ORDERINGS AND VALUATIONS ON *-FIELDS

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1. Introduction to orderings. Let $(D, *)$ be a *-field; that is, a skew field D with an involution $*$ (an anti-automorphism of order 2). Beginning with a definition of Baer, at least four different notions of ordering have been proposed for D [1, 3, 5, 6] with various relationships among them. In this paper we shall work with three of these notions, giving a description of the liftings to D of orderings of the residue class field of a valuation on D. The theorems proved below generalize the commutative theory for orderings and semiorderings (D commutative, $*$ equal to the identity) found in [7, Chapter 7]. The pioneering work with valuations on *-fields was done by Holland [4, 5]. We shall find it convenient to slightly modify some of his definitions in order to arrive at a complete theory of how orderings lift.

Some notation that we shall use throughout this paper includes writing $S(D)$ for the set of symmetric elements in D, namely $\{d \in D \mid d = d^*\}$, $\Pi S(D)$ for the set of all nonzero products of elements from $S(D)$ and, for any subset A of D, writing $A^\times$ for the collection of nonzero elements of A.

DEFINITION 1.1. A *Baer ordering* on $(D, *)$ is a subset P of $S(D)$ satisfying

(a) $P + P \subseteq P$;

(b) $1 \in P$ and, for any nonzero element $d \in D$, $dPd^* \subseteq P$;

(c) $P \cup -P = S(D)^\times$; and

(d) $P \cap -P = \emptyset$

For D commutative with $* =$ identity, a Baer ordering is a semiordering as defined in [7]. The next two definitions give different ways of extending the standard notion of ordering.

DEFINITION 1.2. A *Jordan ordering* of $(D, *)$ is a Baer ordering P

which also satisfies

(e) For any x, $y \in P$, $xy + yx \in P$.

DEFINITION 1.3. A *strong ordering* of $(D, *)$ is a $*$-closed subset P of D which satisfies

(a) $P + P \subseteq P$;

(b) $1 \in P$ and, for any nonzero element $d \in D$, $dPd^* \subseteq P$;

(c) $P \cup -P \supseteq S(D)^{\times}$;

(d) $P \cap -P = \emptyset$;

(e) $P \cdot P \subseteq P$.

The definition of strong ordering which we have just given differs only in form from the definition in [5]. Note that P is a normal subgroup of $D^{\times}$ since, for any $x \in P$ and $d \in D^{\times}$, we have $x^{-1} = x^*(xx^*)^{-1} \in P$ and $dxd^{-1} = (dxd^*)(dd^*)^{-1} \in P$. It is easy to see that if P is a strong ordering, then $P \cap S(D)$ is a Jordan ordering. Conversely, in [6] it is shown that any Jordan ordering gives rise to a strong ordering containing it. We shall see a somewhat more general version of this below (Theorem 2.5).

2. Valuation theory. For general valuation theory on skew fields one can refer to [8]. We need our valuations to also be compatible with the involution $*$. Following [4], we define a $*$-valuation on a $*$-field $(D, *)$ to be a valuation v onto an additively written ordered group Γ with the additional property that $v(x^*) = v(x)$ for all nonzero $x \in D$. It then follows that Γ is abelian since $v(a) + v(b) = v(ab) = v(b^*a^*) = v(b) + v(a)$. We shall write A_v for the valuation ring of v, m_v for its maximal ideal and D_v for the residue class (skew) field. Note that D_v has an induced involution, which we also denote by $*$. For $a \in A_v$, we write $\overline{a}$ for the element $a + m_v$ in D_v. We shall write the induced involution on D_v as $*$ also. As noted in [4], for any nonzero symmetric element $s \in S(D)$, there is an induced involution $\#$ on D_v defined as follows: for $y = \overline{x}$ in D_v, $y^{\#} = \overline{sxs^{-1}}$. In order to lift orderings from the residue class field, we need an additional restriction on the valuations.

DEFINITION 2.1. An element $s \in S(D)^\times$ is called *smooth* if the involution # induced by s on D_v is conjugate to $*$; that is, there exists an automorphism Λ_s of D_v so that

$$(2.2) \qquad \overline{sx^*s^{-1}} = \Lambda_s[(\Lambda_s^{-1}(\overline{x}))^*], \qquad x \in D, \ v(x) = 0.$$

A $*$-valuation v is *smooth* if each equivalence class $v^{-1}(\gamma)$, $\gamma \in \Gamma$, which contains some symmetric element, contains a smooth symmetric element. We say v is *strongly smooth* if, in addition, the element $d + d^*$ is smooth whenever d is a product of smooth symmetric elements.

This definition of smoothness is slightly less restrictive than that found in [4, 5] in that it applies only to symmetric elements. This suffices for lifting Baer orderings. In Example 5.1 we exhibit a valuation on a quaternion algebra which is smooth in our definition but not that of [4]. To lift Jordan orderings, we need the valuation to be strongly smooth.

The following theorem gives a commonly occurring sufficient condition for a valuation to be strongly smooth. In particular, the order valuation [4, 5] arising from looking at archimedean classes with respect to a strong ordering always satisfies this condition [5, Theorem 5.6]. Following [5], we say that a valuation v *collapses* a subgroup G of $D^\times$ if $v(d - 1) > 0$ for all $d \in G$.

THEOREM 2.3. *If a $*$-valuation v collapses the commutator subgroup* $[S(D)^\times, S(D)^\times]$, *then v is strongly smooth. In fact, all automorphisms* Λ_s *can be taken to be the identity.*

PROOF. Let $x \in D^\times, s \in S(D)^\times$. We shall show that the commutator $[s, x] = sxs^{-1}x^{-1}$ reduces to 1 in D_v from which (2.2) follows with Λ_s equal to the identity when $v(x) = 0$. To simplify notation in this proof, we shall write $a \equiv b$ for congruence modulo m_v; i.e., $\overline{a} = \overline{b}$ in D_v. We can write $x = (x+x^*)/2+(x-x^*)/2$. Since v collapses $[S(D)^\times, S(D)^\times]$, we have

$$sxs^{-1}x^{-1} = s(x + x^*)s^{-1}(x + x^*)^{-1}(x + x^*)x^{-1}/2 + s(x - x^*)s^{-1}x^{-1}/2$$
$$\equiv (x + x^*)x^{-1}/2 + s(x - x^*)s^{-1}(x - x^*)^{-1}(x - x^*)x^{-1}/2.$$

If we knew $[s, y] \equiv 1$ for skew elements y, then we would have $sxs^{-1}x^{-1} \equiv (x + x^*)x^{-1}/2 + (x - x^*)x^{-1}/2 = 1$. Thus it suffices to assume that x is skew; i.e., $x^* = -x$ and $x^2 = -xx^* \in S(D)^\times$.

Write $a = [s, x]$, so that $a^{-1} = xsx^{-1}s^{-1}$ and $v(a) = 0$. If $a \equiv 1$ we are done, so assume $a \not\equiv 1$. Since $x^2 \in S(D)^\times$, we have

$$1 \equiv [s, x^2] = sx^2 s^{-1} x^{-2} = ax(sxs^{-1}x^{-1})x^{-1} = axax^{-1},$$

which implies $x^{-1}a^{-1}x \equiv a$. Using this and the definition of a, we obtain

$$sa^* s^{-1} = sx^{-1}s^{-1}x = (x^{-1}a^{-1}s)s^{-1}x \equiv a,$$

whence, $a^* \equiv s^{-1}as$ and $a^{-1*} \equiv s^{-1}a^{-1}s$.

Now $(1 - a^{-1})sx = sx - a^{-1}sx = sx - xs = sx + (sx)^* \in S(D)^\times$, so our hypothesis on v implies

$$\begin{aligned}
1 &\equiv [s^{-1}, (1 - a^{-1})sx] = s^{-1}(1 - a^{-1})s(xsx^{-1}s^{-1})(1 - a^{-1})^{-1} \\
&= s^{-1}(1 - a^{-1})sa^{-1}(1 - a^{-1})^{-1} = s^{-1}(1 - a^{-1})s(a - 1)^{-1} \\
&\equiv (1 - a^{-1*})(a - 1)^{-1} = a^{-1*}(a^* - 1)(a - 1)^{-1}.
\end{aligned}$$

It follows that $a^*(a - 1) \equiv a^* - 1$, or $a^* a + 1 \equiv 2a^*$, whence $\bar{a} \in S(D_v)$. But then $a \equiv a^*$, so that $1 \equiv a^{-1}(a-1)(a-1)^{-1} = a^{-1}$, a contradiction of our assumption that $a \not\equiv 1$. □

It follows from the previous proof that if v collapses $[S(D)^\times, S(D)^\times]$, then, in fact, v collapses $[D^\times, \Pi S(D)]$. This seemingly stronger condition is considered in [5, 4.1]. Our next theorem shows how a strong ordering can be obtained from a Jordan ordering.

We adopt the usual notion of compatibility of an ordering and a valuation. If P is any Baer, Jordan or strong ordering and v is any $*$-valuation of D, then we say that P and v are *compatible* if, whenever $0 < a \le b$ with respect to P, then $v(a) \ge v(b)$. Note that this implies that the residue class field of v has characteristic zero. In fact, the ordering P induces an ordering $\overline{P}$ on the residue class field.

LEMMA 2.4. *Let P be an ordering (any type) of D compatible with a $*$-valuation v.*

(a) *If $a, b \in P$, then $v(a + b) = \min[v(a), v(b)]$.*

(b) *If P is a Jordan ordering and if $d = ab$ with a, b in $S(D)$, then $v(d + d^*) = v(d)$.*

PROOF. (a). For any valuation, we have $v(a + b) \geq \min[v(a), v(b)]$ with equality unless $v(a) = v(b)$. In this latter case, we have $0 < a \leq a + b$, so compatibility implies $v(a + b) \leq v(a) = \min[v(a), v(b)]$.

(b). We first note that, for any x, $y \in S(D)^{\times}$, we have $(xy)^2 + (yx)^2 = x(yxy) + (yxy)x \in P$ (consider the two cases $x \in P$ and $x \in -P$). We have $v(d + d^*) \geq v(d)$, or, equivalently, $v(aba^{-1}b^{-1} + 1) \geq 0$. Write $c = aba^{-1}b^{-1}$ and assume that $v(c + 1) > 0$. Then also $v(c^* + 1) > 0$ since v is a $*$-valuation, and therefore $v(c + c^* + 2) > 0$. But $c + c^* = ab((a^{-1}b^{-1})^2 + (b^{-1}a^{-1})^2)ba \in P$. Now we have $0 < 2 < c + c^* + 2$, so by compatibility we obtain the desired contradiction $0 < v(c + c^* + 2) \leq v(2) = 0$. □

THEOREM 2.5. *Let P be a Jordan ordering compatible with a $*$-valuation v. Set $Q = \{s + k \mid s \in P, k^* = -k, v(k) > v(s)\}$. Then Q is a strong ordering if and only if v collapses $[S(D)^{\times}, S(D)^{\times}]$. When this happens, Q is also compatible with v.*

PROOF. We have $Q = Q^*$ since v is a $*$-valuation. An application of Lemma 2.4(a) shows that $Q + Q \subseteq Q$. The conditions $Q \cap -Q = \emptyset$, $Q \cup -Q \supseteq S(D)^{\times}$ and $dQd^* \subseteq Q$ $(d \in D^{\times})$ follow from the fact that P is a Jordan ordering. Thus Q is a strong ordering if and only if it is closed under multiplication. For $i = 1, 2$, let $s_i + k_i \in Q$, where $s_i \in P \subseteq S(D)^{\times}$, $k_i^* = -k_i$, and $v(k_i) > v(s_i)$. Then the product $(s_1 + k_1)(s_2 + k_2) = s + k$, where

$$s = (s_1 s_2 + s_2 s_1 + k_1 k_2 + k_2 k_1 + k_1 s_2 - s_2 k_1 + s_1 k_2 - k_2 s_1)/2$$

and

$$k = (s_1 s_2 - s_2 s_1 + k_1 k_2 - k_2 k_1 + s_2 k_1 + k_1 s_2 + s_1 k_2 + k_2 s_1)/2.$$

Furthermore, $s^* = s$ and $k^* = -k$. We have $v(s_1 s_2 + s_2 s_1) = v(s_1) + v(s_2)$ by Lemma 2.4(b), which in turn equals $v(s)$ since the remainder of s has larger value. Since P is a Jordan ordering, we have $s_1 s_2 + s_2 s_1 \in P$, and hence $s \in P$ by compatibility. We need $v(k) > v(s)$ for Q to be closed under multiplication. But $v(k) \geq \min[v(s_1 s_2 - s_2 s_1), v(s_i) + v(k_j)$ $(i \neq j)]$, where $v(s_i) + v(k_j) > v(s_1) + v(s_2)$ by hypothesis. Now $v(s_1 s_2 - s_2 s_1) = v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) + v(s_1) + v(s_2) = v(s_1 s_2 s_1^{-1} s_2^{-1} -$

$1) + v(s)$. Therefore $v(k) > v(s)$ if and only if $v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) > 0$. It follows that if v collapses $[S(D)^\times, S(D)^\times]$, then Q is a strong ordering. On the other hand, if $v(aba^{-1}b^{-1} - 1) = 0$ for some $a, b \in S(D)^\times$, then, without loss of generality, $a, b \in P$ and we can set $s_1 = a$, $s_2 = b$, $k_1 = k_2 = 0$, and we see that Q is not closed under multiplication. Using the special form of elements of Q, compatibility of Q and v is an easy consequence of the compatibility of P and v. □

To put the hypotheses in the proper context, we note that, for any strong ordering Q compatible with a $*$-valuation v, if $a \in Q$, then $a = s + k$, where $s = (a + a^*)/2 \in Q \cap S(D)^\times$ and $k = (a - a^*)/2$ is a skew element with $v(k) \geq v(a) = v(s)$ by Lemma 2.4(a). Holland [5, Lemma 5.15] has shown that if one begins with a strong ordering Q and constructs the order valuation v by looking at archimedean classes in D, then $v(k) > v(s)$ whenever s and k are as above.

Moreover, if one begins with a Jordan ordering P and constructs the order valuation v, then defining Q as in Theorem 2.5 gives a maximal strong ordering [6, Corollary 2.3.7]. We shall denote this particular strong ordering containing P by P^s. Thus we have a one-to-one correspondence between Jordan orderings and maximal strong orderings. Our work below on lifting orderings from the residue class field applies equally to Jordan and maximal strong orderings. It will be convenient to frequently shift between the two concepts to take advantage of the fact that P is contained in $S(D)^\times$ or that P^s is closed under multiplication.

LEMMA 2.6. *Let v be a $*$-valuation and let $d \in \Pi S(D)$. Then $v(d + d^*) = v(d)$ if either of the following conditions hold*:

(a) *For $a \in \Pi S(D)$, $b \in S(D)^\times$, $v(1 + aba^{-1*}b^{-1}) \leq 0$,*

(b) *v is compatible with a Jordan ordering.*

PROOF. (a). Write $d = x_1 \cdots x_n$ with each x_i in $S(D)^\times$, and write $a = x_1 \cdots x_{n-1}$ and $b = x_n$. If $v(d + d^*) > v(d)$, then $v(aba^{-1*}b^{-1} + 1) > 0$. This contradicts the hypothesis on v.

(b). Let P be a Jordan ordering compatible with v. Then either $d \in P^s$ or $d \in -P^s$. In the former case, we are done by Lemma 2.4(a).

In the latter case, apply Lemma 2.4(a) to $-d$. □

Given a $*$-valuation $v : D^\times \to \Gamma$, we shall need a *semisection* $s : \Gamma \to D^x$; that is a mapping satisfying $v(s(\gamma)) = \gamma$ for each $\gamma \in \Gamma$. We shall define s somewhat differently than either [**7**] or [**4**]. We shall only need s defined on $S(\Gamma) = \{\gamma \in \Gamma \mid v(a) = \gamma \text{ for some } a \in S(D)^\times\}$. Note that $2\Gamma \subseteq S(\Gamma)$ since $\gamma = v(a)$ implies that $2\gamma = v(aa^*) \in S(\Gamma)$. The set $S(\Gamma)$ may not be a subgroup of Γ (cf. Example 5.3), but Lemma 2.6 can be used to show that it is a subgroup if either of the stated hypotheses is fulfilled. (Given $a_i \in S(D)^\times$ with $v(a_i) = \gamma_i$, we obtain $v(a_1 a_2^{-1} + a_2^{-1} a_1) = \gamma_1 - \gamma_2$.)

THEOREM 2.7. *Let v be a smooth $*$-valuation on $(D, *)$. Then v has a semisection $s : S(\Gamma) \to D^\times$ and automorphisms $\Lambda_d : D_v \to D_v$ satisfying* (2.2) *such that* (a) *through* (g) *hold:*

(a) $s(0) = 1$;

(b) $s(\gamma) \in S(D)^\times$ *for each* $\gamma \in S(\Gamma)$;

(c) *For each* $\gamma \in \Gamma$, *there exists* $d \in D^\times$ *such that* $s(2\gamma) = dd^*$;

(d) *For each* $\gamma \in S(\Gamma)$ *and* $\delta \in \Gamma$, *there exists* $d \in D^\times$ *such that* $s(\gamma + 2\delta) = ds\,(\gamma)d^*$;

(e) $\Lambda_1 = \mathrm{id}$;

(f) *For each* $\gamma \in \Gamma$, $\Lambda_{s(2\gamma)}(\overline{x}) = \overline{dxd^{-1}}$, *where* $s(2\gamma) = dd^*$;

(g) *For each* $\gamma \in S(\Gamma)$ *and* $\delta \in \Gamma$, $\Lambda_{s(\gamma + 2\delta)}(\overline{x}) = \overline{d\Lambda_{s(\gamma)}(x)d^{-1}}$, *where* $s(\gamma + 2\delta) = ds\,(\gamma)d^*$.

If, in addition, v is strongly smooth, then

(h) *If v is compatible with a strong ordering P, then for any γ_1, $\gamma_2 \in S(\Gamma)$, we have* $s(\gamma_1 + \gamma_2) \cdot s(\gamma_1) \cdot s(\gamma_2) \in P$;

(i) *If $v(1 + xyx^{-1*}y^{-1}) \leq 0$ for any $x \in \Pi S(D)$, $y \in S(D)^\times$, then* $s(\gamma_1 + \gamma_2) = a(cs(\gamma_1)s(\gamma_2) + s(\gamma_2)s(\gamma_1)c^*)a^*$, *for some $a \in A_v^\times$ and $c \in [D^\times, S(D)^\times]$.*

PROOF. We first define $s(0) = 1$ and $\Lambda_1 = \mathrm{id}$ so that (a) and (e) hold. For each $\gamma \in \Gamma$, choose an element $d \in D^\times$ such that $v(d) = \gamma$. Then define $s(2\gamma) = dd^*$ so that (c) will always hold. Define the

corresponding automorphism $\Lambda_{s(2\gamma)}$ of D_v by (f). An easy computation shows that (2.2) holds, i.e., dd^* is a smooth symmetric element. We next select a subset $\{\beta_i\}$ from $S(\Gamma)$ in one of two ways.

Case 1. The valuation v does not satisfy either condition (a) or (b) of Lemma 2.6. In this case choose $\{\beta_i\}$ to contain one representative from each nonzero coset of $S(\Gamma)$ modulo 2Γ. For each element β_i, select $s(\beta_i)$ to be any smooth symmetric element in $v^{-1}(\beta_i)$. It comes with an automorphism $\Lambda_{s(\beta_i)}$ by hypothesis. Finally, an arbitrary element of $S(\Gamma)$ has the form $\beta + 2\gamma$ for some $\beta \in \{\beta_i\}$. We define $s(\beta + 2\gamma) = ds\,(\beta)d^*$, where d is the element chosen previously so that $s(\gamma) = dd^*$. Then define $\Lambda_{s(\beta+2\gamma)}$ by (g). The computation that this satisfies Definition 2.1 can be found in [**4**, Lemma 3.1]. We have now verified (a) through (g).

Case 2. The valuation v does satisfy one of the hypotheses of Lemma 2.6. Now $S(\Gamma)$ is a group and we take $\{\beta_i\}$ to be a set of coset representatives for an ordered $\mathbf{Z}/2\mathbf{Z}$-basis of $S(\Gamma)/2\Gamma$. Again use the hypothesis on v to obtain $s(\beta_i)$ and $\Lambda_{s(\beta_i)}$. In this case every element of $S(\Gamma)$ has the form $\gamma = \sum_1^n \beta_i + 2\delta$. We can define $s(\Sigma\beta_i) = s(\beta_1)\cdots s(\beta_n) + s(\beta_n)\cdots s(\beta_1)$ by Lemma 2.6 and the fact that v is strongly smooth. (If the sum is empty, there is the single term $s(0) = 1$; throughout the remainder of the proof, this case requires a similar interpretation which we shall not make explicit.) Then define $s(\gamma) = ds(\Sigma\beta_i)d^*$, where d was chosen earlier so that $v(2\delta) = dd^*$. We obtain the automorphisms $\Lambda_{s(\Sigma\beta_i)}$ and $\Lambda_{s(\gamma)}$ satisfying (g) as in Case 1. To check (h), write $\gamma_i = \Sigma\beta_{j_i} + 2\delta_i$. Then $s(\gamma_1+\gamma_2) = d[s(\beta_1)\cdots s(\beta_n)+s(\beta_n)\cdots s(\beta_1)]d^*$, where the elements β_{j_i} are reordered with duplicates joining the even part of the sum $2(\delta_1+\delta_2)$ reflected in the element d. Using the facts that the sign of a product of symmetric elements is independent of the order and products of squares of symmetric elements are always positive [**5**, Theorem 3.4], it is not hard to see that $s(\gamma_1 + \gamma_2)$ is in P if and only if $s(\gamma_1)$ and $s(\gamma_2)$ have the same sign. Finally, we must check that (i) holds. Write γ_1 and γ_2 as above. Then $s(\gamma_1 + \gamma_2) = d[s(\beta_1)\cdots s(\beta_n) + s(\beta_n)\cdots s(\beta_1)]d^*$. Let $s(2\delta_i) = a_i a_i^*$ and set b equal to the product of elements $s(\beta_{j_i})$ containing one copy of each $s(\beta_{j_i})$ for each β_{j_i} common to γ_1 and γ_2.

Then

$$s(\gamma_1 + \gamma_2) = db^{-1}[bs(\beta_1)\cdots s(\beta_n)b^* + bs(\beta_n)\cdots s(\beta_1)b^*]b^{-1*}d^*$$
$$= db^{-1}[a_1^{-1}a_2^{-1}cs(\gamma_1)s(\gamma_2)a_2^{-1*}a_1^{-1*}$$
$$+ a_1^{-1}a_2^{-1}s(\gamma_2)s(\gamma_1)c^*a_2^{-1*}a_1^{-1*}]b^{-1*}d^*,$$

where c is a product of commutators in $[D^\times, S(D)^\times]$. Setting $a = db^{-1}a_1^{-1}a_2^{-1}$ yields (i). □

We shall call a semisection satisfying Theorem 2.7 a *smooth semisection*. The primary purpose of the automorphisms Λ_s is to take care of complications caused by noncommutativity in pushing down and lifting orderings. The following lemma is fundamental to this.

LEMMA 2.8. *Let v be a smooth $*$-valuation with smooth semisection s. Let $a \in S(D)^\times$ and $d \in D^\times$. Let $\sigma : S(\Gamma)/2\Gamma \to \{\pm 1\}$ be any function. Then we have*

$$\Lambda_{s(v(dad^*))}^{-1}\overline{[dad^*s(v(dad^*))^{-1}\sigma(\overline{v(a)})]}$$
$$= \overline{b\Lambda_{s(v(a))}^{-1}[as(v(a))^{-1}\sigma(\overline{v(a)})]b^*},$$

where $\overline{b} = \Lambda_{s(v(a))}^{-1}\overline{c^{-1}d}$ and $s(v(dad^*)) = cs(v(a))c^*$ (existence of c guaranteed by Theorem 2.7(d)).

PROOF. Set $\gamma = v(a)$. Then we have

$$\Lambda_{s(v(dad^*))}^{-1}\overline{[dad^*s(v(dad^*))^{-1}\sigma(\overline{\gamma})]}$$
$$= \Lambda_{s(\gamma)}^{-1}\overline{c^{-1}[dad^*c^{-1*}s(\gamma)^{-1}c^{-1}\sigma(\overline{\gamma})]c}$$
$$= [\Lambda_{s(\gamma)}^{-1}\overline{c^{-1}d}]\,[\Lambda_{s(\gamma)}^{-1}\overline{ad^*c^{-1*}s(\gamma)^{-1}\sigma(\overline{\gamma})}]$$
$$= \overline{b}\,[\Lambda_{s(\gamma)}^{-1}\overline{s(\gamma)s(\gamma)^{-1}as(\gamma)^{-1}\sigma(\overline{\gamma})}]\,[\Lambda_{s(\gamma)}^{-1}\overline{s(\gamma)d^*c^{-1*}s(\gamma)^{-1}}]$$
$$= \overline{b}\,[\Lambda_{s(\gamma)}^{-1}\overline{as(\gamma)^{-1}\sigma(\overline{\gamma})}]\,\overline{b}^*,$$

where the last step follows from Definition 2.1 for $\Lambda_{s(\gamma)}$. □

3. Lifting Baer orderings. Let $(D, *)$ be a $*$-field. We shall denote the set of all Baer orderings of $(D, *)$ by Y_D and the subset of

Jordan orderings by X_D. Let v be a smooth $*$-valuation of D. Then we shall denote the subset of Y_D compatible with v by Y_D^v and the Jordan orderings therein by X_D^v. In this section we shall determine the relationships among Y_D^v, X_D^v and Y_{D_v}. For $\gamma \in \Gamma$, we write $\overline{\gamma}$ for $\gamma + 2\Gamma$ in $\Gamma/2\Gamma$.

LEMMA 3.1. (Compare [**7**, Lemma 7.5].) *Let $v : D^\times \to \Gamma$ be a smooth $*$-valuation of D with residue class $*$-field D_v and $s : \mathrm{S}(\Gamma) \to D^\times$ a smooth semisection of v. Then every Baer ordering $P \in Y_D^v$ induces mappings*

$$\mathcal{P}_P : \mathrm{S}(\Gamma)/2\Gamma \to Y_{D_v} \ \ and \ \sigma_P : \mathrm{S}(\Gamma)/2\Gamma \to \{\pm 1\}$$

defined by

(i) $\sigma_P(\overline{\gamma})s(\gamma) \in P$ *for all $\gamma \in \mathrm{S}(\Gamma)$ and*

(ii) *for any $\overline{b} \in \mathrm{S}(D_v)^\times$, $\overline{b} \in \mathcal{P}_P(\overline{\gamma}) \Leftrightarrow$ for each lifting γ of $\overline{\gamma}$, $\exists\, a \in P$ with $v(a) = \gamma$ and $\overline{b} = \Lambda_{s(\gamma)}^{-1}[\overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})}]$.*

PROOF. The function σ_P is well-defined by Theorem 2.7(d). To deal with $\mathcal{P}_P(\overline{\gamma})$, we note that if there exists an appropriate $a \in P$ for one lifting of $\overline{\gamma}$, then there exists a positive element for every lifting by Lemma 2.8. Next we claim that, given $\overline{b} \in \mathrm{S}(D_v)^\times$, there exists an element $a \in \mathrm{S}(D)^\times$ such that $\overline{b} = \Lambda_{s(\gamma)}^{-1}[\overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})}]$. A first approximation to a is the element $xs(\gamma)\sigma_P(\overline{\gamma})$ where x is any lifting of $\Lambda_{s(\gamma)}(\overline{b})$; this satisfies the equation but may not be symmetric. Since $\overline{b}$ is symmetric, we have

$$\begin{aligned}
\overline{b} &= \Lambda_{s(\gamma)}^{-1}[\overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})}] \\
&= (\Lambda_{s(\gamma)}^{-1}[\overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})}])^* \\
&= \Lambda_{s(\gamma)}^{-1}[\overline{a^*(s(\gamma)^{-1}\sigma_P(\overline{\gamma})}],
\end{aligned}$$

the last equality following from the defining property of $\Lambda_{s(\gamma)}$. It follows that there exists an element $m \in m_v$ such that $as(\gamma)^{-1} = a^*s(\gamma)^{-1} + m$. Now replace a by $a - ms(\gamma)/2$. This is easily seen to be a symmetric element satisfying $\overline{b} = \Lambda_{s(\gamma)}^{-1}[\overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})}]$. This implies that $\mathcal{P}_P(\overline{\gamma}) \cup -\mathcal{P}_P(\overline{\gamma}) = \mathrm{S}(D_v)^\times$. Next let $d \in \mathrm{S}(D)^\times$ with $v(d) = 0$,

and let $\bar{b} \in \mathcal{P}_P(\bar{\gamma})$. Then there exists $a \in P$ such that $v(a) = \gamma$ and $\bar{b} = \Lambda_{s(\gamma)}^{-1}[as(\gamma)^{-1}\sigma_P(\bar{\gamma})]$. Thus

$$\overline{dbd^*} = \bar{d}\Lambda_{s(\gamma)}^{-1}\overline{[as(\gamma)^{-1}\sigma_P(\bar{\gamma})]}\bar{d}^*$$
$$= \Lambda_{s(\gamma)}^{-1}\overline{[xax^*s(\gamma)^{-1}\sigma_P(\bar{\gamma})]},$$

where x is any lifting of $\Lambda_{s(\gamma)}(\bar{d})$ by Lemma 2.8. Using Lemma 2.4(a), the remainder of the definition of a Baer ordering can be checked for $\mathcal{P}_P(\bar{\gamma})$ by straightforward computation. $\square$

Our next lemma shows how orderings can be lifted from D_v to D.

LEMMA 3.2. (Compare [7, Lemma 7.7].) *Let* $v : D^\times \to \Gamma$ *be a smooth valuation on* D *with smooth semisection* $s : \mathrm{S}(\Gamma) \to D^\times$. *Consider any two functions* $\mathcal{P} : \mathrm{S}(\Gamma)/2\Gamma \to Y_{D_v}$ *and* $\sigma : \mathrm{S}(\Gamma)/2\Gamma \to \{\pm 1\}$ *such that* $\sigma(\bar{0}) = 1$. *These functions induce a Baer ordering* $P \in Y_D^v$ *defined by*

$$(3.3) \qquad a \in P \Leftrightarrow \Lambda_{s(v(a))}^{-1}\overline{[as(v(a))^{-1}\sigma(\overline{v(a)})]} \in \mathcal{P}(\overline{v(a)}).$$

PROOF. First note that if $a \in S(D)^\times$, then $v(a) \in S(\Gamma)$, so $\mathcal{P}$ and σ are defined at $\overline{v(a)}$. Also, the right hand side of (3.3) is symmetric in D_v (cf. (2.2)) so properties (c) and (d) of a Baer ordering follow from the corresponding properties for $\mathcal{P}(\overline{v(a)})$.

For property (b), let $a \in P$ and $d \in D^\times$. Then $v(dad^*) \equiv v(a)$ mod 2Γ and $s(v(dad^*)) = cs(v(a))c^*$ for some $c \in D^\times$. Therefore, using Lemma 2.8,

$$\Lambda_{s(v(dad^*))}^{-1}\overline{[dad^*s(v(dad^*))^{-1}\sigma(\overline{v(a)})]}$$
$$= \bar{b}\Lambda_{s(v(a))}^{-1}\overline{[as(v(a))^{-1}\sigma(\overline{v(a)})]}\bar{b}^*,$$

where $\bar{b} = \Lambda_{s(v(a))}^{-1}\overline{c^{-1}d}$, and thus $dad^* \in P$ since $\mathcal{P}(\overline{v(a)})$ is a Baer ordering.

Finally we check that P is closed under addition. Let $a, b \in P$. If $v(a) = v(b)$, then (3.3) gives us $\Lambda_{s(v(a))}^{-1}\overline{[(a+b)s(v(a))^{-1}\sigma(\overline{v(a)})]}$

in $\mathcal{P}(\overline{v(a)})$. Since we have a unit in D, we must have $v(a + b) = v(s(v(a))) = v(a)$. Replacing $v(a)$ by $v(a + b)$ we obtain $a + b \in P$. On the other hand, if $v(a) \neq v(b)$, we may assume $v(a) < v(b)$. Then $v(a \pm b) = v(a)$ and $bs(v(a))^{-1} \in m_v$. From (3.3) we obtain $a \pm b \in P$. This gives us closure under addition and also compatibility of P with v since $0 < a \leq b$ implies $a - b \notin P$, hence $v(b) \leq v(a)$. $\square$

THEOREM 3.4. *Let* $v : D^\times \to \Gamma$ *be a smooth valuation on a* *-*field* D *with smooth semisection* $s : \mathrm{S}(\Gamma) \to D^\times$. *Then the constructions of Lemmas 3.1 and 3.2 yield an invertible one-to-one correspondence between* Y_D^v *and* $\{P | P : \mathrm{S}(\Gamma)/2\Gamma \to Y_{D_v}\} \times \{\sigma | \sigma : \mathrm{S}(\Gamma)/2\Gamma \to \{\pm 1\}, \sigma(\overline{0}) = 1\}$.

PROOF. Let $P \in Y_D^v$ with $\mathcal{P}$ and σ defined by Lemma 3.1. Let Q be the lifting defined by Lemma 3.2 for $\mathcal{P}$ and σ. Note that if d lies in a Baer ordering, so does $d^{-1} = d^{-1}dd^{-1*}$. Thus, since A_v is a valuation ring, it suffices to show that P and Q agree on A_v. Now $a \in Q \Rightarrow \Lambda_{s(\gamma)}^{-1} \overline{as(\gamma)^{-1}\sigma(\overline{\gamma})} \in \mathcal{P}(\overline{\gamma}), \gamma = v(a) \Rightarrow \exists b \in P$ such that $v(b) = \gamma$ and $\overline{as(\gamma)^{-1}} = \overline{bs(\gamma)^{-1}}$. This implies that there exists an element $m \in m_v$ such that $a = b + ms(\gamma)$. For $a \in A_v$, we have $\gamma \geq 0$, so $v(ms(\gamma)) > v(a)$; by compatibility of P with v, we obtain $a \in P$, which shows that $P = Q$.

Conversely, given mappings $\mathcal{P}$ and σ, define P by Lemma 3.2 and construct $\mathcal{P}_P$ and σ_P as in Lemma 3.1. For any γ in $S(\Gamma)$, we have $\Lambda_{s(\gamma)}^{-1} \overline{[\sigma(\overline{\gamma})s(\gamma)]s(\gamma)^{-1}\sigma(\overline{\gamma})} = \overline{1} \in \mathcal{P}(\overline{\gamma})$, hence $\sigma(\overline{\gamma})s(\gamma) \in P$ by Lemma 3.2. It follows that $\sigma = \sigma_P$. Now $\overline{b} \in \mathcal{P}_P(\overline{\gamma})$ implies by Lemma 3.1 that there exists $a \in P$ with $v(a) = \gamma$ and $\overline{b} = \Lambda_{s(\gamma)}^{-1} \overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})} = \Lambda_{s(\gamma)}^{-1} \overline{as(\gamma)^{-1}\sigma(\overline{\gamma})} \in \mathcal{P}(\overline{\gamma})$, the last containment coming from Lemma 3.2. Thus $\mathcal{P}_P = \mathcal{P}$. $\square$

4. Jordan orderings. We now turn our attention to lifting Jordan orderings. At this point the theory becomes much more complicated than the commutative case in [**7**, Theorem 7.9]. One reason to expect this is that, unlike the commutative situation, there exist noncommutative *-fields for which X_D^v is empty but Y_D^v is nonempty (see Example 5.2 and [**5**]).

LEMMA 4.1. *Let P be a strong ordering. Let $a, b \in \Pi S(D)$. Then $ab \in P$ if and only if $ab + ba \in P$. In particular, if $a, b \in S(D)^{\times}$, then $ab \in P$ if and only if $ab + ba$ lies in the Jordan ordering $P \cap S(D)^{\times}$.*

PROOF. Assume first that $ab \in P$. Strong orderings contain the commutators $[S(D)^{\times}, S(D)^{\times}]$ by [**5**, Theorem 3.4(3)]. Therefore $ba \in P$ and hence $ab + ba \in P$. Conversely, assume $ab \notin P$. Since $a, b \in \Pi S(D)$, we must have $ab \in -P$. But then $ba \in -P$, so that $ab + ba \in -P \cup \{0\}$, hence is not in P. □

LEMMA 4.2. *Let P be a Jordan ordering of D compatible with a smooth $*$-valuation v. Write $\overline{P}$ for $\mathcal{P}_P(0)$. Then $\overline{P}^s$, the maximal strong ordering containing $\overline{P}$, is equal to $\{x + m_v \mid x \in P^s \cap A_v^{\times}\}$.*

PROOF. This will be proved by using the work of Holland and Idris mentioned in the remarks following Theorem 2.5. Let w be the order valuation on D for the ordering P and $\overline{w}$ be the order valuation on D_v for the ordering $\overline{P}$. Explicitly, this means that $A_w = \{x \in D \mid n - xx^* \in P$ for some positive integer $n\}$ and $A_{\overline{w}} = \{\overline{x} \in D_v \mid n - \overline{xx}^* \in \overline{P}$ for some positive integer $n\}$. Letting $\varphi : A_v \to D_v$ denote the canonical homomorphism, it is easy to show that $\varphi^{-1}(A_{\overline{w}}) = A_w$. Now the maximal strong orderings are defined by $P^s = \{s + k \mid s \in P, \ k^* = -k, \ w(k) > 0\}$ and $\overline{P}^s = \{\overline{s} + \overline{k} \mid \overline{s} \in \overline{P}, \ \overline{k}^* = -\overline{k}, \ w(\overline{k}) > 0\}$. Since $\varphi(P^s \cap A_v)$ is a strong ordering of D_v containing $\overline{P}$, it must be contained in $\overline{P}^s$. Conversely, let $\overline{s} + \overline{k}$ be in $\overline{P}^s$, where we may assume $s \in P$ without loss of generality. Also $w(k) > 0$ since $\varphi^{-1}(m_{\overline{w}}) \subseteq m_w$. Thus $s + k \in P^s \cap A_v$, completing the proof. □

THEOREM 4.3. *Let $v : D^{\times} \to \Gamma$ be a strongly smooth valuation on a $*$-field D with smooth semisection $s : S(\Gamma) \to D^{\times}$. For any P in Y_D^v, the following are equivalent:*

(1) $P \in X_D^v$.

(2) (a) $\mathcal{P}_P(\overline{0})$ *is a Jordan ordering of* D_v;

     (b) $S(\Gamma)/2\Gamma$ *is a group and* σ_P *is a character;*

     (c) *For all* $\overline{\gamma} \in S(\Gamma)/2\Gamma$, $\mathcal{P}_P(\overline{\gamma}) = \Lambda_{s(\gamma)}^{-1}(\mathcal{P}_P(\overline{0})^s) \cap S(D_v)$;

(d) *For all* $a \in D^\times$, $b \in S(D)^\times$, $\overline{aba^{-1}b^{-1}} \in \mathcal{P}_P(\overline{0})^s$.

PROOF. (1) $\Rightarrow$ (2). By definition, $\mathcal{P}_P(\overline{0})$ is the image of $P \cap A_v^\times$, hence is a Jordan ordering because P is.

The set $S(\Gamma)$ is a group by the comments following Lemma 2.6. For γ_1, γ_2 in $S(\Gamma)$, $\sigma_P(\overline{\gamma}_1 + \overline{\gamma}_2) = 1 \Leftrightarrow s(\gamma_1 + \gamma_2) \in P$ by definition of σ_P. By Theorem 2.7(h), this holds if and only if $s(\gamma_1)$ and $s(\gamma_2)$ both lie in P or both lie in $-P$, which in turn is equivalent to $\sigma_P(\overline{\gamma}_1)\sigma_P(\overline{\gamma}_2) = 1$. Therefore σ_P is a character and (b) holds.

Next let $\overline{\gamma}$ be any element of $S(\Gamma)/2\Gamma$. By Lemma 3.2, we have $\overline{b} \in \mathcal{P}_P(\overline{\gamma}) \Leftrightarrow \overline{b} = \Lambda_{s(\gamma)}^{-1}\overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})}$ for some $a \in P$ with $v(a) = \gamma$. Thus if $\overline{b} \in \mathcal{P}_P(\overline{\gamma})$, then $\Lambda_{s(\gamma)}\overline{b} \in \overline{(P^s \cap A_v^\times)} = \mathcal{P}_P(\overline{0})^s$ by Lemma 4.2. Therefore $\overline{b} \in \Lambda_{s(\gamma)}^{-1}(\mathcal{P}_P(\overline{0})^s) \cap S(D_v)$. Conversely, assume $\overline{b} \in \Lambda_{s(\gamma)}^{-1}(\mathcal{P}_P(\overline{0})^s) \cap S(D_v)$. Let $c \in P^s \cap A_v^\times$ such that $\Lambda_{s(\gamma)}\overline{b} = \overline{c}$ and set $a = cs(\gamma)\sigma_P(\overline{\gamma})$. Then $\overline{b} = \Lambda_{s(\gamma)}^{-1}\overline{c} = \Lambda_{s(\gamma)}^{-1}\overline{as(\gamma)^{-1}\sigma_P(\overline{\gamma})}$. Now $\overline{b}$ is symmetric, so $\overline{b} = \Lambda_{s(\gamma)}^{-1}\overline{a^*s(\gamma)^{-1}\sigma_P(\overline{\gamma})}$ by (2.2). Therefore, $\overline{b} = \Lambda_{s(\gamma)}^{-1}\overline{[(a+a^*)/2]s(\gamma)^{-1}\sigma_P(\overline{\gamma})}$ where $(a+a^*)/2 \in P^s \cap S(D)^\times = P$, and $v((a+a^*)/2) = v(a) = \gamma$ by Lemma 2.6.

Finally (d) holds by [**5**, Theorem 3.4(3)].

(2) $\Rightarrow$(1). We must check the multiplicative property for the Baer ordering P. Let $a_1, a_2 \in P$. Then, for $i = 1, 2, \Lambda_{s(v(a_i))}^{-1}\overline{a_is(v(a_i))^{-1}\sigma_P(\overline{v(a_i)})}$ $\in \mathcal{P}_P(\overline{v(a_i)})$, and so

$$\overline{a_is(v(a_i))^{-1}\sigma_P(\overline{v(a_i)})} \in \Lambda_{s(v(a_i))}[\mathcal{P}_P(\overline{v(a_i)})] \subseteq \mathcal{P}_P(\overline{0})^s.$$

By condition 2(d) and the fact that $\mathcal{P}_P(\overline{0})^s$ is a strong ordering, the products

$$\overline{a_1a_2s(v(a_1))^{-1}s(v(a_2))^{-1}\sigma_P(\overline{v(a_1)})\sigma_P(\overline{v(a_2)})}$$

and

$$\overline{a_2a_1s(v(a_1))^{-1}s(v(a_2))^{-1}\sigma_P(\overline{v(a_1)})\sigma_P(\overline{v(a_2)})}$$

both lie in $\mathcal{P}_P(\overline{0})^s$. By 2(b), we may replace $\sigma_P(\overline{v(a_1)})\sigma_P(\overline{v(a_2)})$ by $\sigma_P(\overline{v(a_1) + v(a_2)})$. Adding, we obtain

$$\overline{(a_1a_2 + a_2a_1)s(v(a_1))^{-1}s(v(a_2))^{-1}\sigma_P(\overline{v(a_1) + v(a_2)})} \in \mathcal{P}_P(\overline{0})^s.$$

Using Theorem 2.7(i) and condition 2(d), we see that we may replace $s(v(a_1))^{-1}s(v(a_2))^{-1}$ by $s(v(a_1) + v(a_2))^{-1}$, obtaining

$$(a_1, a_2 + a_2 a_1) s(v(a_1) + (v(a_2))^{-1} \sigma_P \overline{(v(a_1) + v(a_2))} \in \mathcal{P}_P(\overline{0})^s.$$

But then

$$\Lambda_{s(v(a_1,a_2))}^{-1} \overline{(a_1 a_2 + a_2 a_1) s(v(a_1 a_2))^{-1} \sigma_P \overline{(v(a_1 a_2))}}$$

lies in $\Lambda_{s(v(a_1 a_2))}^{-1}(\mathcal{P}_P(\overline{0})^s) \cap \mathrm{S}(D_v) = \mathcal{P}_P(\overline{v(a_1 a_2)})$. By Lemma 3.2, we obtain $a_1 a_2 + a_2 a_1 \in P$, and thus $P \in X_D^v$. $\square$

REMARK 4.4. Let P be a Jordan ordering. Then an element $a \in \mathrm{S}(D)^\times$ lies in $P \Leftrightarrow \overline{[as(v(a))^{-1} + s(v(a))^{-1}a]\sigma \overline{(v(a))}}$ lies in $P \Leftrightarrow [as(v(a))^{-1} + s(v(a))^{-1}a]\sigma\overline{(v(a))}$ lies in $\mathcal{P}_P(\overline{0})$. Thus Lemma 3.2 becomes much simpler for Jordan orderings in that P depends only on $\mathcal{P}_P(\overline{0})$. This holds even though Theorem 4.3 shows that the mapping $\mathcal{P}_P : \mathrm{S}(\Gamma)/2\Gamma \to Y_{D_v}$ may not be a constant mapping as it is in the commutative situation. Note however, that $\mathcal{P}_P$ will be a constant mapping if all automorphisms Λ_s are the identity. In particular, this will occur if v collapses $[\mathrm{S}(D)^\times, \mathrm{S}(D)^\times]$ (cf. Theorem 2.3).

COROLLARY 4.5. *Let D be a $*$-field with a $*$-valuation v which collapses $[\mathrm{S}(D)^\times, \mathrm{S}(D)^\times]$. Then there is a bijective correspondence between X_D^v and $X_{D_v} \times \mathrm{Hom}(\mathrm{S}(\Gamma)/2\Gamma, \pm 1)$.*

PROOF. Since v collapses $[\mathrm{S}(D)^\times, \mathrm{S}(D)^\times]$, condition 2(d) of Theorem 4.3 holds (cf. discussion following Theorem 2.3) and $\mathrm{S}(\Gamma)$ is a group (cf. remarks prior to Theorem 2.7). The one-to-one correspondence is given by Theorem 4.3: $P \leftrightarrow (\mathcal{P}_P(\overline{0}), \sigma_P)$. $\square$

This completes the attempt that Holland made to lift strong orderings in [5, Theorem 4.3]. He was only able to show that, for a $*$-valuation v which collapses $[D^\times, \mathrm{S}(D)^\times]$, if $X_{D_v} \neq \emptyset$ then there exists some strong ordering on D. He is able to give examples of the lifting, but only in cases where all $\Lambda_s = \mathrm{id}_{D_v}$.

COROLLARY 4.6. *Let D be a $*$-field with a $*$-valuation v which collapses $[S(D)^\times, S(D)^\times]$. Then $X_D^v = Y_D^v$ if and only if either*

(a) $|S(\Gamma)/2\Gamma| = 1$ *and* $X_{D_v} = Y_{D_v}$, *or*

(b) $|S(\Gamma)/2\Gamma| = 2$ *and* $|X_{D_v}| = |Y_{D_v}| = 1$.

PROOF. As in the proof of Corollary 4.5, $S(\Gamma)$ is a group. Note that every mapping $\sigma : S(\Gamma)/2\Gamma \to \{\pm 1\}$ is a character if and only if $|S(\Gamma)/2\Gamma| \le 2$. Theorem 4.3 shows that if $|S(\Gamma)/2\Gamma| > 1$, then there are non-Jordan orderings unless $|X_{D_v}| = |Y_{D_v}| = 1$; and if $|S(\Gamma)/2\Gamma| = 1$, there are non-Jordan orderings unless $X_{D_v} = Y_{D_v}$. □

In the commutative case with $* = \mathrm{id}$, the conditions of the corollary, when they hold for all real valuations, have numerous other equivalent conditions, giving the so-called SAP fields which have been considered by many authors (for definitions, see [7]). It is not known to what extent these other conditions will generalize to $*$-fields.

For commutative fields, the latter part of (b) becomes $|Y_{D_v}| = 1$ since this implies that $|X_{D_v}| = 1$ (cf. Corollary 4.7 below). This is not true in general. The converse, $|X_{D_v}| = 1 \Rightarrow |Y_{D_v}| = 1$, also holds for commutative fields with $* = \mathrm{id}$; whether it holds in general remains open, but seems unlikely.

COROLLARY 4.7. *Let $(F, *)$ be a commutative $*$-field with $*$-valuation v. If $|Y_F^v| \ne \emptyset$, then $|X_F^v| \ne \emptyset$.*

PROOF. Commutativity implies that the valuation collapses $[S(F)^\times, S(F)^\times]$. Every commutative $*$-field has a place into either $(\mathbf{R}, \mathrm{id})$ or $(\mathbf{C}, *)$, with $* = $ conjugation [5], hence we may assume inductively that F_v has a Jordan ordering. The result then follows from Corollary 4.5. □

5. Examples. In this section, we present a few examples to illustrate the theory above. Details on the construction of the skew fields referred to can be found in [2].

EXAMPLE 5.1. Set $D = \mathbf{C}((x))$, the field of twisted Laurent series over the complex numbers, where $xa = a^*x$ for $a \in \mathbf{C}$ and a^* is the complex conjugate of a. Extend $*$ to D by $(\Sigma a_k x^k)^* = \Sigma (-1)^k a_k^{(*^{k+1})} x^k$, where $a^{(*^r)}$ indicates that a should be conjugated r times; thus $a^{(*^r)}$ equals a if r is even and equals a^* if r is odd. One checks easily that with this definition, $*$ is an involution, $x^* = -x$ and

$$S(D) = \{\Sigma a_k x^{2k} \mid a_k \in \mathbf{R}\} = Z_D, \text{ the center of } D.$$

Therefore D is a standard quaternion $*$-algebra over $S(D)$ generated by x and i.

Define a valuation on D by $v(\Sigma_{k=m}^\infty a_k x^k) = m \in \mathbf{Z} = \Gamma$, where $a_m \neq 0$. This is clearly a $*$-valuation. Here we have $S(\Gamma) = 2\mathbf{Z}$. Since symmetric elements are central, Theorem 2.3 holds and we can take all $\Lambda_s = \text{id}$. Another alternative is to follow Theorem 2.7, setting $s(2k) = x^k x^{k*} = (-1)^k x^{2k}$ and $\Lambda_{2k}(a + bi) = \overline{x^k(a + bi)x^{-k}} = a + (-1)^k bi$.

Note that the skew element x is not smooth since $a^\# = xa^*x^{-1} = a$ for $a \in \mathbf{C}$, and thus $\# = \text{id}$ which is not equivalent to $*$. Similarly, for any element of $v^{-1}(1)$, so v is (strongly) smooth in our definition but not in that of [4].

Since $S(\Gamma)/2\Gamma$ is trivial and $(\mathbf{C}, *)$ has the unique ordering $\mathbf{R}^+$, we see that $(D, *)$ also has a unique Jordan ordering given by Theorems 3.4 and 4.3: for $\Sigma_{k=m}^\infty a_k x^{2k} \in S(D)$, $a_m \neq 0$, $\Sigma a_k x^{2k} \in P \Leftrightarrow (-1)^m a_m \in \mathbf{R}^+$.

EXAMPLE 5.2. Set $D = \mathbf{C}((x))((y))$, Laurent series in two variables with $xy = -yx$. Now define $*$ so that it will be conjugation on $\mathbf{C}$ with $x^* = x$ and $y^* = y$:

$$(\Sigma a_{rs} x^r y^s)^* = \Sigma a_{rs}^* (-1)^{rs} x^r y^s.$$

A computation shows that this does indeed give an involution. Then $Z_D = \mathbf{C}((x^2))((y^2))$, so again we have D being a quaternion algebra over its center; but in this case the involution $*$ is not the standard one. We shall see that this has a marked effect on the orderings, primarily because we must now order a much larger set of symmetric elements. One can check that

$$S(D) = \{\Sigma a_{rs} x^r y^s \mid a_{sr} \in \mathbf{R} \text{ if } rs \equiv 0(\mod 2)$$
$$\text{and } a_{rs} \in \mathbf{R} \cdot i \text{ if } rs \equiv 1(\mod 2)\}.$$

The $*$-field D has a $*$-valuation $v : D^\times \to \Gamma = \mathbf{Z} \times \mathbf{Z}$ (ordered lexicographically) defined by $v(a_{mn}x^m y^n + \cdots) = (n, m)$ with $D_v = \mathbf{C}$. In this case, $S(\Gamma) = \Gamma$ and we can define a semisection by $s(1, 0) = y$, $s(0, 1) = x$ and $s(1, 1) = ixy$. A check of Definition 2.1 shows that we can take all $\Lambda_s = $ id, even though the valuation does not have the nice properties of Theorem 2.3. There are eight choices for $\sigma : S(\Gamma)/2\Gamma \to \{\pm 1\}$ with $\sigma(0, 0) = 1$, and thus $(D, *)$ has eight Baer orderings, none of which are Jordan orderings since $xyx^{-1}y^{-1} = -1$ violates Theorem 4.3(2)(d).

EXAMPLE 5.3. Set $D_0 = \mathbf{R}((x))((y))$, a sub-$*$-field of D in Example 5.2. Then

$$S(D) = \{\Sigma a_{rs} x^r y^s \mid rs \equiv 0 \ (2)\}.$$

The value group $\Gamma_0 = \Gamma = \mathbf{Z} \times \mathbf{Z}$, but $S(\Gamma_0)/2\Gamma_0 = \{(0, 0), (1, 0), (0, 1)\}$ is not a group. In this case there are only four choices for the mapping σ of Theorem 3.4 and thus D_0 has only four Baer orderings. This can also be seen by restricting the eight Baer orderings of D.

REFERENCES

1. M. Chacron, *C-orderable division rings with involution*, J. Algebra **75** (1982), 495-521.

2. P. Cohn, *Skew field constructions*, London Math. Soc. Lecture Notes No. 27, Cambridge Univ. Press, London, 1977.

3. S. Holland, Jr., *Orderings and square roots in $*$-fields*, J. Algebra **46** (1977), 207-219.

4. ———, *$*$-valuations and ordered $*$-fields*, Trans. Amer. Math. Soc. **262** (1980), 219-243.

5. ———, *Strong ordering of $*$-fields*, J. Algebra **101** (1986), 16-46.

6. I. Idris, *$*$-valuated division rings, orderings and elliptic hermitian spaces*, Ph.D. thesis, Carleton University, Ottawa, Canada, 1986.

7. A. Prestel, *Lectures on formally real fields*, Springer Lecture Notes No. 1093, Springer-Verlag, Berlin, 1984.

8. O. Schilling, *The theory of valuations*, Math. Surveys No. 4, Amer. Math. Soc., Providence, R.I., 1950.

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