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PROPER EMBEDDING INTO A UNIT LATTICE

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0. Introduction. An n-dimensional quadratic lattice is a free module of rank n over the rational integer ring $\mathbf{Z}$, which is endowed with a symmetric bilinear form B. Let E_n be a unit lattice, that is an n-dimensional quadratic lattice which has an orthonormal basis with respect to B, i.e.,

$$E_n = \mathbf{Z}e_1 + \cdots + \mathbf{Z}e_n, \quad B(e_i, e_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Let A be a positive integer. A sublattice F of E_n is an r-frame of scale A if

$$F = \mathbf{Z}f_1 + \cdots + \mathbf{Z}f_r, \quad B(f_i, f_j) = A\delta_{ij}.$$

A frame F in E_n is proper if $B(F, e_j) \neq \{0\}$ for each j. In this situation we have a problem:

($*$) When does E_n contain a proper r-frame of scale A?

We shall give a complete answer in the case of $r = 2$. Why proper? The Siegel Mass Formula can answer the question: When does E_n contain an r-frame of scale A?

This problem leads to diophantine equations in the following:

(#) E_n contains a proper 1-frame of scale A if and only if there are integers $x_1, \ldots, x_n$ in $\mathbf{Z}$ satisfying

$$x_1^2 + \cdots + x_n^2 = A, \quad x_1 \neq 0, \ldots, x_n \neq 0;$$

(##) E_n contains a proper 2-frame of scale A if and only if there are integers $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathbf{Z}$ satisfying

$$x_1^2 + \cdots + x_n^2 = y_1^2 + \cdots + y_n^2 = A, \quad x_1 y_1 + \cdots + x_n y_n = 0,$$
$$x_1 \neq 0 \text{ or } y_1 \neq 0, \ldots, x_n \neq 0 \text{ or } y_n \neq 0.$$

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T. Ono proposed an interesting problem - a skew hanging of picture frames - which leads to the above problem $(*)$ in the case of $n = 3$ and $r = 2$.

We can write $A = 2^e A_1 A_3$, where $p \equiv j \bmod 4$ for all prime divisors p of $A_j (j = 1, 3)$.

1. 1-frames. This case is to characterize the set S_n of sums of n non-zero squares by $(\#)$.

THEOREM 1. *Let $n \neq 3$. E_n contains a proper 1-frame of scale A if and only if one of the following is satisfied:*

(1) $n = 1$, *A is a square;*

(2) $n = 2$, *A_3 is a square, e is odd or $A_1 > 1$;*

(3) $n = 4$, $A \neq 1, 3, 5, 9, 11, 17, 29, 41, 2 \cdot 4^k, 6 \cdot 4^k, 14 \cdot 4^k$;

(4) $n = 5$, $A \neq 1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18, 33$;

(5) $n \geq 6$, $A \neq 1, 2, \ldots, n-1, n+1, n+2, n+4, n+5, n+7, n+10, n+13$.

PROOF. It is classical for $n = 1, 2$. Assume $n \geq 4$. All the conditions are clearly necessary by direct calculations. In case $n = 4$, it suffices to show that $A \in S_4$ if $A \not\equiv 0 \bmod 8$ and $A \neq 1, 3, 5, 9, 11, 17, 29, 41, 2, 6, 14$. When $A \not\equiv 1 \bmod 4$, we define an integer C by $A = a^2 + C$, where $a = 1$ (if $A \equiv 4, 7 \bmod 8$), $a = 2$ (if $A \equiv 2 \bmod 8$), and $a = 4$ (if $A \equiv 3, 6 \bmod 8$). When $A \equiv 1 \bmod 4$, we define an integer C by $A = a^2 + 4C$, where $a = 1$ (if $A \equiv 13, 25 \bmod 32$), $a = 3$ (if $A \equiv 1, 21 \bmod 32$), $a = 5$ (if $A \equiv 5, 17 \bmod 32$), and $a = 7$ (if $A \equiv 9, 29 \bmod 32$). Then C is a positive integer with $C \equiv 3, 6 \bmod 8$; Hence we have $C \in S_3$ by a classical result. Therefore $A \in S_4$. In case $n \geq 5$, we use induction on n. Put $T_n = \{n, n+3, n+6, n+8, n+9, n+11, n+12\} \cup \{m \in \mathbf{Z} : m \geq n+14\}$. Take $A \in T_5$ with $A \neq 33$. If $A \leq 45$, then we have $A \in S_5$ by direct calculations. If $A > 45$, then either $A - 1^2$ or $A - 2^2$ is an odd positive integer > 41. Hence it is a sum of four non-zero squares from the case of $n = 4$. Then we have $A \in S_5$. Take $A \in T_n$ with $n \geq 6$. Then we have $A - 1^2 \in T_{n-1}$. By the inductive hypothesis, we have $A - 1^2 \in S_{n-1}$ (except $n = 6$ and $A = 34$). Thus $A \in S_n$ (and

$34 = 2^2 + 2^2 + 2^2 + 2^2 + 3^2 + 3^2$). $\square$

REMARK. In case $n = 3$, the problem is: what A is a sum of three non-zero squares? This is a famous and open problem. We may assume that $A \equiv 1, 2, 5 \bmod 8$ (because we know the following: $A \in S_3$ if and only if $4A \in S_3$, $A \in S_3$ if $A \equiv 3, 6 \bmod 8$, $A \notin S_3$ if $A \equiv 7 \bmod 8$). Under the notation $A = 2^e A_1 A_3$, we may assume that A_3 is a square (because we know that $A \in S_3$ if A_3 is not a square). If A contains an odd square > 1, then we have

$$A = u^2 + v^2 + w^2 \text{ or } A = u^2 = v^2 + w^2$$

for some non-zero integers u, v, w by Theorem 2 (in case $n = 3$) in the next section. If A is the first, then we have $A \in S_3$. If A is the second, then u contains a prime $p \equiv 1 \bmod 4$ since $vw \neq 0$. First suppose that $A \neq p^2$. The same argument implies that $A/p^2 \in S_3$ or $A/p^2 = v_1^2 + w_1^2$ with $v_1 w_1 \neq 0$. Hence $A \in S_3$ or $A = (pv_1)^2 + (pw_1)^2 = (pv_1)^2 + ((s^2 - t^2)w_1)^2 + (2stw_1)^2 \in S_3$, since we can write $p = s^2 + t^2$ with $s > t > 0$. Let $A = p^2 = (s^2 - t^2)^2 + (2st)^2$. If $s \equiv 0$ or $t \equiv 0$ or $s^2 \equiv t^2 \bmod 5$, then we have $A \in S_3$ since $(5r)^2 = (3r)^2 + (4r)^2$. Otherwise, we have $s^2 \equiv -t^2 \bmod 5$, so $p = 5$. We note that $25 \notin S_3$. After all we may consider the case where A is square-free, $A > 1$, and $p \equiv 1 \bmod 4$ for all prime divisors of A. In this case, we have the known formulas

$$\#\{(a, b) \in \mathbf{Z}^2 : a^2 + b^2 + c^2 = A\} = 2^{t+1}$$

and

$$\#\{(a, b, c) \in \mathbf{Z}^3 : a^2 + b^2 + c^2 = A\} = 12h(-A) \geq 12 \cdot 2^{t-1},$$

where t is the number of distinct primes in $4a$, and $h(-A)$ is the ideal class number of quadratic field $\mathbf{Q}(\sqrt{-A})$. Hence we have $A \in S_3$ if and only if $h(-A) = 2^{t-1}$. This result shows that $A \in S_3$ if and only if each genus contains only one ideal class in the quadratic field $\mathbf{Q}(\sqrt{-A})$. A numerical example shows that if $A < 1376256$, then it occurs if and only if $A = 1, 2, 5, 10, 13, 25, 37, 58, 85, 130$.

2. 2-frames.

THEOREM 2. *E_n contains a proper 2-frame of scale A if and only if one of the following is satisfied:*

(1) $n = 2$, *A_3 is a square;*

(2) $n = 3$, *A_3 is a square, A contains an odd square* > 1;

(3) $n \geq 4$, *n is even,* $A \geq n/2$;

(4) $n = 5$, $A \neq 1, 2, 3, 5, 6, 9, 21$;

(5) $n = 7$, $A \neq 1, 2, 3, 4, 7$;

(6) $n \geq 9$, *n is odd,* $A \geq (n+3)/2$.

PROOF. Consider the Gaussian integer ring $\Gamma = \mathbf{Z}[i]$ with $i = \sqrt{-1}$. Putting $z_j = x_j + y_j i$ in (##), the lattice E_n contains a proper 2-frame of scale A if and only if

($) there are Gaussian integers $z_1, \ldots, z_n \in \Gamma$ satisfying

$$z_1^2 + \cdots + z_n^2 = 0, \quad N(z_1) + \cdots + N(z_n) = 2A, \quad z_1 \neq 0, \ldots, z_n \neq 0,$$

where $N(z) = z\overline{z}$ is the norm of z. If the condition ($) holds, then we have $n \geq 2$.

(1). *Case* $n = 2$. By ($), we have $z_2^2 = -z_1^2$, so $N(z_1) = N(z_2)$. Hence $A = N(z_1)$, which shows A_3 is a square. Conversely assume that A_3 is a square. Then we have $A = N(z_1)$ with $0 \neq z_1 \in \Gamma$. Putting $z_2 = iz_1$, we see that the condition ($) holds.

(2). *Case* $n = 3$. Assume that the condition ($) holds. By δ we denote a G.C.D. of z_1, z_2 and z_3. Put $z_j = \delta w_j$ with $w_j \in \Gamma$. Take a prime $\pi = 1 + i$. We may suppose that $w_1 \equiv w_2 \equiv 1,\ w_3 \equiv 0 \bmod \pi$, since $w_1^2 + w_2^2 + w_3^2 = 0$. Then we have $w_3^2 \equiv 0 \bmod 4$, by noticing that $w_1^2 \equiv \pm 1,\ w_2^2 \equiv \pm 1,\ w_3^2 \equiv 0,\ 2i \bmod 4$ and $w_1^2 + w_2^2 + w_3^2 \equiv 0 \bmod 4$. Hence we see that

$$(iw_1 + w_2)/2 = \beta^2 \varepsilon, \quad (iw_1 - w_2)/2 = \gamma^2 \varepsilon^{-1}, \quad w_3 = 2\beta\gamma$$

with $\beta, \gamma \in \Gamma$ and $\varepsilon \in \{\pm 1, \pm i\}$. Hence we have

$$w_1 = -i(\beta^2 \varepsilon + \gamma^2 \varepsilon^{-1}), \quad w_2 = \beta^2 \varepsilon - \gamma^2 \varepsilon^{-1},$$

whence $2A = N(\delta)(N(w_1) + N(w_2) + N(w_3)) = 2N(\delta)(N(\beta) + N(\gamma))^2$. Using the fact

$$1 \equiv w_2 \equiv \beta^2 - \gamma^2 \equiv \beta - \gamma \bmod \pi \ \text{ and } \ \beta\gamma \neq 0,$$

we see that $N(\beta)+N(\gamma)$ is odd and greater than 1, and that A contains an odd square > 1. We note that $N(\delta)$ is of form $2^k a_1 a_3$ with a_3 a square (see the end of §0).

Conversely assume that A_3 is a square and A contains an odd square $a^2 > 1$. Thus we can write $A = N(\delta)a^2$ with $\delta \in \Gamma$. A classical result shows that $a = x_1^2 + x_2^2 + x_3^2 + x_4^2$ with $x_1, x_2, x_3, x_4 \in \mathbf{Z}$ and $x_1 x_3 \neq 0$. Put $\beta = x_1 + ix_2$, $\gamma = x_3 + ix_4$, $w_1 = -i(\beta^2 + \gamma^2)$, $w_2 = \beta^2 - \gamma^2$, $w_3 = 2\beta\gamma$ and $z_j = \delta w_j$. Then we have $w_1 w_2 w_3 \neq 0$ from the fact that a is odd. Hence the condition ($) holds.

(3). *Case* $n \geq 4$, even. The necessity is clear. Assume that $2A \geq n$. Then we have $A+2-n/2 = a_1^2+a_2^2+a_3^3+a_4^2$ with $a_1, a_2, a_3, a_4 \in \mathbf{Z}$ and $a_1 a_2 \neq 0$, since $A+2-n/2 \geq 2$. Put $z_1 = a_1 + ia_3, z_2 = a_2 + ia_4, z_3 = a_3 - ia_1, z_4 = a_4 - ia_2, z_5 = \cdots = z_m = 1$ and $z_{m+1} = \cdots = z_n = i$, where $m = 2 + n/2$. Thus the condition ($) holds.

(4). *Case* $n = 5$. The necessity is clear. We shall show the sufficiency. By Lemma 4 below, if $A \neq 5, 7, 9, 15, 21, 39, 4^k, 2 \cdot 4^k, 3 \cdot 4^k, 6 \cdot 4^k$, then we can write

$$A = a^2 + b^2 + g^2(c^2 + d^2),$$

where $a, b, c, d \in \mathbf{Z}$, $ac \neq 0$, $g > 1$ and g odd. From the case $n = 3$, there are z_3, z_4, z_5 in Γ satisfying

$$z_3^2 + z_4^2 + z_5^2 = 0, \quad N(z_3) + N(z_4) + N(z_5) = 0, \quad z_3 z_4 z_5 \neq 0.$$

Putting $z_1 = a + bi$ and $z_2 = b - ai$, we see that the condition ($) holds.

We write $z = (z_1, z_2, z_3, z_4, z_5)$ shortly. If $A = 7, 15, 39$, then we put $z = (2, 1+2i, 1-i, 1-i, i), (3+i, 2-2i, 1+i, 1, 3i), (6, 1-3i, 1+2i, 1+i, 5i)$ respectively. If $A = 4^k, 2 \cdot 4^k, 3 \cdot 4^k, 6 \cdot 4^k$ with $k \geq 1$, then we put $z = 2^{k-1}(2, i, i, i, i), 2^{k-1}(2-2i, 1+i, 1+i, 1+i, 1+i), 2^{k-1}(3, 1, 1+2i, 1-2i, 2i), 2^{k-1}(4-2i, 2+3i, 2+i, i, 3i)$ respectively. Thus ($) holds.

(5). *Case* $n = 7$. The necessity is clear. If $A - 1 \neq 0, 1, 2, 3, 5, 6, 9, 21$, then ($) holds, using the z_j's in the case $n = 5$ for $A-1$ and putting $z_6 = 1, z_7 = i$. If $A - 2 \neq -1, 0, 1, 2, 3, 5, 6, 9, 21$, then we use the z_j's in the case $n = 5$ for $A-2$ and put $z_6 = 1+i, z_7 = 1-i$. Hence if $A \neq 1, 2, 3, 4, 7$, then ($) holds. We shall give another proof without using the case $n = 5$. If $A = 5, 6, 8, 9, 10$, then we put $z = (1, 1, 1, 1, 1, i, 2i), (1, 1, 1, 1, 2i, 1+i, 1-i), (1, 1, 1, 1, 2i, 2i, 2), (1, 1, 1, 1, 1, 2, 3i), (1, 1, 1+i, 1-i, 1+2i, 1-2i, 2)$ respectively. If $A \geq 11$ then we put $z = (a_1 + a_3 i, a_2 + a_4 i, -a_1 i,$

$a_4 - a_i, 2+2i, 2-i, 1-2i)$, where $A - 9 = a_1^2 + a_2^2 + a_3^2 + a_4^2$ with $a_1 a_2 \neq 0$.

(6). *Case* $n \geq 9$, odd. If ($) holds, then we have $2A \geq n$, so $2A \geq n+1$. If $2A = n+1$, then we may suppose that $N(z_1) = 2, N(z_2) = \cdots = N(z_n) = 1$. Hence $z_1 = \pm 1 \pm i$ and $z_j = \pm 1$ $(j \geq 2)$, which contradicts the fact $z_1^2 + \cdots + z_n^2 = 0$. Thus $A \geq (n+3)/2$. We shall prove the sufficiency. If $A = (n+3)/2$, then we put $z_1 = 2i, z_2 = \cdots = z_{(n-3)/2} = i$, and $z_{(n-1)/2} = \cdots = z_n = 1$. If $A = (n+5)/2$ then we put $z_1 = 2i, z_2 = 1+i, z_3 = 1-i, z_4 = \cdots = z_{(n-1)/2} = i$, and $z_{(n+1)/2} = \cdots =, z_n = 1$. If $A \geq (n+9)/2$, then we put $m = A - (n-7)/2$. Then we have $m \geq 8$, so there are $z_1, \ldots, z_7$ in Γ such that $z_1^2 + \cdots + z_7^2 = 0, N(z_1) + \cdots + N(z_7) = 2m, z_1 \neq 0, \ldots z_7 \neq 0$ by the case $n = 7$. We put $z_8 = \cdots = z_{(n+7)/2} = 1$ and $z_{(n+9)/2} = i$. Thus the condition ($) holds. □

LEMMA 3. *Let m be a positive integer $\equiv \pm 1 \bmod 5$. Then there are integers $a, b, c, d \in \mathbf{Z}$ such that*

$$m = a^2 + 25(b^2 + c^2 + d^2), \quad ac \neq 0,$$

if and only if $m \notin G$, where G is the set defined by

$$G = \{19, 21, 31, 39, 49, 69, 71, 81, 119, 121, 179, 191, 211, 239, 379, 391\} \\ \cup \{4^k, 9 \cdot 4^k, 11 \cdot 4^k, 6 \cdot 4^k, 14 \cdot 4^k, 46 \cdot 4^k, 94 \cdot 4^k : k \geq 0\}.$$

PROOF. Put $T = \{m = a^2 + 25(b^2 + c^2 + d^2) : a, b, c, d \in \mathbf{Z},\ ac \neq 0,\ m \equiv \pm 1 \bmod 5\}$. We note that if $m \equiv 0 \bmod 8$, then $m \in T$ if and only if $m/4 \in T$. The necessity follows from direct calculations. Take a positive integer $m \notin G$ such that $m \not\equiv 0 \bmod 8$ and $m \equiv \pm 1 \bmod 5$. Then we can find an integer a such that $m \equiv a^2 \bmod 25$ with $1 \leq a \leq 24$. We can assume that $m - a^2 \equiv 1, 2, 3, 5, 6 \bmod 8$ (replacing a by $25 - a$ if necessary). Thus, if $m > a^2$, then we have $(m - a^2)/25 = b^2 + c^2 + d^2$ with $c \neq 0$, which proves that $m \in T$ if $m > 24^2$. When $m < 24^2$, then we see that $m \in T$ by a direct calculation.

LEMMA 4. *For a positive integer m, there are integers a, b, c, d, g in $\mathbf{Z}$ such that*

$$m = a^2 + b^2 + g^2(c^2 + d^2), \quad ac \neq 0, \quad g > 1, \ g \text{ odd},$$

if and only if $m \neq 5, 7, 9, 15, 21, 39, 4^k, 2 \cdot 4^k, 3 \cdot 4^k, 6 \cdot 4^k$

PROOF. We lose nothing by supposing $m \not\equiv 0 \bmod 8$. The necessity follows from a direct calculation. For the sufficiency, it suffices to show that m is of the desired form if $m \not\equiv 0 \bmod 8$ and $m \neq 1, 2, 3, 4, 5, 6, 7, 9, 12, 15, 21, 39$.

(1). *Case* $m \not\equiv 0 \bmod 3$. Putting $c = 2$ (if $m 1 \mod 4$) or $c = 1$ (otherwise), we have $m - 9c^2 \not\equiv 0, 4, 7 \bmod 8$. This implies that $m - 9c^2 = a^2 + b^2 + d^2$ with $a, b, d \in \mathbf{Z}$ if $m > 9c^2$ (that is $m \neq 1, 2, 4, 5, 7, 13, 17, 25, 29$). We may assume that $d \equiv 0 \bmod 3$, since $m \not\equiv 0 \bmod 3$. Thus we can write $m = a^2 + b^2 + 9(c^2 + d^2)$ with $ac \neq 0$, noticing that if $a = b = 0$ then we would have $m \equiv 0 \bmod 3$. A direct calculation shows that it is true for $m = 13, 17, 25, 29$.

(2). *Case* $m \equiv 0 \bmod 9$. If $m > 9$, then we can write $m/9 = a^2 + b^2 + c^2 + d^2$ with $ac \neq 0$, so $m = (3a)^2 + (3b)^2 + 9(c^2 + d^2)$.

(3). *Case* $m \equiv 0 \bmod 3$ with $m \not\equiv 0 \bmod 9$. Put $m = 3h$ with $h \in \mathbf{Z}$ so $h \not\equiv 0 \bmod 3$.

(i) If $h \equiv 0 \bmod 25$, then m is of the desired form by a similar argument like the case $m \equiv 0 \bmod 9$.

(ii) Suppose that $h \equiv 0 \bmod 5$ with $h \not\equiv 0 \bmod 25$. Putting $d_1 = 10$ (if $h \equiv 1 \bmod 4$) or $d_1 = 5$ (otherwise), we have $h - d_1^2 \not\equiv 0, 4, 7 \bmod 8$, which implies that $h - d_1^2 = a^2 + b^2 + c_1^2$ with $a, b, c_1 \in \mathbf{Z}$ if $h > d_1^2$ (that is $h \neq 10, 20, 65, 85$). The two integers $c = (3c_1 \pm 4d_1)/5$ and $d = (4c_1 \pm 3d_1)/5$ are prime to 5, taking a suitable sign. Hence we have $h = a^2 + b^2 + c^2 + d^2$ with $abcd \not\equiv 0 \bmod 5$. This is also true for $h = 10, 20, 65, 85$. Now we may assume, since $h \equiv 0 \bmod 5$, that $a \equiv d \equiv 1$, $b \equiv c \equiv 2 \bmod 5$ by changing the signs if necessary. Putting

$$a_1 = b - c + d, \quad b_1 = c - a + d, \quad c_1 = a - b + d, \quad d_1 = a + b + c,$$

we have $m = 3h = a_1^2 + b_1^2 + c_1^2 + d_1^2$ with $a_1 \equiv 1 \bmod 5, c_1 \equiv d_1 \equiv 0 \bmod 5$. If $c_1 = d_1 = 0$, then we would have $h = 3(a^2 + b^2)$, which is a contradiction. Hence m is of the desired form.

(iii) Suppose that $h \equiv \pm 1 \bmod 5$. By Lemma 3, if $h \notin G$, then we have $h = a^2 + 25(b^2 + c^2 + d^2)$ with $ac \neq 0$. Using a similar argument as in (ii), we can write $25(b^2+c^2) = b_1^2 + c_1^2$ with $b_1c_1 \not\equiv 0 \bmod 5$. Hence we have $h = a^2 + b^2 + c^2 + 25d^2$ with $abc \not\equiv 0 \bmod 5$. We may assume, since $h \equiv \pm 1 \bmod 5$, that $a \equiv b \equiv 2c \bmod 5$. Putting $a_1 = b - c + d, \dots$ as in (ii), we have $m = 3h = a_1^2 + b_1^2 + c_1^2 + d_1^2$ with $a_1 \not\equiv 0$, $c_1 \equiv d_1 \equiv 0 \bmod 5$. If $c_1 = d_1 = 0$ then we would have $h \equiv 0 \bmod 3$. Thus m is of the desired form. If $h \in G$ with $h \neq 1, 4$, then we see that m is of the desired form by a direct calculation.

(iv) Suppose that $h \equiv \pm 2 \bmod 5$. Then $m = 3h \equiv \pm 1 \bmod 5$. By Lemma 3, if $h \notin G$, then we have $m = a^2 + (5b)^2 + 25(c^2 + d^2)$ with $ac \neq 0$. If $m \in G$, then $m = 6, 21, 39, 69$, since $m \equiv 0 \bmod 3$. Notice that $69 = 2^2 + 4^2 + 49(1^2 + 0^2)$.

3. 3-frames.

For 3-frames, we give the next theorem without a proof. But the problem is open for $n = 5, 6$.

THEOREM 5. *Let $n \neq 5, 6$. Then E_n contains a proper 3-frame of scale A if and only if one of the following is satisfied:*

(1) $n = 3, A$ *is a square*;

(2) $n = 4$, $A > 1$;

(3) $n \geq 7$, $n \equiv 1 \bmod 3$, $A \geq (n+2)/3$;

(4) $n \geq 8$, $n \equiv 2 \bmod 3$, $A \geq (n+4)/3$;

(5) $n \geq 9$, $n \equiv 0 \bmod 3$, $A \geq n/3$.

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