suspension.) It follows easily from the techniques of [3] that $S(S^n/\alpha_n) \approx S^{n+1}$. Also, since $S^n - \alpha_n$ is not 1-connected, $S(S^n/\alpha_n) - S(\alpha_n)$ is not 1-connected. If $S^{n+1} - \alpha_{n+1}$ were projectively 1-connected at α_{n+1}, there would be a neighborhood V of α_{n+1} in S^{n+1} such that every loop in $V - \alpha_{n+1}$ could be shrunk to a point in $S^{n+1} - \alpha_{n+1}$. Thus, any loop in $S(S^n/\alpha_n) - S(\alpha_n)$ could be pushed up the product structure into $V - \alpha_{n+1}$ and then be shrunk to a point, which would imply that $S(S^n/\alpha_n) - S(\alpha_n)$ is 1-connected. Hence, $S^{n+1} - \alpha_{n+1}$ is not projectively 1-connected at α_{n+1}. The arc α_n can be considered to be in E^n by removing a point of S^n not on α_n and so α_n satisfies Example 4.

Lemma 7. *Let $A \subset X \subset Y$ be topological spaces such that $Y - X$ is not 1-LC at any point of A. Let $W \subset Z \subset E^1$ where W is open in E^1. Then, $(Y \times E^1) - (X \times Z)$ is not 1-LC at any point of $A \times W$.*

The proof of Lemma 7 is similar to that of Lemma 2 and so we omit it.

Example 5. *For $n \geqq 4$, there is a closed 1-string $X_n{}^1$ in E^n such that $E^n - X_n{}^1$ fails to be 1-LC at every point of $X_n{}^1$. (Hence, $X_n{}^1$ fails to be locally flat at every point and is everywhere wild by Lemma* 3.)

Let $\alpha_{n-1} \subset E^{n-1}$ be the arc of Example 4. Then, $(E^{n-1}/\alpha_{n-1}) - \alpha_{n-1}$ fails to be 1-LC at α_{n-1}. Let $X_n{}^1 = \alpha_{n-1} \times E^1 \subset (E^{n-1}/\alpha_{n-1}) \times E^1$. By [3], $(E^{n-1}/\alpha_{n-1}) \times E^1 \approx E^n$, and by Lemma 7, $E^n - X_n{}^1$ fails to be 1-LC at every point of $X_n{}^1$.

Example 6. *For $n \geqq 4$, there is an arc β_n in E^n such that $E^n - \beta_n$ fails to be 1-LC at every interior point of β_n. (Hence, β_n is everywhere wild.) Furthermore, β_n is cellular.*

In the notation of the preceding paragraph and by the same reasoning, $\beta_n = \alpha_{n-1} \times [-1, 1] \subset (E^{n-1}/\alpha_{n-1}) \times E^1 \approx E^n$ is the desired arc. It is cellular by Lemma 4.

Example 7. *For integers n and k such that $n \geqq 3$, $0 < k < n$, and $n - k \neq 2$, there is a k-cell $D_n{}^k$ in E^n such that $E^n - D_n{}^k$ fails to be 1-LC at every interior point of $D_n{}^k$. (Hence, $D_n{}^k$ is everywhere wild.) Furthermore, $D_n{}^k$ is cellular.*

Let $D_3{}^2$ be any 2-cell on the wild sphere constructed in [**13**]. Then, $E^3 - D_3{}^2$ is not 1-LC at any interior point of $D_3{}^2$, however it is easy to see that $D_3{}^2$ satisfies the cellularity criterion of [**16**] and so is cellular in E^3. We will now assume that Example 7 holds in dimension n and show that it holds in dimension $n + 1$. The existence of D^1_{n+1} follows from Example 6. Let $k + 1$ be such that $1 < k + 1 < n + 1$ and $(n + 1) - (k + 1) \neq 2$. By assumption there is a k-cell $D_n{}^k \subset E^n$ which satisfies Example 7. Let $D^{k+1}_{n+1} = D_n{}^k \times [-1, 1] \subset E^n \times E^1 = E^{n+1}$. It follows from Lemma 7 that $E^{n+1} - D^{k+1}_{n+1}$ fails to be 1-LC at every interior point of D^{k+1}_{n+1} and it follows from Lemma 4 that D^{k+1}_{n+1} is cellular.

EXAMPLE 8. *For integers n and k such that* $n \geqq 3$, $0 < k < n$, *and* $n - k \neq 2$, *there is a closed k-string* $X_n{}^k$ *in* E^n *such that* $E^n - X_n{}^k$ *fails to be* 1-LC *at every point of* $X_n{}^k$. (*Hence,* $X_n{}^k$ *is everywhere wild.*)

One can obtain $X_3{}^2$ by one-point compactifying E^3 and then removing a point of the 2-sphere constructed in [**5**] or a point of the 2-sphere constructed in [**13**]. The existence of X^1_{n+1} follows from Example 5. By the reasoning of the preceding paragraph the existence of appropriate X^{k+1}_{n+1} follows by letting $X^{k+1}_{n+1} = X_n{}^k \times E^1 \subset E^n \times E^1 = E^{n+1}$ and applying Lemma 7.

EXAMPLE 9. *For integers n and k such that* $n \geqq 3$, $0 < k < n$, *and* $n - k \neq 2$, *there is an everywhere wild k-sphere* $S_n{}^k$ *in* E^n.

To get $S_n{}^k$, one-point compactify the pair $(X_n{}^k, E^n)$ of Example 8 and remove a point not on the resulting k-sphere.

6. **Noncellular everywhere wild cells.** Note that it follows from the corollary to Theorem 8 of [**16**] and Lemma 4 that product cells constructed by the methods of §§3, 4 and 5 will always be cellular. However, the next theorem can be used to generate noncellular everywhere wild cells.

THEOREM 2. *Let* $D^k \subset S^n$ *be a k-cell.*

(i) *If* $D^k - S^n$ *is not simply connected, then* $S(S^k) - S(D^k)$ *is not simply connected.* (*Hence,* $S(D^k)$ *is not cellular in* $S(S^n) \approx S^{n+1}$.)

(ii) *If* $n - k \neq 2$ $(n - k = 2)$ *and* $S^n - D^k$ *is not* 1-LC (1-SS, 1-*LA*) *at any point of* D^k, *then* $S(S^n) - S(D^k)$ *is not* 1-LC (1-SS, 1-*LA*) *at any point of* Int $S(D^k)$. (*Hence,* $S(D^k)$ *is everywhere wild in* $S(S^n) \approx S^{n+1}$.)

Part (i) of Theorem 2 is clear and part (ii) can be established by a proof similar to that of Lemma 2. Let us consider some examples of cells which are appropriate to plug into Theorem 2. First, if D^2 is some disk on Bing's hooked rug [5], then its complement is not simply connected and $S^3 - D^2$ is not 1-LC at any point of Int D^2. Following the statement of Example 4, we constructed an arc in S^n, $n \geqq 4$, whose complement was not 1-connected. It is also easy to see that this arc fails to be 1-LC at interior points.

7. **Everywhere wild arcs which pierce.** In [9], Morton Brown showed that if a noncellular arc in S^n, $n > 3$, is such that every proper subarc is cellular, then it can pierce no locally flat $(n-1)$-sphere. This motivated the author to ask the following question of his topology class at the University of Utah. *Can an everywhere wild arc in* S^n *pierce a locally flat* $(n-1)$*-sphere*? Orville Bierman, a student, originated the following example. Let Σ^{n-2} be a locally flat $(n-2)$-sphere in S^{n-1}, $n \geqq 4$, let α be an arc of Example 1 contained in one complementary domain of Σ^{n-2} and let β be another copy of α which is contained in the other complementary domain of Σ^{n-2}. Then, it follows from [3], that $S(S^{n-1}/\{\alpha, \beta\}) \approx S^n$. If v is one of the suspension points, then it follows from this paper or from [9] that $v * \alpha$ and $v * \beta$ are everywhere wild arcs in S^n. Hence, $A = (v * \alpha) \cup (v * \beta)$ is an everywhere wild arc. Let D^{n-1} be the closure of the complementary domain of Σ^{n-2} which contains α and let $\Sigma^{n-1} = D^{n-1} \cup (v * \Sigma^{n-2})$. It is clear that the $(n-1)$-sphere Σ^{n-1} is locally flat everywhere but at v, hence by [10], it is locally flat there. Then, A pierces the locally flat $(n-1)$-sphere Σ^{n-1} at v. Furthermore, it follows from [9] that A can pierce a locally flat $(n-1)$-sphere at no other point. C. E. Burgess pointed out that a similar example can be constructed in S^3 by running the construction of [1] up to a point v from two different directions. It is shown in [14] that the resulting arc can pierce a locally flat 2-sphere at no other point. Hence, we have the following example.

EXAMPLE 10. *For* $n \geqq 3$, *there is an everywhere wild arc* α *in* S^n *which can pierce locally flat* $(n-1)$*-spheres at one and only one point.*

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