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STABILITY IN WITT RINGS AND ABELIAN SUBGROUPS OF PRO-2-GALOIS GROUPS

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Let F denote a field of characteristic not two, let $F(2)$ be the maximal 2-extension (i.e., quadratic closure) of F, and let $G_F(2) = \mathrm{Gal}(F(2)/F)$. In this note we introduce and study an invariant, $a(F)$, of F defined to be the largest integer m such that $G_F(2)$ has a closed torsion free abelian subgroup of rank m. [If there is no largest m, we define $a(F) = \infty$]. Recall that the (absolute) *stability index* of F, introduced in [**5**], is $\mathrm{st}\,(F) = \min\{n \mid I^{n+1}F = 2I^nF\}$, where IF is the fundamental ideal of even dimensional forms in the Witt ring, WF, of anisotropic quadratic forms over F and I^nF is the n^{th} power of this ideal. The connection between $a(F)$ and $\mathrm{st}(F)$ will be investigated and it will be shown that if F is a pythagorian field or a finitely generated extension of a hereditarily euclidean or hereditarily quadratically closed base field, then $a(F) = \mathrm{st}\,(F)$. First we present a few examples.

EXAMPLES. (1). $a(F) = 0$ if and only if F is either euclidean (i.e., formally real with $|\dot{F}/\dot{F}^2| = 2$) or quadratically closed if and only if $F(\sqrt{-1})$ is quadratically closed.

(2). If F is a finite, local, or global field, then $a(F) = 1$.

(3). Let F be a rigid field (i.e., every element $a \notin \pm\dot{F}^2$ satisfies $F^2 + aF^2 \subseteq F^2 \cup aF^2$) with $|\dot{F}/\dot{F}^2| = 2^m > 2$. Then $a(F) = m$ if and only if F is nonreal and $F(\sqrt{-1})$ contains all 2-power roots of unity: $a(F) = m - 1$, otherwise. Special cases: $a(\mathbf{R}((t_1))\dots((t_m))) = a(\mathbf{C}((t_1))\dots((t_m)) = a(\mathbf{F}_p((t_1))\dots((t_m))) = m$.

(4). If $F = \mathbf{R}(t_1, \dots, t_n)$ or $\mathbf{C}(t_1, \dots, t_n)$, then $a(F) = n$.

PROOF. (1) is clear and (4) will follow from Theorem 3.

(2). This is clear if F is finite. If F is local or global it can be proved using standard algebraic number theory (cf. [**8**] for a corresponding result for the absolute Galois group). We give an argument that uses the Witt ring:

Let A be a closed torsion free abelian subgroup of $G_F(2)$ and let L be the fixed field of A. If rank $A = 2$ then, by [**15**, Theorem 3.6], L contains all 2-power roots of unity and $WL \cong \mathbf{Z}/2\mathbf{Z}[V]$, with $|V| = 4$. Hence, by Theorem 3.3 of [**17**], there is a valuation v on L with residue field L_v such that

$$|\dot{L}_v/\dot{L}_v^2| \leq 2 \text{ and } 1 + M_v \subseteq L^2$$

where M_v is the maximal ideal of the valuation ring of v. Also, because L/F is algebraic, $|\Gamma_v/2\Gamma_v| = 2$, where Γ_v is the value group. This forces $|\dot{L}_v/\dot{L}_v^2| = 2$ (so v is non dyadic) and implies that L_v contains all 2-power roots of unity. As L_v is necessarily an algebraic extension of a finite field, this is impossible. Hence $a(F) = 1$, as claimed.

(3). By [**15**], $G = G_F(2)$ is a metabelian pro-2-group, and, by [**9**, Theorem 4.12], there is a split exact sequence $1 \to A \to G \to \overline{G} \to 1$ with A free abelian and $\overline{G} \cong \mathbf{Z}_2$, the additive group of 2-adic integers, or $\overline{G} \cong \mathbf{Z}/2\mathbf{Z}$. In [**15**] it is shown that $G_F(2)$ is abelian if and only if F contains all 2-power roots of unity (as $m > 1$). Hence $a(F) = m$ in this case. Similarly, if F is nonreal and $F(\sqrt{-1})$ contains all 2-power roots of unity, then, by [**15**, Theorem 1.5], rank $G_F(2) =$ rank $G_{F(\sqrt{-1})}(2)$. Since $G_{F(\sqrt{-1})}$ has maximal rank among abelian subgroups of $G_F(2)$ and $m =$ rank $G_F(2)$ we have $a(F) = m$ in this case, as well. If $\overline{G} \cong \mathbf{Z}/2\mathbf{Z}$, then F is formally real, $A \cong G_{F(\sqrt{-1})}(2)$, and $a(F) =$ rank $A = m - 1$ (cf. [**2**, Chapter III]).

Thus it remains to consider the case when $\overline{G} \cong \mathbf{Z}_2$ and $F(\sqrt{-1})$ does not contain all 2-power roots of unity. If rank $A > 1$, then the fixed field L of A contains $F(\mu)$, where μ is the group of all 2-power roots of unity. Since $\text{Gal}(L/F) \cong \mathbf{Z}_2$ it follows that $\text{Gal}(F(\mu)/(F) \cong \mathbf{Z}_2$ or $\mathbf{Z}/2\mathbf{Z}$. If $\text{Gal}(F(\mu)/F) \cong \mathbf{Z}/2\mathbf{Z}$, then $F(\mu) = F(\sqrt{-1})$ (see, for example, [**9**, Lemma 4.1]), a case we have excluded. Hence $\text{Gal}(F(\mu)/F) \cong \mathbf{Z}_2$ and we obtain a (split) exact sequence $1 \to B \to G_F(2) \to \mathbf{Z}_2 \to 1$ where $B = \text{Gal}(F(2)/F(\mu))$. By [**15**], B is abelian and B is maximal among all abelian subgroups of $G_F(2)$. Hence $a(F) = a(B) =$ rank $G_F(2) - 1 = m - 1$. Finally, if rank $A = 1$ then rank $G_F(2) = 2$ and the only possible cases occur as pro-2-Galois groups over nondyadic local fields (see [**9**; Table 5.2, Lemma 4.1, and the remark following the proof of Lemma 4.3]). Since $a(F) = 1$ and $|\dot{F}/\dot{F}^2| = 4$ for a nondyadic local field this completes the proof. □

If G is an arbitrary pro-2-group we define $a(G) = \max\{m \mid G$ has a closed torsion free abelian subgroup of rank $m\}$. Recall that the *cohomological dimension* of a pro-2-group G is $\mathrm{cd}(G) = \min\{n | H^n(G, \mathbf{Z}/2\mathbf{Z}) \neq 0\}$ [**14**, I-17].

PROPOSITION. *Let G be a pro-2-group.*

(i) *If G is a torsion free abelian pro-2-group and if one of the numbers $a(G)$, $\mathrm{cd}(G)$ is finite, then both are finite and $a(G) = \mathrm{cd}(G)$.*

(ii) $a(G) \leq \mathrm{cd}(G)$, *in general.*

(iii) *If H is a closed subgroup of finite index in G, then $a(H) = a(G)$.*

(iv) *If N if a closed normal subgroup of G such that G/N is a torsion group, then $a(N) = a(G)$.*

(v) *Suppose $1 \to N \to G \to \overline{G} \to 1$ is an exact sequence of pro-2-groups with $\overline{G} \subseteq G_k(2)$ for some field k. Then $a(G) \leq a(\overline{G}) + a(N)$.*

PROOF (i). It is a consequence of Pontryagin duality that a torsion free abelian pro-2-group is isomorphic to $\mathbf{Z}_2^I$ for some index set I (where $\mathbf{Z}_2^I$ denotes the direct product of $|\mathbf{I}|$ copies of the additive group $\mathbf{Z}_2$ of 2-adic integers). Since $\mathrm{cd}(\mathbf{Z}_2) = 1$, (i) follows from [**14**, I-32,33] and induction.

(ii). This follows from (i) and [**14**, I-20].

(iii). Let A be a closed torsion free abelian subgroup of G. Then $(A : A \cap H) \leq (G : H) < \infty$. We may assume rank $A \cap H < \infty$. Then rank $A < \infty$ (as $|A/A \cap H| < \infty$) and, by (i), $a(A \cap H) = \mathrm{cd}(A \cap H)$, $a(A) = \mathrm{cd}(A)$. By [**14**, I-20], $\mathrm{cd}(A) = \mathrm{cd}(A \cap H)$, proving (iii).

(iv). Let A be a finitely generated closed torsion free abelian subgroup of G. It suffices to prove that $\operatorname{rank} A = \operatorname{rank} A \cap N$. Since G/N is a torsion group so is $A/A \cap N$, and because A is finitely generated (as a pro-2-group) this implies that $A/A \cap N$ is finite. Hence (iv) follows from (iii).

(v). Let A be a closed torsion free abelian subgroup of G. It suffices to show that $a(A) \leq a(A \cap N) + a(A/A \cap N)$. Since $A/A \cap N$ is a closed abelian subgroup of the pro-2-Galois group $G_k(2)$ there are only two possibilities: either $A/A \cap N$ is torsion free or $|A/A \cap N| = 2$. In the first case the sequence $1 \to A \cap N \to A \to A/A \cap N \to 1$ splits, so $a(A) = a(A \cap N) + a(A/A \cap N)$, and in the second, (iii) gives $a(A) = a(A \cap N)$. □

COROLLARY *If K/F is a finite 2-extension, then $a(F) = a(K)$.*

REMARKS 1. Concerning the inequality $a(G) \leq \mathrm{cd}(G)$, one can have $a(G) < \mathrm{cd}(G)$. For example, $\mathrm{cd}(\mathbf{Z}/2\mathbf{Z}) = \infty$ and $a(\mathbf{Z}/2\mathbf{Z}) = 0$; if F is a local or global field, then $\mathrm{cd}(G_F(2)) = 2$, while $a(G_F(2)) = 1$.

2. The inequality in (v) can be strict. For example, if $F = \mathbf{Q}_3$, then the 3-adic valuation on F induces an exact sequence $1 \to \mathbf{Z}_2 \to G_F(2) \to G_{\mathbf{F}_3}(2) \to 1$ and $G_{\mathbf{F}_3}(2) = \mathbf{Z}_2$, so the corresponding inequality is $1 < 1 + 1$.

3. There are finite extensions K/F with $a(K) > a(F)$. Let $F = \mathbf{Q}(2)$, the quadratic closure of the rationals. In [**11**, p. 219] it is shown that if K is any proper extension of F, then K is not quadratically closed. Hence $a(K) > 0$.

4. I suspect that $a(F) \leq a(K)$ for any finite extension K/F but I have not been able to prove this.

THEOREM 1. *Suppose F has a nondyadic 2-henselian valuation v with residue field F_v and value group Γ with $1 \leq |\Gamma/2\Gamma| < \infty$. Then*

(i). *$a(F) = \log_2 |\Gamma/2\Gamma| + a(F_v)$ if and only if either $G_{F_v}(2) \ncong \mathbf{Z}_2$ or $F_v(\mu) \neq F_v(2)$, where μ is the group of all 2-power roots of unity.*

(ii). *$a(F) = \log_2 |\Gamma/2\Gamma|$ if and only if $F_v(\mu) = F_v(2)$.*

PROOF. We need a lemma:

LEMMA. *Let $G = G_F(2)$. If there exists a split exact sequence $1 \to A \to G \to B \to 1$ of pro-2-groups with A, B abelian and rank $B > 1$, then G is abelian.*

PROOF. By Theorem 2.5 in [**9**] there is a field k such that $G_k(2) \cong B$ and $WF \cong Wk[A/A^2]$. Since $\operatorname{rank} B > 1$, $Wk \ncong \mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}$, so, by the Realization theorem of [**1**] (also see [**9**], the proof of Theorem 2.1), one can find a nondyadic 2-henselian valuation $v : F \to \Gamma$ with residue field F_v such that $A \cong \operatorname{Gal}(F_{nr}/F)$, where F_{nr} is the maximal nonramified 2-extension of F with respect to v. By valuation theory, $B \cong G_{F_v}(2)$ and $\operatorname{rank} B > 1$ implies F_v contains all 2-power roots of unity [**15**]. Since v is 2-henselian, F also contains all 2-power roots of unity. This implies that the metabelian group $G = G_F(2)$ is abelian [**16**].

(i). The 2-henselian valuation v gives rise to a split exact sequence $1 \to A \to G_F(2) \to G_{F_v}(2) \to 1$, where $A = G_{F_{nr}}(2)$ is abelian and $\operatorname{rank} A = \log_2 |\Gamma/2\Gamma|$. By Proposition, part (v), $a(F) \leq \log_2 |\Gamma/2\Gamma| + a(F_v)$. If $a(F_v) > 1$, then consider the split sequence $1 \to A \to G_0 \to B \to 1$, where $B \subseteq G_{F_v}(2)$ is a closed abelian subgroup of maximal rank and $G_0 \subseteq G_F(2)$. By the lemma, G_0 is abelian and $G_0 \cong A \times B$. Hence $a(F) \geq \log_2 |\Gamma/2\Gamma| + a(F_v)$, completing the proof in this case. If $a(F_v) = 0$, then A is a maximal abelian subgroup of $G_F(2)$ so $a(F) = \log_2 |\Gamma/2\Gamma| = \log_2 |\Gamma/2\Gamma| + a(F_v)$. Thus it remains to consider the case $a(F_v) = 1$.

Let μ be the group of all 2-power roots of unity and let $L = F(\mu)$. Since $\mu \subseteq F_{nr}$ we have a split sequence $1 \to A \to G_L(2) \to G_k(2) \to 1$, where $k = F_v(\mu)$. If $a(k) = 1$, then a closed torsion free abelian subgroup $B \subseteq G_k(2)$ with rank $B = 1$ gives rise to a split sequence $1 \to A \to G_0 \to B \to 1$ with $G_0 \subseteq G_L(2)$. As L contains μ, the metabelian group G_0 is abelian and $G_0 \cong A \times B$. Hence $a(F) \geq a(L) \geq \operatorname{rank} G_0 = \log_2 |\Gamma/2\Gamma| + 1 = \log |\Gamma/2\Gamma| + a(k_v)$, with Proposition (v) giving equality. If $a(k) = 0$, then $F_v(\mu) = F_v(2)$ and $G_{F_v}(2) \cong \mathbf{Z}_2$ (if $G_{F_v}(2) \cong \mathbf{Z}/2\mathbf{Z}$ or 1, then $a(F_v) = 0$). This proves that $a(F) = \log_2 |\Gamma/2\Gamma| + a(F_v)$, unless $F_v(\mu) = F_v(2)$ and $G_{F_v}(2) \cong \mathbf{Z}_2$. Finally we show that, conversely, these conditions imply that $a(F) \neq \log_2 |\Gamma/2\Gamma| + a(F_v)$.

Given $F_v(\mu) = F_v(2)$ and $G_{F_v}(2) \cong \mathbf{Z}_2$ it follows that $G_F(2)$ is a nonabelian, metabelian pro-2-group and $a(F_v) = 1$. Moreover, $\operatorname{rank} G_F(2) = \operatorname{rank} A + 1 = \log_2 |\Gamma/2\Gamma| + 1$. By Example (3), $a(F) = \operatorname{rank} G_F(2) - 1$. Hence $a(F) = \log_2 |\Gamma/2\Gamma| \neq \log_2 |\Gamma/2\Gamma| + a(F_v)$, completing the proof of (i).

(ii). This is contained in the proof of (i). □

COROLLARY 1. *Under the hypothesis of Theorem* 1, *either* $a(F) = \log_2 |\Gamma/2\Gamma| + a(F_v)$ *or* $a(F) = \log_2 |\Gamma/2\Gamma|$, *and if* $|\dot{F}_v/\dot{F}_v^2| \neq 2$, *then* $a(F) = \log_2 |\Gamma/2\Gamma| + a(F_v)$.

COROLLARY 2. *Suppose* F *has a nondyadic valuation* v *with residue field* F_v *and value group* Γ. *Then*

(i). $a(F) \geq \log_2 |\Gamma/2\Gamma|$.

(ii). *If* $G_{F_v}(2) \supsetneq \mathbf{Z}_2$ *or* $F_v(\mu) \neq F_v(2)$ (*in particular, if* $|\dot{F}_v/\dot{F}_v^2| \neq 2$) *then* $a(F) \geq \log_2 |\Gamma/2\Gamma| + a(F_v)$.

PROOF. Let (L, w) be a 2-henselization of (F, v) [**2**, Chapter II], [**4**]. Then (L, w) has the same value group and the same residue as (F, v). Since L/F is a 2-extension, $a(F) \geq a(L)$, so (i) and (ii) follow from Theorem 1. □

COROLLARY 3. $a(k(t_1, \ldots, t_n)) \geq n + a(k)$, *unless* $k(\mu) = k(2)$ *and* $G_k(2) \cong \mathbf{Z}_2$.

COROLLARY 4. *Let* F/k *be a finitely generated field extension with* $\operatorname{tr}\deg_k F = n$. *Then* $a(F) \geq n$.

PROOF. This follows from Corollary 2 and [**7**, Lemma 1]. □

REMARK. We will see that the inequality in Corollary 3 can be strict. For example, if k is quadratically closed but not hereditarily quadratically closed (e.g., $k = Q(2)$) it will follow from Theorem 3 that $a(k(t_1, \ldots, t_n)) > n$.

Recall that the *ν-invariant* of F, introduced by Elman and Lam in [**6**], is $\nu(F) = \min\{m \mid I^m F \text{ is torsion free}\}$ and the *reduced stability index* of F, defined by Bröcker [**3, 4**], is $\operatorname{st}_{\text{red}}(F) = \min\{n | 2^n W_{\text{red}} F \subseteq C(X, \mathbf{Z})\}$, where $W_{\text{red}} F$ is the reduced Witt ring of F and $C(X, \mathbf{Z})$ is the ring of continuous functions on the topological space, X, of orderings on F.

THEOREM 2. (i) *$a(F) \leq \nu(F(\sqrt{-1}) - 1$ and if F is nonreal, then $a(F) \leq \nu(F) - 1$.*

(ii) $\mathrm{st}_{\mathrm{red}}(F) \leq a(F) \leq \mathrm{st}(F) + 1$.

PROOF (i). Observe that if F is nonreal, then $\nu(F)$ is the index of nilpotence of the fundamental ideal IF. If $\nu(F(\sqrt{-1})) = 1$, then $WF(\sqrt{-1}) \cong \mathbf{Z}/2\mathbf{Z}$ so $F(\sqrt{-1})$ is quadratically closed and $a(F) = 0$. Hence, if $a(F) = 1$, then $\nu(F(\sqrt{-1})) \geq 2$ so we may assume $a(F) > 1$. Let A be a closed torsion free abelian subgroup of $G_F(2)$ of rank $m \geq 2$ and let L be its fixed field. By **15**, $WL \cong \mathbf{Z}/2\mathbf{Z}[\dot{L}/\dot{L}^2]$ and $\log_2 |\dot{L}/\dot{L}^2| = m$. Hence the index of nilpotence of the ideal IL is $m+1$. Since $\sqrt{-1} \in L$, [**6**, Theorem 6.3] implies that $\nu(F(\sqrt{-1})) \geq \nu(L)$, proving the first part of (ii). The second part of (ii) also follows from [**6**, Theorem 6.3].

(ii). The inequality $a(F) \leq \mathrm{st}\,(F) + 1$ follows from (i) and the inequality $\nu(F(\sqrt{-1})) - 2 \leq \mathrm{st}\,(F)$ [**6**, Corollary 4.7].

In [**2**, p. 143], Becker showed that $\mathrm{st}_{\mathrm{red}}(F) + 1$ is the largest integer n (or ∞) such that there is a 2-extension K/F with K pythagorean and $G_{K(\sqrt{-1})}(2)$ abelian, $\log_2 |\dot{K}/\dot{K}^2| = n$, and $K = FK^2$. As K is pythagorean, $\mathrm{rank}\, G_{K(\sqrt{-1})}(2) = n - 1$ so $a(F) \geq n - 1 = \mathrm{st}_{\mathrm{red}}\,(F)$.

REMARKS 1. It can happen that $a(F) = \mathrm{st}(F) - 1$; for example, if $F = \mathbf{Q}$, then $a(F) = 1$ and $\mathrm{st}(F) = 2$. However, I know no example where $a(F) \notin \{\mathrm{st}(F), \mathrm{st}(F) - 1\}$. In particular, I suspect that the inequality in (i) can be improved to $a(F) \leq \mathrm{st}(F)$. By Theorem 2(i), this is true if $\sqrt{-1} \in F$.

2. The inequality $\mathrm{st}_{\mathrm{red}}(F) \leq a(F)$ can be strict. In fact, let $F_1 = \mathbf{R}$ and let F_2 be the power series field $\mathbf{C}((\Gamma))$, where $\Gamma = \mathbf{Z}^{(I)}$ is an infinite direct sum of copies of $\mathbf{Z}$ ordered lexicographically (see, for example, [**2**; pp. 66, 119]). Then $WF_1 \cong \mathbf{Z}$, $WF_2 \cong \mathbf{Z}/2\mathbf{Z}\,[\Gamma/2\Gamma]$, $G_{F_1}(2) \cong \mathbf{Z}/2\mathbf{Z}$, and $G_{F_2}(2) \cong \mathbf{Z}_2^I$. By a construction of Kula [**10**], there is a field F such that $WF \cong WF_1 \times WF_2$, and, by a theorem of Jacob (see [**9**, Theorem 3.4]), $G_F(2)$ is the free product (in the category of pro-2-groups) of $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}_2^I$. In particular, $a(F) = \infty$. Since $WF \cong \mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}[\Gamma/2\Gamma]$, $W_{\mathrm{red}}F \cong \mathbf{Z}$ and $\mathrm{st}_{\mathrm{red}}(F) = 0$.

Theorem 2 has several corollaries:

COROLLARY 1. *Let F be a pythagorean field. Then* $a(F) = \mathrm{st}(F) = \nu(F(\sqrt{-1}) - 1$.

PROOF. By [**12**, Proposition 13.1], $\mathrm{st}(F) = \mathrm{st}_{\mathrm{red}}(F)$ and, because $\dot{F}/\dot{F}^2 \to F(\sqrt{-1})/F(\sqrt{-1})^2$ is surjective, $\mathrm{st}(F) = \nu(F(\sqrt{-1}) - 1$.

It appears to be an open question whether $\mathrm{st}(F) = \mathrm{cd}(G_{F(\sqrt{-1})}(2))$ for an arbitrary pythagorian field F, although it has been shown by Minac [**13**] when $|\dot{F}/\dot{F}^2| < \infty$. In any event, Corollary 1 and Proposition (ii) yield

COROLLARY 2. *Let F be a pythagorian field. If* $H^n(G_{F(\sqrt{-1})}(2), \mathbf{Z}/2\mathbf{Z}) = 0$, *then* $I^n F(\sqrt{-1}) = 0$.

COROLLARY 3. *If F is a formally real field with $a(F) \leq 1$, then F satisfies* SAP (*see* [**5**, **12**, **4**]).

COROLLARY 4. *Let F be a pythagorian field. Then $a(F) \leq 1$ if and only if $G_{F(\sqrt{-1})}(2)$ is a free pro-2-group.*

PROOF. Apply Corollary 3 and [**16**, Proposition 3.2].

COROLLARY 5. *Let F be a nonreal field. Then* $a(F) \leq \log_2 |\dot{F}/\dot{F}^2|$.

PROOF. We may assume $\log_2 |\dot{F}/\dot{F}^2| = n < \infty$. A result of Kneser states that every quadratic form of dimension $> 2^n$ is isotropic [**11**, p. 317], whence, by a theorem of Pfister, $I^{n+1}F = 0$ [**11**, p. 317]. Hence $a(F) \leq n$, as desired. □

REMARK. Corollary 5 should hold for formally real fields, as well. In fact, if F is pythagorian, then the inequality can be improved to $a(F) \leq \log_2 |\dot{F}/\dot{F}^2| - 1$. On the other hand, using Theorem 2 and [**11**, Theorem 3.4, p. 202], I have only been able to obtain the inequality $a(F) \leq 2\log_2|\dot{F}/\dot{F}^2| - 2$, in general.

THEOREM 3. (cf. [**7**]) *Let F/k be a finitely generated extension with* $\mathrm{tr\,deg}_k F = n > 1$. *Then $a(F) = n$ if and only if $k(\sqrt{-1})$ is hereditarily quadratically closed.*

PROOF. Assume $a(F) = n$ and let S be a simple algebraic extension of $k(\sqrt{-1})$. It suffices to show that S is quadratically closed. By [**7**, Lemma 1] there exists a k-valuation v of F with value group $\mathbf{Z}^n$ and residue field F_v such that $S \subseteq F_v$ and $[F_v : S] < \infty$. Let (L, w) be a 2-henselization of (F, v). Then $a(L) \leq n$ and we have a split exact sequence $1 \to A \to G_L(2) \to G_{F_v}(2) \to 1$ where A is abelian of rank n. By Theorem 1, $a(L) = n + a(F_v)$, unless $F_v(\mu) = F_v(2)$ and $G_{F_v}(2) \cong \mathbf{Z}_2$. If $a(L) = n + a(F_v)$, then $a(L) \leq n$ implies $a(F_v) = 0$; i.e., F_v is quadratically closed (since $\sqrt{-1} \in S$). On the other hand, suppose $F_v(\mu) = F_v(2)$. As $n > 1$, L contains the group μ of 2-power roots of unity and since (L, w) is 2-henselian with residue field F_v, F_v also contains μ. Hence F_v is quadratically closed in all cases, and, by a well-known theorem of Diller and Dress [**11**, p. 254], S is also quadratically closed.

Conversely, assume $k(\sqrt{-1}))$ is hereditarily quadratically closed. By Corollary 4 to Theorem 1, $a(F) \geq n$. However, by [**7**, Theorem] $\nu(F(\sqrt{-1})) = n + 1$. So, by Theorem 2, $a(F) \leq n$. □

REMARK. The field $F = \mathbf{F}_p(t)$ shows that the assumption $n > 1$ is necessary. However, it is only needed to prove the implication $a(F) = n \Rightarrow k(\sqrt{-1})$ is hereditarily quadratically closed.

COROLLARY 1. *If F/k is a finitely generated extension with $k(\sqrt{-1})$ hereditarily quadratically closed, then* $a(F) = \mathrm{st}(F) = \nu(F(\sqrt{-1})) - 1$.

PROOF. Apply Theorem 3 and [**7**, Theorem]. □

Elman and Wadsworth [**7**] prove, under the hypotheses of Corollary

1, that the cohomological 2-dimension of the *absolute* Galois group of F is equal to the stability index of F. I have not been able to determine whether $\mathrm{st}(F) = \mathrm{cd}(G_{F(\sqrt{-1})}(2))$. However, from Corollary 1 and Proposition, part (ii), we have

COROLLARY 2. *If F/k is a finitely generated extension with $k(\sqrt{-1})$ hereditarily quadratically closed, then $\mathrm{st}(F) \leq \mathrm{cd}(G_{F(\sqrt{-1})}(2))$. Hence, if $H^n(G_{F(\sqrt{-1})}(2), \mathbf{Z}/2\mathbf{Z}) = 0$ then $I^nF(\sqrt{-1}) = 0$.*

REFERENCES

1. J. Arason, R. Elman and B. Jacob, *Rigid elements, valuations, and realization of Witt rings*, J. Algebra, **110** (1987), 449-467.

2. E. Becker, "Hereditarily Pythagorean Fields and Orderings of Higher Level", Monografias de Matematica **29**, IMPA, Rio de Janeiro, 1978.

3. L. Bröcker, *Zur Theorie der quadratischen Formen über formal reelen Körpern*, Math. Ann. **210**, (1974) 233-256.

4. ———, *Characterization of fans and hereditarily pythagorean fields*, Math. Zeit. **151** (1976), 149-163.

5. R. Elman and T.Y. Lam, *Quadratic forms over formally real fields and pythagorean fields*, Amer. J. Math. **94** (1972), 1155-1194.

6. ——— and ———, *Quadratic forms under algebraic extensions*, Math. Ann. **219** (1976), 21-42.

7. ——— and A. Wadsworth, *Hereditarily quadratically closed fields*, J. Algebra **111** (1987), 475-482.

8. W.D. Geyer, *Unendliche algebraische Zahlkörper, über denen jede Gleichung auflösbar von beschränkter Stufe ist*, J. Number Theory **1** (1969), 346-374.

9. B. Jacob and R. Ware, *A recursive description of the maximal pro-2 Galois group via witt rings*, Math. Zeit, to appear.

10. M. Kula, *Fields with prescribed quadratic form schemes*, Math. Zeit. **167** (1979), 201-212.

11. T.Y. Lam, *The Algebraic Theory of Quadratic Forms*, Benjamin, New York, 1973.

12. ———, "Orderings, Valuations, and Quadratic Forms", CBMS, No. 52, AMS, 1983.

13. J. Minac, *Stability and cohomological dimension*, C.R. Math. Rep. Acad. Sci. Canada VIII (1986), 13-18.

14. J.P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math. **5**, Springer-Verlag, New York/Berlin, 1965.

15. R. Ware, *When are Witt Rings groups rings*? II, Pacific J. Math. **76** (1978), 541-564.

16. ———, *Quadratic forms and profinite 2-groups*, J. Algebra **58** (1979), 227-237.

17. ———, *Valuation rings and rigid elements in fields*, Canad. J. Math. **33** (1981), 1328-1325.

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