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FINITE SPACES OF SIGNATURES: RESEARCH ANNOUNCEMENT

VICTORIA POWERS

Mulcahy's Spaces of Signatures [**3**] provide an abstract setting for Becker and Rosenberg's reduced Witt rings of higher level [**1**], along the same lines as Marshall's Spaces of Orderings setting for the ordinary reduced Witt ring.

This note concerns a generalization of Marshall's result that a finite Space of Orderings arises in connection with a field [**2**].

DEFINITION. A Space of Signatures (SOS) is a pair (X, G), where G is an abelian group of even exponent, and $X \subseteq G^* = \mathrm{Hom}\,(G, \mu)$ (μ being the complex roots of 1), which satisfies certain axioms (see [**3**]).

(X, G) is said to be realizable when $(X, G) = (X_T, \dot{K}/\dot{T})$ for a preordered field $\{K, T\}$.

Let (X, G) be a SOS such that $|G| = 2^s$ for some $s \in \mathbf{N}$. Our main result is

THEOREM 1. *(X, G) is realizable.*

The general idea of the proof of Theorem 1 is Marshall's: We show that (X, G) can be built up from smaller SOS's, and since the building operations preserve realizability, we can use induction.

DEFINITION. (X, G) is said to be a *group extension* of an SOS (X_0, G_0) if $G_0 \subseteq G$ and $X = \{\sigma \in G^* : \sigma|_{G_0} \in X_0\}$. If we just say that (X, G) is a group extension, we mean that (X, G) is a group extension of an SOS (X_0, G_0) where $G_0 \neq G$.

An SOS (X, G) is said to be a *direct sum* of the SOS's (X_1, G_1) and

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(X_2, G_2) if $G \simeq G_1 \times G_2$ and $X = X_1 \cup X_2$, where X_1 and X_2 have the obvious embeddings in $G^* \simeq G_1^* \times G_2^*$.

The following is proven in [**4**]:

THEOREM 2. *A group extension of a realizable* SOS *is realizable, and a direct sum of realizable* SOS *'s is realizable.*

To prove Theorem 1 we need only show that (X, G) is either a group extension or a direct sum, for then we can use induction. (The smallest possible SOS is trivially realizable.)

We must first classify certain small SOS 's by brute force.

PROPOSITION 3. *If G has ≤ 3 cyclic summands, then (X, G) is either a direct sum or a group extension.*

The proof of Proposition 3 is long and complicated. We use Mulcahy's Rigidity Theorem [**3**, 5.5]: if G contains x such that both x and $-x$ are rigid, then (X, G) is a group extension. Note that in Marshall's case, i.e., when $G^2 = 1$, Proposition 3 is trivial.

If $\sigma, \tau \in X$ we say σ is *equivalent* to τ, written $\sigma \sim \tau$, when there exists $\alpha \in X \backslash \{\sigma^{-1}, \tau^{-1}\}$ such that $\sigma\tau\alpha \in X$.

This is actually an equivalence relation on X, but the proof that $\sim$ is transitive requires most of the work needed for the proof of our next result.

THEOREM 4. *If X has only one equivalence class w.r.t. $\sim$, then (X, G) is a group extension.*

Finally we have

THEOREM 5. *If X has more than one equivalence class w.r.t. $\sim$, then (X, G) is a direct sum.*

Putting everything together Theorem 1 is now proven.

Of course we would like to prove Theorem 1 for finite SOS 's without the 2-power assumption on the order of G. Unfortunately many of our methods do not seem to extend to the general case.

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII AT MANOA, HONOLULU, HI 96822