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COMBINATORIAL TECHNIQUES AND ABSTRACT WITT RINGS II

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1. Introduction and terminology. We will follow mainly the notation and terminology set up in [**2**]. Thus (R, G_R, B_R, q_R) is an abstract Witt ring as defined in [**4**] and recall that $q_R : G_R \times G_R \to B_R$ is a symmetric bilinear mapping with G_R and B_R being groups of exponent 2. G_R has a distinguished element -1 satisfying $q(a, -a) = 1$, and q_R satisfies

$$\begin{aligned}&\text{For all } a, b, c, d \in G_R, q_R(a, b) = q_R(c, d) \text{ implies}\\ &\text{there exists } x \in G_R \text{ with } q_R(a, b) = q_R(a, x)\\ (L)\qquad &\qquad\qquad\qquad\qquad = q_R(c, x) = q_R(c, d).\end{aligned}$$

Denote by Q_R the image of q_R in B_R and when there is no confusion write $G = G_R$, $B = B_R$, $q = q_R$ and $Q = Q_R$. For $a \in Q_R$ set $Q(a) = \{q(a, x) \mid x \in G_R\}$. Y_R will denote the collection $\{Q(a) \mid a \in G_R \backslash \{1\}\}$ and $\{Q_i\}_{i=1}^n$ is the collection of distinct elements of Y_R. For a subgroup of Q of B_R, the subgroup $\{x \in G_R \mid Q(x) \subseteq Q\}$ of G_R will be denoted by $H(Q)$. We let $H_i = H(Q_i)$ and $h_i = |H_i|$. The value set of $\langle 1, x\rangle$ is $D\langle 1, x\rangle = \{y \in G_R \mid q(-x, y) = 1\}$, and, for any subgroup K of G_R, let $\dot{K}$ denote $K \backslash \{1\}$. Finally set $g = |G_R|$.

In §2 construct a quotient quaternionic mapping $\overline{q}$ and give some technical conditions under which $\overline{q}$ satisfies (L). This quotient technique together with the counting technique of [**2**], proves to be quite useful in §3 where we classify Witt rings having a simple Hasse diagram. Specifically, we generalize Cordes' classification [**1**] of Witt rings with ≤ 4 quaternion algebras by classifying all non-degenerate Witt rings with $|Y_R| \leq 4$ (See [**3**;Chapter 5, §10]) for a statement and proof of Cordes' classification using the notation and terminology used here.)

2. Quotients. Let R be an arbitrary abstract Witt ring with associated linked quaternionic mapping $q : G_R \times G_R \to B_R$. For an arbitrary subgroup Q of B_R set $H = H(Q)$ and define $\overline{q} : G_R/H \times G_R/H \to B_R/Q$ by $\overline{q}(\overline{a}, \overline{b}) = q(a, b)Q$, where $\overline{a} = aH$.

PROPOSITION 2.1. *$\overline{q}$ is a non-degenerate quaternionic mapping.*

PROOF. $\overline{q}$ is clearly bilinear and symmetric. To show $\overline{q}$ is well defined suppose $a = a'h$ for some $a, a' \in G_R$ and $h \in H$. Then $q(a,b) = q(a'h, b) \in q(a', b)Q(h) \subseteq q(a', b)Q$. Finally, notice that $\overline{q}$ is non-degenerate, for, if $\overline{q}(aH, bH) = Q\ \forall b \in G_R$, then $q(a,b) \in Q\ \forall b \in G_R$. Hence $Q(a) \subseteq Q$ and $a \in H$. □

Consequently, $\overline{q}$ is a linked quaternionic mapping if and only if $\overline{q}$ satisfies (L). In this case we say that *reduction at Q is possible.* Note that $Q(\overline{a}) = \{\overline{q}(\overline{a}, \overline{b}) \mid \overline{b} \in G_R/H\} = \{q(a,b)Q \mid a, b \in G_R\} = Q(a)Q/Q$. Thus $Y_{\overline{R}} = \{Q_iQ/Q \mid Q_i \in Y_R\}$.

PROPOSITION 2.2. *Suppose reduction at $Q(x)$ is possible. Then $\overline{R} \cong R/I(H)$ where $H = H(Q(x))$ and $I(H) = (\{\langle 1, -h\rangle \mid h \in H\})$.*

PROOF. It suffices to show that the kernel of the natural mapping $\alpha : R \to \overline{R}$ is contained in $I(H)$. Let $\varphi \in \ker \alpha$. If $\varphi = \langle\langle 1, -a\rangle\rangle$, then $\overline{a} = 1$ implies $a \in H$ and $\langle\langle 1, -a\rangle\rangle \in I(H)$. If $\varphi = \langle\langle -a, -b\rangle\rangle$, then $q(a,b) \in Q(x)$, and thus $\langle\langle -a, -b\rangle\rangle = \langle\langle -x, -c\rangle\rangle$ for some $c \in G_R$. Since $\langle\langle -x, -c\rangle\rangle = \langle\langle 1, -x\rangle\rangle\langle\langle 1, -c\rangle\rangle$ and $x \in H$, $\varphi \in I(H)$. In general, suppose $\varphi = \langle\langle a_1, a_2, \ldots, a_n\rangle\rangle$. There exists a sequence of forms $\overline{\varphi}, \overline{\varphi}_1, \overline{\varphi}_2, \ldots, \overline{\varphi}_h = 0$ in $\overline{R}$ with each pair $\overline{\varphi}_i$, $\overline{\varphi}_{i+1}$ of the form $\overline{\varphi}_i = \langle\langle \overline{c}_1, \overline{c}_2, \overline{c}_3, \ldots, \overline{c}_n\rangle\rangle$, $\overline{\varphi}_{i+1} = \langle\langle \overline{d}_1, \overline{d}_2, \overline{c}_3, \ldots, \overline{c}_n\rangle\rangle$ with $\overline{q}(\overline{c}_1, \overline{c}_2) = \overline{q}(\overline{d}_1, \overline{d}_2)$ and $\overline{c}_1\overline{c}_2 = \overline{d}_1\overline{d}_2$. Now $0 = \overline{\varphi}_i - \overline{\varphi}_{i+1} = \langle\langle \overline{c}_1, \overline{c}_2, -\overline{d}_1, -\overline{d}_2\rangle\rangle = \overline{c}_1\langle\langle \overline{c}_1\overline{c}_2, -\overline{c}_1\overline{d}_1\rangle\rangle \perp -\overline{d}_2\langle\langle 1, -\overline{c}_1\overline{c}_2\overline{d}_1\overline{d}_2\rangle\rangle$. Since $\overline{c}_1\overline{c}_2 = \overline{d}_1\overline{d}_2$ and $\overline{q}(\overline{c}_1, \overline{c}_2) = \overline{q}(\overline{d}_1, \overline{d}_2)$, $\langle\langle 1, -\overline{c}_1\overline{c}_2\overline{d}_1\overline{d}_2\rangle\rangle$ and $\langle\langle \overline{c}_1\overline{c}_2 - \overline{c}_1\overline{d}_1\rangle\rangle$ are both equal to 0. As above, $\langle\langle 1, -c_1c_2d_1d_2\rangle\rangle$, $\langle\langle c_1c_2 - c_1d_1\rangle\rangle \in I(H)$. Consequently, $\varphi_i - \varphi_{i+1} \in I(H)$ and since $\varphi = (\varphi_1 - \varphi_2) + (\varphi_2 - \varphi_3) + \cdots + (\varphi_{k-1} - \varphi_k)$, $\varphi \in I(H)$ as desired. □

Note that reduction at an arbitrary subgroup Q of B_R is not always possible. What follows is an example of such a Q and some rather technical conditions under which reduction is possible. We suspect that reduction at every $Q(x)$ is possible but the proof eludes us. In any case, some of what follows turns out to be sufficient and quite useful for our classification in the next section.

EXAMPLE. Let L be a Witt ring of local type with $|G_L| \geq 16$, $-1 \neq 1$, and take R to be $L[\{1,t\}]$. Pick $a \in G_L$, $a \notin D\langle\langle 1,1\rangle\rangle$ and set $Q = \{1, q(a,t)\}$. Note that $H(Q) = \{1\}$. Choose $x \in G_L$, $x \neq -1$, with $q_L(x,a) \neq 1$, then pick $y \in G_L$ with $q_L(y,a) = 1$ and $q_L(x,y) \neq 1$. This is possible for otherwise $q_L(x,y) = 1 \,\forall\, y \in D\langle\langle 1,-a\rangle\rangle$ and hence $D\langle\langle 1,-a\rangle\rangle = D\langle\langle 1,-x\rangle\rangle$, a contradiction. Now $q_R(xt,y) = q_R(x,y)q_R(t,y) = q_R(x,y)q_R(t,ay)q_R(t,a) = q_R(a,ay)q_R(t,ay)q_R(t,a)$ since $q_R(x,y) \neq 1$ and $q_R(a,ay) \neq 1$.

Consequently, $q_R(xt,y) = q_R(at,ay)q_R(t,a)$ and $\overline{q}_R(\overline{xt},\overline{y}) = \overline{q}_R(\overline{at},\overline{ay})$. Assuming $\overline{R}$ satisfies (L), there exists $w \in G_R$ such that $\overline{q}_R(\overline{xt},\overline{y}) = \overline{q}(\overline{xt},\overline{w}) = \overline{q}(\overline{at},\overline{w}) = \overline{q}(\overline{at},\overline{ay})$ and thus $\overline{q}_R(\overline{xt},\overline{yw}) = 1$ and $\overline{q}_R(\overline{at},\overline{ayw}) = 1$. We have $q_R(xt,yw) = 1$ or $q_R(t,a)$ and $q_R(at,ayw) = 1$ or $q_R(t,a)$. Assume first that $q_R(xt,yw) = 1$. Then $yw \in D\langle\langle 1,-xt\rangle\rangle$ hence $yw = 1$ or $-xt$. If $yw = 1$, then $w = y$ and $q_R(at,a) = 1$ or $q_R(t,a)$. Both choices lead to a contradiction. If $yw = -xt$, then $w = -xty$ and $q_R(at,-axt) = 1$ or $q_R(t,a)$. That is $q_R(at,x) = 1$ or $q_R(t,a)$. Again both choices lead to a contradiction. Consequently, $q_R(xt,yw) = q_R(t,a)$. Applying (L), obtain $q_R(xt,yw) = q_R(xt,v) = q_R(a,v) = q_R(a,t)$ for some $v \in G_R$. The last equality implies $v = \alpha t$ for some $\alpha \in D\langle\langle 1,-a\rangle\rangle$. The middle equality implies $axt \in D\langle\langle 1,-\alpha t\rangle\rangle$. So $axt = 1$ or $-\alpha t$. $axt = 1$ is clearly impossible and $axt = -\alpha t$ implies $-ax = \alpha \in D\langle\langle 1,-a\rangle\rangle$, again a contradiction. □

PROPOSITION 2.3. *Let Q be a subgroup of B_R. If for every $x,y \in \dot{G}_R$, with $Q(x) \subsetneq Q$, $Q(y) \subsetneq Q$, we have $Q(x)Q(y) \cap Q \subseteq Q(x) \cup Q(y) \cup Q(xy)$ then reduction at Q is possible.*

PROOF. By Proposition 2.1 it is sufficient to show that (L) holds for $\overline{q}$. Suppose that $q(a,b) = q(c,d)\rho$, for some $\rho \in Q$. First assume $a \notin H(Q)$ and $c \notin H(Q)$, hence $Q(a) \subsetneq Q$ and $Q(c) \subsetneq Q$. Now $\rho \in Q(A) \cdot Q(c) \cap Q$, so, by our assumption, $\rho \in Q(a) \cup$

$Q(c) \cap Q(ac)$. If $\rho \in Q(a)$ then write $\rho = q(a,e)$ for some $e \in G_R$. We have $q(c,d) = q(a,b)\rho = q(a,be)$. Applying linkage, there exists $z \in G_R$ with $q(c,d) = q(c,z) = q(a,z) = q(a,be)$. Since $q(a,be) = q(a,b)\rho \in q(a,b)Q$ we have $q(c,d), q(c,z), q(a,z) \in q(a,b)Q$. Consequently, $\overline{q}(\overline{a},\overline{b}) = \overline{q}(\overline{a},\overline{z}) = \overline{q}(\overline{c},\overline{z}) = \overline{q}(\overline{c},\overline{d})$. A similar argument applies if $\rho \in Q(c)$, so assume $\rho \in Q(ac)$. Write $\rho = q(ac,e)$ for some $e \in G_R$. Here we have $q(a,b) = q(c,d)\rho = q(c,d)q(ac,e)$, hence $q(a,be) = q(c,de)$. Applying linkage, there exists $z \in G$ with $q(a,be) = q(a,z) = q(c,z) = q(c,de)$, then $q(a,b) = q(a,ze) = q(c,z)q(a,e) = q(c,de)q(a,e)$. Now, $q(c,z)q(a,e) = q(c,z)q(a,e)q(ac,e)q(ac,e) = q(c,ze)q(ac,e) = q(c,ze)\rho$ and $q(c,de)q(a,e) = q(c,d)\rho$. Consequently, $q(a,b) = q(a,ze) = q(c,ze)\rho = q(c,d)\rho$ and thus $\overline{q}(\overline{a},\overline{b}) = \overline{q}(\overline{a},\overline{ze}) = \overline{q}(\overline{c},\overline{ze}) = \overline{q}(\overline{c},\overline{d})$. □

We say that a subgroup Q of B_R is *comparable* with a subgroup Q' of B_R if either $Q \subseteq Q'$ or $Q' \subseteq Q$. Let $\mathrm{Cmp}(Q) = \{Q(x) \in Y_R \mid Q(x)$ is comparable with $Q\}$.

COROLLARY 2.4. *Let Q be a subgroup of B_R and suppose that $\forall\, x,y,z \in \dot{G}_R$ with $Q(x), Q(y), Q(z) \notin \mathrm{Cmp}\,(Q)$ and at least two of $Q(x), Q(y), Q(z)$ comparable. Then reduction at Q is possible.*

PROOF. Let $x, y \in \dot{G}_R$ with $Q(x) \subsetneq Q$, $Q(y) \subsetneq Q$. Take $z = xy$. Show that $Q(x)Q(y) \cap Q \subseteq Q(x) \cup Q(y) \cup Q(z)$. We assume $Q(x), Q(y), Q(z) \notin \mathrm{Cmp}\, Q$, otherwise it is trivial. By assumption, at least two of $Q(x), Q(y), Q(z)$ are comparable. If $Q(x)$ and $Q(y)$ are comparable, then $Q(x)Q(y) \subseteq Q(x)$ or $Q(y)$ and it is proved. If $Q(x)$ is comparable with $Q(z)$ then either $Q(x) \subseteq Q(xy)$ or $Q(xy) \subseteq Q(x)$. In the first case $Q(x)Q(y) \subseteq Q(xy)$ and in the second $Q(x)Q(y) \subseteq Q(x)$. Consequently, in either case $Q(x)Q(y) \cap Q \subseteq Q(x) \cup Q(y) \cup Q(xy)$. A similar argument works for $Q(y)$ comparable to $Q(z)$. □

PROPOSITION 2.5. *For every $x \in G_R$ with $|Q(x)| = 2$ satisfying $\forall\, a,b,c,d \in G_R$, $Q(x) \subseteq Q(a)Q(c) \Rightarrow \{b,d\} \cup D\langle\langle 1,-a\rangle\rangle \cup D\langle\langle 1,-c\rangle\rangle \cup D\langle\langle 1,-ac\rangle\rangle \subseteq D\langle\langle 1,-x\rangle\rangle$, reduction at $Q(x)$ is possible.*

PROOF. To show reduction at $Q(x)$ is possible it suffices to show that (L) holds for $\overline{q}$. Suppose $q(a,b) = q(c,d)q(x,y)$. Assume $q(x,y) \neq 1$ for otherwise the result follows from the linkage of q. By the hypothesis, there exists $z \in \{b,d\} \cup D\langle\langle 1,-c\rangle\rangle \cup D\langle\langle 1,-a\rangle\rangle \cup D\langle\langle 1,-ac\rangle\rangle$ with $z \notin D\langle\langle 1,-x\rangle\rangle$. If $z = d$, then $q(x,d) \neq 1$, and since $|Q(x)| = 2$ we have $q(x,d) = q(x,y)$. Applying linkage to $q(a,b) = q(cx,d)$ we obtain $q(a,b) = q(a,t) = q(cx,t) = q(cx,d)$ for some $t \in G_R$. Consequently, $\overline{q}(\overline{a},\overline{b}) = \overline{q}(\overline{a},\overline{t}) = \overline{q}(\overline{c},\overline{t}) = \overline{q}(\overline{c},\overline{d})$ and (L) holds. A similar argument works if $z = b$. For the remainder of the proof we can then assume that $b,d \in D\langle\langle 1,-x\rangle\rangle$. If $z \in D\langle\langle 1,-c\rangle\rangle$ then $q(a,b) = q(c,dz)q(x,dz)$. By linkage, $q(a,b) = q(a,t) = q(cx,t) = q(cx,dz)$ for some $t \in G_R$. Note that $q(cx,dz) = q(c,d)q(x,z)q(c,z)q(x,d)$. Since $q(c,z) = q(x,d) = 1$ we have $\overline{q}(\overline{a},\overline{b}) = \overline{q}(\overline{a},\overline{t}) = \overline{q}(\overline{c},\overline{t}) = \overline{q}(\overline{c},\overline{d})$ and (L) holds. Again, a similar argument works if $z \in D\langle\langle 1,-a\rangle\rangle$. Finally we can assume $D\langle\langle 1,-a\rangle\rangle \cup D\langle\langle 1,-c\rangle\rangle \subseteq D\langle\langle 1,-x\rangle\rangle$. Suppose $z \in D\langle\langle 1,-ac\rangle\rangle$. Here $q(a,b) = q(c,d)q(acx,z)$ so $q(a,bz) = q(cx,d)q(cx,z) = q(cx,dz)$. Applying linkage, $q(a,bz) = q(a,t) = q(cx,t) = q(cx,dz)$ for some $t \in G_R$. Consequently, $q(a,b) = q(a,tz) = q(cx,t)q(az) = q(cx,dz)q(a,z)$. Since $q(a,z) = q(c,z)$ we have $q(a,b) = q(a,tz) = q(c,tz)q(x,t) = q(c,d)q(x,dz)$, and (L) follows. □

COROLLARY 2.6. *For every $x \in G_R \backslash D\langle\langle 1,1\rangle\rangle$ with $|Q(x)| = 2$, reduction at $Q(x)$ is possible.*

PROOF. Observe that if $\{b,d\} \cup D\langle\langle 1,-a\rangle\rangle \cup D\langle\langle 1,-c\rangle\rangle \cup D\langle\langle 1,-ac\rangle\rangle \subseteq D\langle\langle 1,-x\rangle\rangle$, then in particular $-a,-c \in D\langle\langle 1,-x\rangle\rangle$. But then $ac,-ac \in D\langle\langle 1,-x\rangle\rangle$ and it follows that $-1 \in D\langle\langle 1,-x\rangle\rangle$, a contradiction. □

COROLLARY 2.7. *Suppose the mapping $a \to D\langle\langle 1,a\rangle\rangle$ is an injection. Reduction at $Q(x)$ is possible for every $x \in G_R$ with $|Q(x)| = 2$.*

PROOF. Suppose $q(a,b) = q(c,d)q(x,y)$. As before we may assume $q(x,y) \neq 1$, for otherwise the result follows from linkage on q. By Proposition 2.5 we may say $D\langle\langle 1,-a\rangle\rangle \subset D\langle\langle 1,-x\rangle\rangle$. Remark that we may make the assumption that $Q(x) \subseteq Q(a)$. Otherwise $q(x,y) = q(a,z)$ for some $z \in G_R$ and thus $q(a,bz) = q(c,d)$. Applying linkage we obtain $q(a,bz) = q(a,t) = q(c,t) = q(c,d)$ for some $t \in G_R$. (L) on $\bar{q}$ now follows. Further, $D\langle\langle 1,-ax\rangle\rangle \subseteq D\langle\langle 1,-x\rangle\rangle$ for otherwise $G_R = D\langle\langle 1,-x\rangle\rangle D\langle\langle 1,-ax\rangle\rangle$ since $D\langle\langle 1,-x\rangle\rangle$ has index 2 in G_R. But then, for an arbitrary $g \in G_R$, $g = \alpha\beta$ for some $\alpha \in D\langle\langle 1,-x\rangle\rangle$, $\beta \in D\langle\langle 1,-ax\rangle\rangle$. Consequently, $q(x,g) = q(x,\alpha\beta) = q(x,\beta) = q(\beta,a)$. That is, $Q(x) \subseteq Q(a)$, another contradiction. Then $D\langle\langle 1,-a\rangle\rangle$, $D\langle\langle 1,-ax\rangle\rangle \subseteq D\langle\langle 1,-x\rangle\rangle$. Now $D\langle\langle 1,-a\rangle\rangle = D\langle\langle 1,-a\rangle\rangle \cap D\langle\langle 1,-x\rangle\rangle \subseteq D\langle\langle 1,-ax\rangle\rangle$ and $D\langle\langle 1,-ax\rangle\rangle = D\langle\langle 1,-ax\rangle\rangle \cap D\langle\langle 1,-x\rangle\rangle \subseteq D\langle\langle 1,-a\rangle\rangle$, thus $D\langle\langle 1,-a\rangle\rangle = D\langle\langle 1,-ax\rangle\rangle$, contradicting our hypothesis. □

3. The Hasse diagram of $\mathbf{Y_R}$. In this section we will take R to be non-degenerate and consider the usual Hasse diagram of the partially ordered set Y_R. This is constructed as follows: Represent each element of Y_R as a point (vertex). If $Q(a) \subseteq Q(b)$ then draw an arrow from the point representing $Q(a)$ to that of $Q(b)$. Make the following simplifications. The arrows from a point to itself are omitted. All arrows that are implied by the transitive property of inclusion are omitted. Last, arrange the diagram so that all arrows point upward and replace the arrows with lines.

EXAMPLE. Suppose $Y_R = \{Q_1, Q_2, Q_3, Q_4\}$ with $Q_1 \subset Q_2 \subset Q_4$, $Q_1 \subset Q_3 \subset Q_4$ and no other containments. Then the Hasse diagram of Y_R is Diagram 1.

We take a moment to clarify some of the notation which follows. By diagram 2 we will mean any Hasse diagram with $\dot{1}$ on the bottom. Similarly, diagram 3 represents any Hasse diagram with diagram 4 on the bottom. The counting coefficients c_j are the same as defined in [**2**].

All Witt rings having $|Y_R| \leq 4$ will be classified but we first proceed with the following applications of the block design counting in [**2**].

THEOREM 3.1. *Suppose R has Hasse diagram 5. If the counting coefficients c_j are non-negative then $|Q_1| = 2$.*

PROOF. This is a restatement of [**2**, Theorem 13]. □

COROLLARY 3.2. *Suppose $Y_R = \{Q_i\}_{i=1}^n$ with $Q_1 \subset Q_2 \subset \cdots \subset Q_n$. Then R is of local type.*

PROOF. Show $n = 1$, and the result follows from [**2**, Theorem 5]. By Theorem 3.1 (as in [**2**, Corollary 14]), $|Q_1| = 2$. Suppose $n = 2$. Reduction at Q_1 is possible by Corollary 2.4 and $Y_{\overline{R}} = \{Q_2/Q_1\}$. By [**2**, Theorem 5], $|Q_2/Q_1| = 2$, thus $|Q_2| = 4$. By Cordes' classification [**1**] no such R can exist. Consequently $n \neq 2$. In general, reduction at Q_1 is possible by Corollary 2.4 and $Y_{\overline{R}} = \{Q_2/Q_1,\ Q_3/Q_1, \ldots, Q_n/Q_1\}$. Here we have $|Y_{\overline{R}}| = n-1$ and $Q_2/Q_1 \subset Q_3/Q_1 \subset \cdots \subset Q_n/Q_1$. By induction, if such a chain of length $n \geq 2$ exists then there is a Witt ring having a chain of length 2 which is a contradiction. □

(3.2) answers affirmatively a question posed by M. Kula and he is thanked for bringing it to our attention. The proof above, using both the counting formula and quotient formation, was the basic motivation behind this paper and [**2**].

LEMMA 3.3. *Let $Y_R = \{Q_i\}$ and let a be an element of $\dot{H}_1 = \dot{H}(Q_1)$. Let $i_j = |D\langle\langle 1, -a\rangle\rangle \cap H_j|$. Then, for any number $h \geq |Q_1|$:*

(i) $\left[\frac{1}{|Q_1|^2} - \frac{1}{h|Q_1|}\right] h_1 \leq \left[\frac{1}{|Q_1|} - \frac{1}{h}\right] i_1,$

(ii) $\left[\frac{1}{|Q_1|\,|Q_j|} + \frac{1}{h|Q_1 \cap Q_j|} - \frac{2}{h|Q_1|}\right] h_j \leq \left[\frac{1}{|Q_j|} - \frac{1}{h}\right] i_j,$

for any j with $|Q_j| \leq \frac{h}{2}$,

(iii) $\frac{h_j}{|Q_1|} \leq i_j$ *for every j.*

PROOF. (i). Since $D\langle\langle 1, -a\rangle\rangle \cdot H_1 \subseteq G_R$, $(|D\langle\langle 1, -a\rangle\rangle|/i_1)h_1 \leq g$, then $(|D\langle\langle 1, -a\rangle\rangle|/g)h_1 \leq i_1$. Now $(|D\langle\langle 1, -a\rangle\rangle|/g)h_1[g/|Q_1| - g/h] \leq i_1[g/|Q_1| - g/h]$ since $h \geq |Q_1|$ and (i) follows since $|D\langle\langle 1, -a\rangle\rangle|/g = 1/|Q_1|$.

(ii). Since $|Q_1| \geq |Q_1 \cap Q_j|$ and $|Q_j| \leq h/2$ we have

$$\frac{1}{|Q_1 \cap Q_j|}\left[\frac{1}{|Q_j|} - \frac{1}{h} - \frac{1}{h}\right] \geq \frac{1}{|Q_1|}\left[\frac{1}{|Q_j|} - \frac{2}{h}\right],$$

so
$$\frac{1}{|Q_i \cap Q_j|}\left[\frac{1}{|Q_j|} - \frac{1}{h}\right] \geq \frac{1}{|Q_1 \cap Q_j|h} + \frac{1}{|Q_1|\,|Q_j|} - \frac{2}{h|Q_1|}$$
and
$$\frac{h_j}{|Q_1 \cap Q_j|}\left[\frac{1}{|Q_j|} - \frac{1}{h}\right] \geq \left[\frac{1}{|Q_1 \cap Q_j|h} + \frac{1}{|Q_1|\,|Q_j|} - \frac{2}{h|Q_1|}\right]h_j.$$
It is sufficient to show that
$$\frac{h_j}{|Q_1 \cap Q_j|}\left[\frac{1}{|Q_j|} - \frac{1}{h}\right] \leq \left[\frac{1}{|Q_j|} - \frac{1}{h}\right]i_j.$$
But $h_j/i_j \leq |Q_1 \cap Q_j|$ since the mapping $y \to q(a,y)$ from H_j to $Q_1 \cap Q_j$ has kernel $D\langle\langle 1,-a\rangle\rangle \cap H_j$. Consequently, $h_j/|Q_1 \cap Q_j| \leq i_j$ and
$$\frac{h_j}{|Q_1 \cap Q_j|}\left[\frac{1}{|Q_j|} - \frac{1}{h}\right] \leq i_j\left[\frac{1}{|Q_j|} - \frac{1}{h}\right].$$

(iii). Since $D\langle\langle 1,-a\rangle\rangle \cdot H_j \subseteq G_R$, and $(|D\langle\langle 1,-a\rangle\rangle|/i_j)h_j \leq g$ then $g/|Q_1| \cdot h_j/i_j \leq g$ and $h_j/|Q_1| \leq i_j$. □

THEOREM 3.4. *Suppose R has Hasse diagram 6. Assume $|Q_1| \leq |Q_j|$ for $j \leq s$ and suppose there is an $a \in \dot{H}_1$ satisfying $aH_j \cap H_k = \{0\}$ for $2 \leq j,\ k \leq s$. If the counting coefficients $c_j,\ j = s+2, \ldots, n$, are non-negative then $|Q_1| \leq s$.*

PROOF. First recall more notation from [**2**]. H_j' will denote the set
$$H_j \backslash \cup_{H_k \in T_j} H_k$$
where $T_j = \{H_i \,|\, H_i \subsetneq H_j\}$, and let $h_j' = |H_j'|$. Further $i_j = |D\langle\langle 1,-a\rangle\rangle \cap H_j|$, $i_j' = |D\langle\langle 1,-a\rangle\rangle \cap H_j'|$ and $p_j = |Q_1 \cap Q_j|$. Now, by [**2**, Proposition 8],
$$\sum_{y \neq 1,a} \frac{1}{|Q(a) \cap Q(ay)|} \cdot \frac{1}{|Q(y)|} = \left[\sum_{z \in D\langle\langle 1,-a\rangle\rangle} \frac{1}{|Q(z)|}\right] - \frac{2}{|Q(a)|}.$$

If $y \in H_1 \backslash \{1, a\}$, $Q(a) \cap Q(ay) = Q_1$ and $Q(y) = Q_1$. If $y \in H_j$, $2 \leq j \leq s$, then $Q(a) \cap Q(ay) = Q_1$ since $aH_j \cap H_k = \{0\}$ for $2 \leq j$, $k \leq s$, $ay \in H_{s+1}$ and $Q_1 \subseteq Q_{s+1}$. If $y \in H_{s+1} \cap aH_j$, $2 \leq j \leq s$, then $Q(a) \cap Q(ay) = Q_1 \cap Q_j$. If $y \in H_{s+1} \backslash \cup_{j=1}^{s} aH_j$ then $Q(a) \cap Q(ay) = Q_1$. Finally, if $y \in H'_j$, $j > s+1$, then $Q(a) \cap Q(ay) = Q_1$. The above equation becomes

$$(h_1 - 2)\frac{1}{|Q_1|^2} + \sum_{j=2}^{s}(h_j - 1)\frac{1}{|Q_1|\,|Q_j|} + \sum_{j=2}^{s}(h_j - 1)\frac{1}{p_j|Q_{s+1}|}$$
$$+ \left(h_{s+1} - h_1 - 2\left[\sum_{j=2}^{s} h_j\right] + 2s - 2!\right)\frac{1}{|Q_1|\,|Q_{s+1}|} + \sum_{j=s+2}^{n} h'_j \frac{1}{|Q_1|\,|Q_j|}$$
$$= \sum_{j=1}^{s}(i_j - 1)\frac{1}{|Q_j|} + \left(i_{s+1} - \sum_{j=1}^{s} i_j + (s-1)\right)\frac{1}{|Q_{s+1}|}$$
$$+ \sum_{j=s+2}^{n} i'_j \frac{1}{|Q_j|} + 1 - \frac{2}{|Q_1|}.$$

After simplifying we obtain

$$\left[\frac{-2}{|Q_1|^2} - \sum_{j=2}^{s}\frac{1}{|Q_1|\,|Q_j|} - \sum_{j=2}^{s}\frac{1}{p_j|Q_{s+1}|} + \frac{(2s-2)}{|Q_1|\,|Q_{s+1}|}\right]$$
$$+ \left[\frac{1}{|Q_1|^2} - \frac{1}{|Q_1|\,|Q_{s+1}|}\right]h_1 + \sum_{j=2}^{s}\left[\frac{1}{|Q_1|\,|Q_j|} + \frac{1}{P_j|Q_{s+1}|}\right.$$
$$(*1) \qquad \left. - \frac{2}{|Q_1|\,|Q_{s+1}|}\right]h_j + \sum_{j=s+1}^{n} h'_j \frac{1}{|Q_1|\,|Q_j|}$$
$$= \left[-\sum_{j=1}^{s}\frac{1}{|Q_j|} + \frac{s-1}{|Q_{s+1}|} - \frac{2}{|Q_1|+1}\right] + \left[\sum_{j=1}^{s}\frac{1}{|Q_j|} - \frac{1}{|Q_{s+1}|}\right]i_j$$
$$+ \sum_{j=s+1}^{n} i'_j \frac{1}{|Q_j|}$$

where we redefined h'_{s+1} and i'_{s+1} to be h_{s+1} and i_{s+1} respectively.

Recall [**2**, Lemma 12] that $\sum_{j=s+1}^{n} h'_j/|Q_j| = \sum_{j=s+1}^{n} c_j h_j$ and $\sum_{j=s+1}^{n} i'_j/|Q_j| = \sum_{j=s+1}^{n} c_j i_j$. By Lemma 3.3(iii), and since $c_j \geq 0$

for $j \geq s+1$, we have

$$\sum_{j=s+1}^{n} h'_j \frac{1}{|Q_1|\,|Q_j|} = \sum_{j=s+1}^{n} \frac{1}{|Q_1|} c_j h_j \leq \sum_{j=s+1}^{n} c_j i_j = \sum i'_j \frac{1}{|Q_j|}.$$

It follows from (*1) and Lemma 3.3 that

$$\frac{-2}{|Q_1|^2} - \sum_{j=2}^{s} \frac{1}{|Q_1|\,|Q_j|} - \sum_{j=2}^{s} \frac{1}{p_j |Q_{s+1}|} + \frac{(2s-s)}{|Q_1|\,|Q_{s+1}|}$$

$$(*2) \qquad \geq -\sum_{j=1}^{s} \frac{1}{|Q_j|} + \frac{(s-1)}{|Q_{s+1}|} - \frac{2}{|Q_1|} + 1.$$

Since $1/|Q_{s+1}| \leq 1/|Q_j|$ and $1/|Q_1| \leq 1/p_j$, then

$$-\sum_{j=2}^{s} \frac{1}{|Q_1|\,|Q_j|} - \sum_{j=2}^{s} \frac{1}{p_j |Q_{s+1}|} + \frac{(2s-2)}{|Q_1|\,|Q_{s+1}|} \leq 0.$$

Since the left hand side is negative, the right hand side is also. We have

$$-\sum_{j=1}^{s} \frac{1}{|Q_j|} + \frac{(s-1)}{|Q_{s+1}|} - \frac{2}{|Q_1|} + 1 \leq 0,$$

$$1 - \frac{3}{|Q_1|} \leq \sum_{j=2}^{s} \frac{1}{|Q_j|} - \frac{(s-1)}{|Q_{s+1}|}.$$

Since $|Q_1| \leq |Q_j|$, then $1 - 3/|Q_1| \leq (s-1)/|Q_1| - (s-1)/|Q_{s+1}|$ and $1 - (s+2)/|Q_1| < 0$. Consequently, $1 < (s+2)/|Q_1|$ and $|Q_1| < s+2$. It remains only to show that $|Q_1| \neq s+1$. Suppose not. Then, by (*2),

$$\frac{-2}{(s+1)^2} - \sum_{j=2}^{s} \frac{1}{(s+1)|Q_j|} - \sum_{j=2}^{s} \frac{1}{p_j |Q_{s+1}|} + \frac{(2s-2)}{(s+1)|Q_{s+1}|}$$

$$\geq -\sum_{j=1}^{s} \frac{1}{|Q_j|} + \frac{(s-1)}{|Q_{s+1}|} - \frac{2}{s+1} + 1$$

and

$$\left[\frac{-2}{(s+1)^2} + \frac{3}{s+1} - 1\right] + \left[\sum_{j=2}^{s} \frac{1}{|Q_j|}\left(\frac{s}{s+1}\right)\right]$$

$$+ \frac{1}{|Q_{s+1}|}\left[-\sum_{j=2}^{s} \frac{1}{p_j} + \frac{2s-2}{s+1} - (s-1)\right] \geq 0.$$

On the other hand, $|Q_1| \leq |Q_j|$, so $1/|Q_j| \leq 1/|Q_1| = 1/(s+1)$ and

$$\begin{aligned}
&\left[-\frac{2}{(s+1)^2}+\frac{3}{s+1}-1\right]\\
&+\sum_{j=2}^{s}\frac{1}{|Q_j|}\left(\frac{s}{s+1}\right)+\frac{1}{|Q_{s+1}|}\left[-\sum_{j=2}^{s}\frac{1}{p_j}+\frac{2s-2}{s+1}-(s-1)\right]\\
\leq&\left[-\frac{2}{(s+1)^2}+\frac{3}{s+1}-1\right]+\frac{s-1}{s+1}\cdot\frac{s}{s+1}\\
&+\frac{1}{|Q_{s+1}|}\left[-\sum_{j=2}^{s}\frac{1}{p_j}-\frac{s^2-2s+1}{s+1}\right]\\
=&\frac{1}{|Q_{s+1}|}\left[-\sum_{j=2}^{s}\frac{1}{p_j}-\frac{s^2-2s+1}{s+1}\right]<0,
\end{aligned}$$

a contradiction. □

For a subgroup Q of B_R let $W(Q) = \{Q_i \in Y_R \mid Q_i \subseteq Q\}$.

LEMMA 3.5. *Suppose* $Q_R = \cup_{i=1}^{n} W(Q_i)$ and $W(Q_i) \cap W(Q_j) = \emptyset$ *for* $i \neq j$. *If* $n > 1$, *then* $|G_R| \leq n(n-2)+1$.

PROOF. For $i = 1, \ldots, n$, let $H_i = H(Q_i)$. Fix $a \in H_k$, $a \neq 1$. To show that $|aH_i \cap H_j| \leq 1$ for all i and j with $j \neq i$, $j \neq h$, $i \neq h$, suppose $ax_i = x_j$ and $ax_i' = x_j'$ for some $x_i, x_i' \in H_i$, x_j, $x_j' \in H_j$. Then $x_i x_i' = x_j x_j' \in H_i \cap H_j$ and $Q(x_i x_i') \subseteq Q_i \cap Q_j$. By assumption, $x_i x_i' = 1$ so $x_i = x_i'$ and $x_j = x_j'$. Now see that, for all $i \neq k$, $aH_i \cap H_i = \emptyset$ and $aH_i \cap H_k = \{a\}$ since $H_i \cap H_k = \{1\}$. Since $G_R = \cup_{i=1}^{n} H_i$, $aH_i \subseteq \cup_{j \neq i} H_j$. Thus $|H_i| \leq n-1$ and $|G_R| = \sum_{i=1}^{n}(|H_i|-1)+1 \leq n(n-2)+1$. □

Let $\mathrm{Max}(Y_R) = \{Q_i \mid Q_i \subseteq Q_j \Rightarrow i = j\}$.

LEMMA 3.6. $|\mathrm{Max}(Y_R)| \neq 2$.

PROOF. Suppose $\mathrm{Max}(Y_R) = \{Q_1, Q_2\}$ and let $x \in G_R$. $Q(x) \subset Q_1$

or $Q(x) \subset Q_2$, thus $x \in H(Q_1) \cup H(Q_2)$. But G_R is not the union of two proper subgroups so $G_R = H(Q_1)$ or $G_R = H(Q_2)$. This shows that $Q_2 \subseteq Q_1$ or $Q_1 \subseteq Q_2$, a contradiction. □

THEOREM 3.7. *Let R be a finitely generated non-degenerate Witt ring.*

(1) *If $|Y_R| = 1$ then R is a Witt ring of local type with* $\dot{1}$ *as associated Hasse diagram.*

(2) $|Y_R| \neq 2$.

(3) *If $|Y_R| = 3$ then R is a product of two Witt rings of local type and R has associated Hasse diagram* 7.

(4) *If $|Y_R| = 4$ then $R \cong S[t]$ with $|G_s| = 4$ and S is totally degenerate. R has associated Hasse diagram* 8.

PROOF. (1). follows from [**2**, Theorem 5].

(2). By Lemma 3.6, there must be an inclusion relation. This is impossible by Corollary 3.2.

(3). First show that diagram 9 is the associated Hasse diagram of R. Notice that $Q_1 \subset Q_2 \subset Q_3$ is not possible by Corollary 3.2. If there are no inclusion relations then $|G_R| \leq 4$ by Lemma 3.5. Witt rings with $|G_R| \leq 4$ have been classified and none of them satisfy the criteria. Thus assume $Y_R = \{Q_1, Q_2, Q_3\}$ with $Q_1 \subset Q_3$, $Q_2 \subsetneq Q_1$ and $Q_3 \subsetneq Q_2$. If $Q_2 \subsetneq Q_3$ then Lemma 3.6 is contradicted. Consequently the diagram must be diagram 10.

By Theorem 3.4, assume $|Q_1| = 2$. Reduction at Q_1 is possible by Corollary 2.4 and $Y_{\overline{R}} = \{Q_3/Q_1, Q_1Q_2/Q_1\}$. By part (2) of this theorem, $Q_3 = Q_1Q_2$, and, by part (1) of this theorem, $|Q_3| = 2|Q_1| = 4$. Consequently $|Q_3| = 2|Q_1| = 2|Q_2|$ and we finish by appealing to Cordes' classification [**1**].

(4) Show that the associated Hasse diagram must be diagram 11. There is no chain of length 4 by Corollary 3.2. Suppose that the associated Hasse diagram contains a chain of length 3, say $Q_1 \subset Q_2 \subset Q_3$. If $Q_4 \subsetneq Q_3$ we contradict Lemma 3.6. Three possible cases follow (see Diagram 12):

Case (i). Reduction at Q_4 is possible by Corollary 2.4. In $Y_{\overline{R}}$ we have $Q_1Q_4/Q_4 \subseteq Q_2Q_4/Q_4 \subseteq Q_3/Q_4$. By Corollary 3.2, there is equality at each spot. That is, $Q_1Q_4 = Q_2Q_4 = Q_3$. Now reduction at Q_1 is also possible by Corollary 2.4. In the resulting Witt ring $\overline{R}$, $|Y_{\overline{R}}| \leq 3$ and $Q_2/Q_1 \subsetneq Q_3/Q_1$. By part (2) of this theorem, $|Y_{\overline{R}}| = 3$ and thus $\overline{R}$ is the product of two Witt rings of local type. In particular $|Q_2/Q_1| = |Q_1Q_4/Q_1| = 2$ and $|Q_3/Q_1| = 4$. But recall $Q_1Q_4 = Q_3$, a contradiction.

Case (ii). Reduction at Q_4 is possible. The resulting Witt ring has $Y_{\overline{R}} = \{Q_1Q_4/Q_4, Q_2/Q_4, Q_3/Q_4\}$ with $Q_1Q_4/Q_4 \subseteq Q_2/Q_4 \subseteq Q_3/Q_4$. By Corollary 3.2 we have equality in both spots. But then $Q_2 = Q_3$, a contradiction.

Case (iii). For convenience, relabel the diagram 13. By [**2**, Theorem 13], $|Q_1| = 2$. Reduction at Q_1 is possible and the resulting Witt ring has $Y_{\overline{R}} = \{Q_2/Q_1, Q_3/Q_1, Q_4/Q_1\}$. By part (3) of this theorem, $\overline{R}$ is a product of two Witt rings of local type. In particular, $|Q_2| = 2|Q_1| = 4$, $|Q_3| = 2|Q_1| = 4$ and $|Q_4| = 2|Q_2| = 8$. Fix $a \in H_2 \backslash H_1$. Applying [**2**, Proposition 8] we obtain

$$\begin{aligned}(h_1 - 1)\frac{1}{8} &+ (h_2 - 2h_1)\frac{1}{16} + (h_2 - 1)\frac{1}{8} + (h_3 - h_1)\frac{1}{16} \\ &+ (h_4 - 2h_3 - h_2 + 2h_1)\frac{1}{32} + (h_3 - h_1)\frac{1}{16} \\ = (i_1 - 1)\frac{1}{2} &+ (i_2 - i_1)\frac{1}{4} + (i_3 - i_1)\frac{1}{4} + (i_4 - i_3 - i_2 + i_1)\frac{1}{8} + 1 - \frac{1}{2}.\end{aligned}$$

After simplifying and using $i_4 = D\langle\langle 1, -a\rangle\rangle = g/|Q_2| = g/4 = h_4/4$ we obtain

$$\frac{1}{32}(2h_1 + h_2 + 2h_3) - \frac{1}{4} = \frac{1}{8}(i_1 + i_2 + i_3).$$

Now, the mapping $H_j \to Q_2 \cap Q_j$ via $h \to q(a,,h)$ has kernel $D\langle\langle 1, -a\rangle\rangle \cap H_j$. Consequently, $h_1 \leq 2i_1$, $h_2 \leq 4i_2$ and $h_3 \leq 2i_3$ since $Q_2 \cap Q_1 = Q_1$, $Q_2 \cap Q_2 = Q_2$ and $Q_2 \cap Q_3 = Q_1$. It follows that $(2h_1 + h_2 + 2h_3)/32 \leq (i_1 + i_2 + i_3)/8$. From the above equation $-1/4 \geq 0$, a contradiction. Thus the associated Hasse diagram has no chain of length 3. If $|\mathrm{Max}(Y_R)| = 1$, say $Q_1, Q_2, Q_3 \subset Q_4$, then there are no inclusions among Q_1, Q_2, Q_3 or there would be a chain of length ≥ 3. We then have diagram 14 as desired. Recall that $|\max(Y_R)| \neq 2$ by Lemma 3.6. Suppose $|\mathrm{Max}(Y_R)| = 3$. We have three possible cases (see diagram 13). In each case reduction at Q_4 is possible by Corollary 2.4 yielding each time a Witt ring $\overline{R}$ with $|Y_{\overline{R}}| = 3$ and no inclusion relations, contradicting part (3) of this theorem. Finally, if $|\mathrm{Max}(Y_R)| = 4$, then Lemma 3.5 implies $|G| \le 8$. Such Witt rings have been classified and none have this Y_R. Consequently, diagram 16 is the associated Hasse diagram of R.

To finish the proof it suffices to show that $|Q_4| \le 4$, for the result follows from Cordes' classification [**1**]. Without loss of generality assume $|Q_1| \le |Q_2| \le |Q_3|$. Fix $a \in \dot{H}_1$. Note $|aH_2 \cap H_3| \le 1$ as in Lemma 3.5 since $H_2 \cap H_3 = \{1\}$. If $|aH_2 \cap H_3| = 0$ then $|Q_1| = 2$ by Theorem 3.4. If $|aH_2 \cap H_3| = 1$ then, by [**2**, Proposition 8],

$$
\begin{aligned}
&(h_1 - 2)\frac{1}{|Q_1|^2} + \frac{1}{|Q_1 \cap Q_3|}\frac{1}{|Q_2|} + (h_2 - 2)\frac{1}{|Q_1|\,|Q_2|} + \frac{1}{|Q_1 \cap Q_2|}\frac{1}{|Q_3|} \\
&\quad + (h_3 - 2)\frac{1}{|Q_1|\,|Q_3|} + (h_2 - 2)\frac{1}{|Q_1 \cap Q_2|}\frac{1}{|Q_4|} \\
&\quad + (h_3 - 2)\frac{1}{|Q_1 \cap Q_3|}\frac{1}{|Q_4|} + (h_4 - h_1 - 2h_2 - 2h_3 + 6)\frac{1}{|Q_1|\,|Q_3|} \\
&= (i_1 - 1)\frac{1}{|Q_1|} + (i_2 - 1)\frac{1}{|Q_2|} + (i_3 - 1_{\frac{1}{|Q_3|}} \\
&\quad + (h_4 - h_1 - h_2 - h_3 + 2)\frac{1}{|Q_4|} + 1 - \frac{2}{|Q_1|}.
\end{aligned}
$$

The following is obtained by simplifying:

$$
\begin{aligned}
&\left[\frac{1}{|Q_1|^2} - \frac{1}{|Q_1|\,|Q_4|}\right]h_1 + \left[\frac{1}{|Q_1|\,|Q_2|} + \frac{1}{|Q_1 \cap Q_2|\,|Q_4|} - \frac{2}{|Q_1|\,|Q_4|}\right]h_2 \\
&\quad + \left[\frac{1}{|Q_1|\,|Q_3|} + \frac{1}{|Q_1 \cap Q_3|\,|Q_4|} - \frac{2}{|Q_1|\,|Q_4|}\right]h_3 \\
(*3)\quad &\quad + \frac{1}{|Q_1|\,|Q_4|}h_4\left[-\frac{2}{|Q_1|^2} - 2\sum_{j=2}^{3}\frac{1}{|Q_1|\,|Q_j|} - 2\sum_{j=2}^{3}\frac{1}{|Q_1 \cap Q_j|\,|Q_4|}\right. \\
&\qquad\qquad \left. + \frac{1}{|Q_1 \cap Q_3|\,|Q_2|} + \frac{1}{|Q_1 \cap Q_2|\,|Q_3|} + \frac{6}{|Q_1|\,|Q_4|}\right] \\
&= \left[\frac{1}{|Q_1|} - \frac{1}{|Q_4|}\right]i_1 + \left[\frac{1}{|Q_2|} - \frac{1}{|Q_4|}\right]i_2 + \left[\frac{1}{|Q_3|} - \frac{1}{|Q_4|}\right]i_3 + \frac{1}{|Q_4|}i_4 \\
&\qquad + \left[-\frac{3}{|Q_1|} - \frac{1}{|Q_2|} - \frac{1}{|Q_3|} + \frac{2}{|Q_4|} + 1\right].
\end{aligned}
$$

Notice that reduction at Q_1 is possible and the resulting Witt ring $\overline{R}$ has $|Y_{\overline{R}}| \le 3$. First eliminate the case when $|Y_{\overline{R}}| = 3$. Here we obtain

$4|Q_1| = |Q_4| = 2|Q_1Q_2| = 2|Q_1Q_3|$ and thus $|Q_1 \cap Q_2| = |Q_2|/2$, $|Q_1 \cap Q_3| = |Q_3|/2$. Reduction at Q_2 is also possible yielding a Witt ring $\overline{R}$. If $|Y_{\overline{R}}| = 1$ then $Q_1Q_2 = Q_4$, a contradiction. Thus $|Y_{\overline{R}}| = 3$ yielding $4|Q_3| = |Q_4|$ and reduction at Q_3 is possible yielding $4|Q_3| = |Q_4|$. We have $|Q_4|/4 = |Q_1| = |Q_2| = |Q_3|$ and $|Q_1 \cap Q_2| = |Q_1 \cap Q_3| = |Q_1|/2$. Now $\dot{H}_1\dot{H}_2 \subset \dot{H}_3$. Otherwise, there exists $x_1 \in H_1$, $x_2 \in H_2$ with $x_1x_2 \notin H_3$. Consequently, $x_1x_2 \notin H_1 \cup H_2 \cup H_3$ so $Q(x_1x_2) = Q_4$. But $Q(x_1x_2) \subseteq Q(x_1)Q(x_2) = Q_1Q_2$, another contradiction. In the same way $\dot{H}_1\dot{H}_3 \subset \dot{H}_2$. Now $|a\dot{H}_2| \geq 1$, $a\dot{H}_2 \subseteq aH_2 \cap H_3$ and $|aH_2 \cap H_3| \leq 1$. So $a\dot{H}_2 = aH_2 \cap H_3$ and $|a\dot{H}_2| = 1$. Similarly $|a\dot{H}_3| = 1$. So $a\dot{H}_2 = aH_2 \cap H_3$ and $|a\dot{H}_2| = 1$. Similarly $|a\dot{H}_3| = 1$. Thus $|H_2| = |H_3| = 2$. Also $\dot{H}_1\dot{H}_3 \subseteq \dot{H}_2$ implies $|H_1| = 2$.

By $(*3)$ with $h_4 = g$ and $i_4 = g/|Q_1|$ we have

$$\frac{3}{|Q_1|^2} + \frac{g}{4|Q_1|^2} = \frac{3}{4|Q_1|}i_1 + \frac{3}{4|Q_1|}i_2 + \frac{3}{4|Q_1|}i_3 + \frac{g}{4|Q_1|^2} - \frac{9}{2|Q_1|} + 1$$
$$\frac{12}{|Q_1|} + \frac{g}{|Q_1|} = 3i_1 + 3i_2 + 3i_3 + \frac{g}{|Q_1|} - 18 + 4|Q_1|$$
$$\frac{12}{|Q_1|} = 3i_1 + 3i_2 + 3i_3 - 18 + 4|Q_1|.$$

Consequently $|Q_1|$ is a 2-power which divides 12, i.e., $|Q_1| = 2$ or 4. If $|Q_1| = 2$ then $16 = 3(i_1 + i_2 + i_3)$ which is impossible, and if $|Q_1| = 4$, then $5 = 3(i_1 + i_2 + i_3)$ which is again impossible. Therefore assume $|Y_{\overline{R}}| = 1$. Here $|Q_4| = 2|Q_1|$, $Q_1Q_2 = Q_1Q_3 = Q_4$ and then $|Q_2| = 2|Q_1 \cap Q_2|$ and $|Q_3| = 2|Q_1 \cap Q_3|$. Suppose now that this is reduced at Q_2. The resulting Witt ring $\overline{\overline{R}}$ has $|Y_{\overline{\overline{R}}}| \leq 3$. If $|Y_{\overline{\overline{R}}}| = 3$ then $|Q_4/Q_2| = 2|Q_1Q_2/Q_2|$. But $Q_1Q_2 = Q_4$, a contradiction. Consequently $|Y_{\overline{\overline{R}}}| = 1$ and $|Q_4| = 2|Q_2|$. Reduction at Q_3 yields a Witt ring of local type and $|Q_4| = 2|Q_3|$. We have

$$(*4) \qquad \frac{1}{2}|Q_4| = |Q_1| = |Q_2| = |Q_3| \text{ and}$$
$$|Q_1 \cap Q_2| = |Q_1 \cap Q_3| = \frac{1}{2}|Q_1|.$$

Recall our purpose is to show that $|Q_4| \leq 4$. Consequently, it suffices to show $|Q_1| = 2$. Recall that if $|aH_2 \cap H_3| = 0$ then $|Q_1| = 2$ by

Theorem 3.4, so assume $|aH_2 \cap H_3| = 1$. By (*3) and Lemma 3.3 it follows that

$$\begin{aligned} & -\frac{2}{|Q_1|^2} - 2\sum_{j=2}^{3} \frac{1}{|Q_1|\,|Q_j|} - 2\sum_{j=2}^{3} \frac{1}{|Q_1 \cap Q_j|\,|Q_4|} \\ & \quad + \frac{1}{|Q_1 \cap Q_3|\,|Q_2|} + \frac{1}{|Q_1 \cap Q_2|\,|Q_3|} + \frac{6}{|Q_1|\,|Q_4|} \\ & \geq -\frac{3}{|Q_1|} - \frac{1}{|Q_2|} - \frac{1}{|Q_3|} + \frac{2}{|Q_4|} + 1. \end{aligned}$$

Using (*4) and simplifying yields $-3/|Q_1|^2 \geq -4/|Q_1| + 1$, so $|Q_1|^2 - 4|Q_1| + 3 \leq 0$ and $(|Q_1| - 3)(|Q_1| - 1) \leq 0$. This implies $|Q_1| \leq 3$ and since $|Q_1|$ is a 2-power, $|Q_1| = 2$. □

LIST OF DIAGRAMS

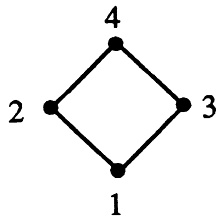


DIAGRAM 1.

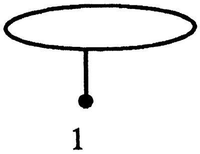
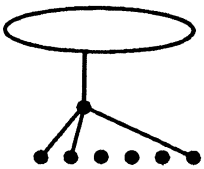


DIAGRAM 2.



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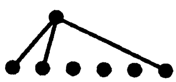

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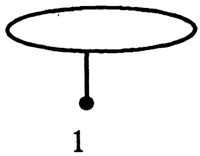

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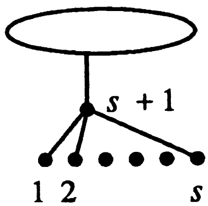

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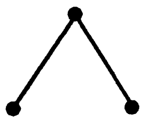

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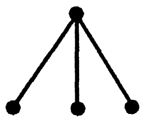

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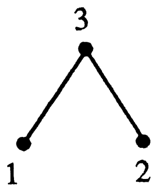

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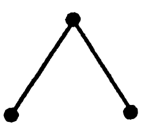

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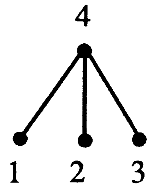

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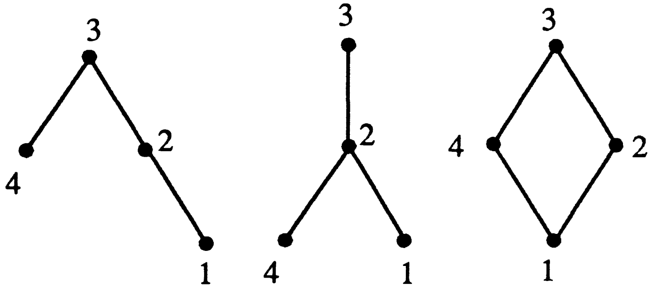

DIAGRAM 12. (i), (ii) and (iii)

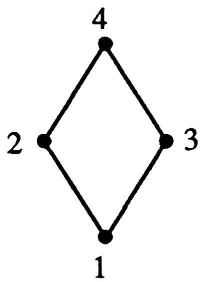

DIAGRAM 13.

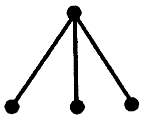

DIAGRAM 14.

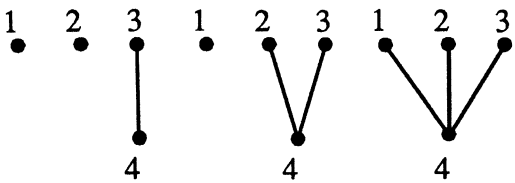

DIAGRAM 15. (i), (ii) and (iii)

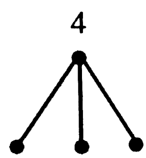

DIAGRAM 16.

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