ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

# THETA SERIES OF DEGREE 2 OF QUATERNARY QUADRATIC FORMS

R. SCHULZE-PILLOT

**Introduction .** The purpose of this note is to investigate linear combinations of theta series of degree two of quaternary positive definite integral quadratic forms from a representation theoretic point of view, using the expression of theta series with the help of the oscillator or Weil representation of the metaplectic group. In the case that the quadratic form is the norm form of a definite quaternion algebra we find that certain linear combinations of the theta series of the forms in a given (similitude) genus are related to the Maass spezialschar (also called Saito-Kurokawa space). Such a connection did (in special cases) already appear in the work of Yoshida [**15, 16**]. Using Andrianov's zeta functions [**1, 2**] one can use these results to deduce relations between representation numbers. As always, the representation theoretic formulation of the results generalizes beyond the original setting and allows number fields and indefinite forms whenever the involved integrals converge.

I wish to thank S. Rallis, who helped me understand the representation theoretic situation considered here. I also wish to thank the University of Maryland for its hospitality.

**1. Basics.** Let $V$ be a vector space over $\mathbf{Q}$ of dimension $m$ with positive definite quadratic form $q$ and bilinear form $B(x,y) = q(x+y) - q(x) - q(y)$. The group of proper similitudes of $(V,q)$ will be denoted by $\mathrm{GSO}(V)$. For a lattice $L$ on $V$ ($q(L) \subseteq \mathbf{Z}$) the theta series of degree $n$ is

$$\theta(L,z) = \sum_{x\in L^n} \exp(2\pi i \operatorname{tr}(q(x)Z))$$

(where $Z \in \mathcal{H}_n$, the Siegel upper half space of genus $n$, $x = (x_1,\dots,x_n)$ and $q(x) = 1/2(B(x_i,x_j))$ is the matrix of scalar products of $x_1,\dots,x_n$ with respect to $B$).

Work done at the University of Maryland, College Park, and supported by a Forschungsstipendium of the Deutsche Forschungsgemeinschaft.

It is now standard (see, e.g. [**16**]) that $\theta(L,Z)$ can be expressed via the Weil representation of the metaplectic group: If $\psi$ is a nontrivial character of $\mathbf{Q}\backslash\mathbf{Q}_A, W = W' \oplus W''$ a $2n$-dimensional nondegenerate symplectic space with polarization $W', W''$, let $w_\psi$ denote the Weil representation of $\tilde{\mathrm{Sp}}(W\otimes V)_A$ acting on the Schwartz space

$$S((W'\otimes V)_A) \simeq S(V_A^n)$$

(for formulas see, e.g., [**16**, §2] or [**8**, II. 6]) and put

$$\tilde{\theta}_f(g) = \int_{\mathrm{so}(v)\backslash \mathrm{so}(v)\mathrm{so}_A(L)} w_\psi((g,1)) \sum_{x\in V^n} f(h^{-1}x)dh.$$

With $f_p$ the characteristic function of $L_p^n$ for all finite $p$ and $f_\infty(x)$ $= \exp(-2\pi tr q(x))$ one then has (with $g_\infty = \begin{pmatrix} T & X\cdot{}^tT^{-1} \\ 0 & {}^tT^{-1}\end{pmatrix}, T^tT = Y$, $X+iY \in \mathcal{H}_n, g_p = \mathrm{Id}_{2n}$ for all finite $p$)

$$\tilde{\theta}_f(g) = \frac{1}{|O(L)|}\tilde{\theta}(L, X+iY) \quad [\mathbf{16},\ 2.15\ ].$$

In this language, Siegel's theorem becomes the assertion that

$$\int_{\mathrm{so}(v)\backslash \mathrm{so}_A(V)} w_\psi((g,1)) \sum_{x\in v^n} f(h^{-1}x)dh$$

is an Eisenstein series [**14**] and assertions about other linear combinations of theta series are transformed into assertions about theta liftings

$$\theta_f(\varphi) = \int_{\mathrm{so}(v)\backslash \mathrm{so}_A(v)} \varphi(h) w_\psi((g,1)) \sum_{x\in V^n} f(h^{-1}x)dh$$

of an automorphic form $\varphi$ on $\mathrm{SO}_A(V)$ to $\tilde{\mathrm{Sp}}(W)$. For example, the statements about theta series of a spinor genus (Satz 3 of [**12**]) and the difference of theta series of forms belonging to the same spinor genus (Satz 4 of [**12**]) can be derived from Proposition 20 of [**13**]. Also, replacing $f_p$ by the characteristic function of $\{x \in L_p | x \equiv x_0 \bmod p^{\nu_p}\}$ for finitely many $p$ and some $x_0$ gives results about theta series with congruence conditions.

**2. The theta correspondence for the orthogonal group of a quaternionic norm form.** Let now $V = D$ be a definite quaternion algebra with norm form $q$. GSO$(D)$ can then [**3** §5.4] be identified with $\{(a,a)|a \in F^x\}\backslash D^x \times D^x$ by$(d_1,d_2)(x) = d_1 x d_2^{-1}$, the special orthogonal group SO$(D)$ is then identified with the image of $\{(d_1,d_2) \in D_A^x \times D^x | q(d_1) = q(d_2)\}$. As in [**15, 16**] we will consider liftings of functions of the type $\varphi_1 \otimes \varphi_2$ on $\mathrm{SO}_A(D)$ where $\varphi_1, \varphi_2$ are automorphic forms on $PD_A^x = D_A^x/\mathbf{Q}_A^x$ with

$$\int_{PD^x\backslash PD_A^x} \varphi_1(x)\varphi_2(x)dx = 0.$$

Corollary 5.4 of [**4**] guarantees then that $\theta_f(\varphi_1 \otimes \varphi_2)$ is cuspidal. We assume $\varphi_i$ to be taken from the space of an irreducible automorphic representation $\pi_i$ of $\mathrm{PD}_A^x$ (acting by right translation). $\pi_i$ factors into a tensor product $\pi_i \simeq \otimes_v \pi_v^{(i)}$ over all places $v$ of $\mathbf{Q}$ where $\pi_v^{(i)}$ is unramified for almost all $v$, i.e., $\pi_v^{(i)}$ is induced from an unramified character $\mu_v^{(i)}$ of the torus of $\mathrm{PD}_v^x(\simeq \mathrm{PGL}_2(\mathbf{Q}_v)$ for almost all $v)$. Since the element $\mathrm{diag}(s_1, s_2, s_1^{-1}, s_2^{-1})$ of the torus of SO$(D_v)$ maps onto $(\mathrm{diag}(s_1, s_2^{-1}), \mathrm{diag}(s_1, s_2))$ in the torus of $\mathrm{PGL}_2(\mathbf{Q}_v) \times \mathrm{PGL}_2(\mathbf{Q}_v)$, the given representation of SO$(D_v)$ is induced from the character

$$\mathrm{diag}(s_1, s_2, s_1^{-1}, s_2^{-1}) \to \mu_v^{(1)}\mu_v^{(2)}(s_1)\mu_v^{(1)}(\mu_v^{(2)})^{-1}(s_2)$$

of the standard torus of SO$(D_v)$.

It should be noted here that we get the same character on the torus of SO$(D_v)$ if we replace $\mu_v^{(1)}, \mu_v^{(2)}$ by $\mu_v^{(1)}\chi_v, \mu_v^{(2)}\chi_v$ with a quadratic character $\chi_v$.

Using a result of Rallis [**11**] (see also [**7**, Theorem 2.5]) we see that the theta lift of $\pi_1 \otimes \pi_2$ (restricted to SO$(D_A)$) is factored as $\otimes_v \tau_v$ where $\tau_v$ is induced from the character

$$\mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1}) \to \mu_v^{(1)}\mu_v^{(2)}(t_1)\mu_v^{(1)}(\mu_v^{(2)})^{-1}(t_2)$$

at almost all $v$.

In order to compare with Piatetski-Shapiro's characterization of the Saito-Kurokawa space [**9**; Lemma 7.1 and Theorem 2.2], we have to consider which characters of the torus of $\mathrm{PGSp}_2 \simeq \mathrm{SO}(3,2)$ restrict to the character of the torus of $\mathrm{Sp}_2$ given above. Since the element $\operatorname{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})$ of the torus of $\mathrm{Sp}_2$ maps onto $\operatorname{diag}(t_1 t_2^{-1}, t_1 t_2, t_1^{-1} t_2^{-1}, t_1^{-1} t_2, 1)$ in the torus of $\mathrm{SO}(3,2)$, all characters

$$\operatorname{diag}(u_1, u_2 u_1^{-1}, u_2^{-1}, 1) \to \chi_v \mu_v^{(1)}(u_1) \chi_v \mu_v^{(2)}(u_2)$$

($\chi_v$ a quadratic character) have this property.

As $\mu_v^{(1)}$ and $\mu_v^{(2)}$ are determined only up to multiplication with (the same) quadratic character anyway, this ambiguity has to be expected. Also, it is not hard to see that there is a quadratic character $\chi$ on $\mathbf{Q}_A^x/\mathbf{Q}^x$ such that $\chi_v$ is the restriction of $\chi$ to $\mathbf{Q}_A^x$ at almost all places.

If $\pi_1$ is the trivial representation we have, in particular,

THEOREM. *Let $\varphi$ be in the space of an irreducible cuspidal representation $\pi$ of $\mathrm{PD}_A^x/\mathrm{PD}^x$. Then the theta lift $\theta_f(1 \otimes \varphi)$ lies in a representation space of $\mathrm{Sp}_2(\mathbf{Q}_A)$ which is obtained by restricting to $\mathrm{Sp}_2(\mathbf{Q}_A)$ a representation of $\mathrm{PGS}p_2(\mathbf{Q}_A)$ of the form $\chi \otimes \rho$ where $\rho$ is in the Saito-Kurokawa space and $\chi$ is a (possibly trivial) quadratic character of $\mathbf{Q}_A^x/Qx$.*

**3. Concluding remarks.** The ambiguity in the statement of the theorem should not surprise. In fact, in the general situation (e.g., $L \simeq \langle 1, 5, 5, 5^2 \rangle$) the genus of the lattice will represent some binary forms of a given discriminant and not represent others (e.g, $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$), so that even the theta series of the genus (an Eisenstein series) does not satisfy the Maas relations. Also, the same problem arises when one generalizes Yoshida's methods. In order to obtain eigenfunctions of the operators $T(1,1,p,p)$ it is not enough to look at linear combinations of theta series of lattices in a fixed genus. Instead one has to consider lattices in a similitude genus. Depending on the character of the norms of similitudes introduced into the summation, one obtains the different possibilities of the theorem, which, in terms of $T(p)$ eigenvalues, says that the lifted function (if it is an eigenfunction) has $T(1,1,p,p)$ eigenvalue $\chi(p)(p+1+\lambda_p)$ where $\lambda_p$ is the $T(p)$ eigenvalue of $\varphi$. It appears likely that the ambiguity could be resolved with the help of an appropriate generalization of the results of Rallis and Kudla to the lifting from $\mathrm{GSO}_A(D)$ to $\mathrm{GSp}_2$, since the different possibilities of summation of the theta series mentioned above correspond to different extensions to $\mathrm{GSO}_A(D)$ of the form on $\mathrm{SO}_A(D)$ to be lifted.

REMARK. (added at revision, August 1987). A calculation given in [**10**] (Proof of Theorem 1.1) does indeed afford this generalization in the present situation. In particular, the theta lift of $1 \otimes \varphi$ is now seen to be in the Saito-Kurokawa space if the representations $1 \otimes \pi_v$ of the group of proper similitudes are induced from a character trivial on the similitude norm).

Finally, as in [**15, 16**] our theorem allows us to compute almost all Euler factors of the zeta functions associated to the lifted function by Andrianov [**1, 2**] and thus to relate the Fourier coefficients of $T$ and $T'$ where either $T'$ is similar to $T$ [**1**] or represented by $T$ [**2**]. As a possible application consider the case when $\varphi$ is a quadratic character of the spinor norm. The theta lift of $1 \otimes \varphi$ is then the difference of the degree two theta series of two half genera, and from the arithmetic results [**6, 5**], it is well known to have nonzero Fourier coefficients only for binary matrices $N$ with discriminant in a fixed square class. The eigenvalue of $\theta_f(1 \otimes \varphi)$ under the Hecke operator $T(1,1,p,p)$ is $2p+2$ or 0 for almost all $p$, depending on whether $\varphi_p(p) = +1$ or $-1$. If the discriminant of $N$ is prime to $p$ and we denote by $a(N)$ the Fourier coefficient of $\theta_f 1 \otimes \varphi)$ at the matrix $N$, we see immediately that

$$a(pN) = \begin{cases} 2p\ a\ (N) & \text{if } \varphi_p(p) = +1 \\ 0 & \text{if } \varphi_p(p) = -1 \end{cases}$$

for almost all $p$.

If $\theta_f(1 \otimes \varphi)$ is in the Saito-Kurokawa space, we might expect the Maass relation to hold true at least when restricted to $N$ with the g.c.d. of the coefficients prime to the level of the lattice $L$. Under this assumption, we obtain, from the above,

$$a\begin{pmatrix} 1 & \frac{pr}{2} \\ \frac{pr}{2} & p^2mn \end{pmatrix} = \varphi_p(p)p \cdot a \begin{pmatrix} 1 & r/2 \\ r/2 & mn \end{pmatrix}$$

for almost all $p$ and $\gcd(n,m,r)$ prime to the level; a formula similar to the result of [**12**] on ordinary theta series of half genera of ternary quadratic forms.

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INSTITUT FÜR MATHEMATIK II, FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 3 D-100 BERLIN 33, WEST GERMANY