GREEN'S LEMMA FOR GROUPOIDS

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Introduction. A set G together with a fixed mapping μ from $G \times G$ into G is called a groupoid. The product ab, for a, b in G, is defined as $\mu((a, b))$. When the product is associative $(a(bc) = (ab)c$ for any a, b, c in G), G is called a *semigroup*. In this latter case a rich theory can be developed [3]. The equivalence relations of Green, Green's Lemma [3, p. 49] and its immediate corollaries are important tools in much of this theory. They are fundamental results whose proof depends on little more than the associativity of the product. However, when we remove this requirement, we find ourselves hard pressed to obtain similar results for groupoids. Groupoids can be created in any whimsical fashion by simply filling in the spaces in the Cayley (multiplication) table for any given set with elements from that set.

In order to restrict this almost total arbitrariness (we could have chosen not to fill in some of the spaces in the Cayley table) we will examine only groupoids satisfying some additional conditions. In §1 we show that these are sufficient for the formulation of Green's equivalences and permit us to quickly prove one-half of Green's Lemma. In §2 we find an additional condition sufficient to demonstrate the other half.

However, if these conditions are satisfied globally the groupoid is associative. This perhaps indicates that semigroups are the simplest "interesting" algebraic structure.

1. Preliminaries. We first define two relations (after Green) on a given groupoid G.

(1.1) DEFINITION. For a, $b \in G$ define $a \mathcal{R} b$ if either $a = b$ or there exist x, $y \in G$ such that $ax = b$ and $by = a$. Dually, define $c \mathcal{L} d$ if either $c = d$ or there exist u, $v \in G$ such that $uc = d$ and $vd = c$. When these are equivalence relations we will write R_a for the $\mathcal{R}$-class of a, and L_c for the $\mathcal{L}$-class of c.

It is easy to check that when G is associative $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations; indeed: $a \mathcal{R} b$ iff $aG \cup \{a\} = bG \cup \{b\}$. (The left-right dualization is also true; however, we will henceforth not mention it explicitly.) We note that for arbitrary groupoids these two subsets of G need not have any particular relationship even though $a \mathcal{R} b$. In order to get around this difficulty we make the following definition.

(1.2) DEFINITION. 1. A groupoid G is said to be *left consistent* if $H(xy) = (Hx)y$ for any $x, y \in G$ and any *subgroupoid* (a subset closed with respect to multiplication) H of G. G is said to be *weakly left consistent* if the above holds just for $H = G$.

2. A groupoid G is said to be *right consistent* if, dually, we have $(xy)H = x(yH)$ for any $x, y \in G$ and any subgroupoid H of G. G is said to be *weakly right consistent* if the above holds just for $H = G$.

3. A groupoid G is said to be [*weakly*] *consistent* if it is both [weakly] left and [weakly] right consistent.

4. A groupoid G is said to be *intra-consistent* if $(xH)y = x(Hy)$ for any $x, y \in G$ and any subgroupoid H of G. G is said to be *weakly intra-consistent* if the above holds just for $H = G$.

(1.3) PROPOSITION. *Let G be a weakly right consistent or a weakly intra-consistent groupoid. Then $a\mathscr{R}b$ if and only if $aG \cup \{a\} = bG \cup \{b\}$ for $a, b \in G$.*

PROOF. Let G be weakly right consistent. Assume $a\mathscr{R}b$. If $a = b$ the result is immediate. Otherwise there exist $x, y \in G$ such that $ax = b$ and $by = a$. Then $aG = (by)G = b(yG) \subseteq bG = (ax)G = a(xG) \subseteq aG$. Hence $aG = bG$ and the result follows since $a \in bG$ and $b \in aG$. If G is weakly intra-consistent and $a\mathscr{R}b$ and $a \neq b$ we can show in a similar manner that $a \in aG = bG$, and $b \in bG$, and the result follows immediately. The converse, which does not depend on either consistency condition, is trivial.

Since the subset formulation clearly defines an equivalence relation we have the following

(1.4) COROLLARY. *If G is either a weakly consistent or a weakly intra-consistent groupoid $\mathscr{R}$ and $\mathscr{L}$ are equivalence relations. Indeed if G is weakly consistent then $\mathscr{R}$ is a left congruence and $\mathscr{L}$ is a right congruence.*

We consider the following

(1.5) EXAMPLE. Let $G = \{0, 1, 2, 3, \cdots\}$ where multiplication is defined by the table below.

It is clear that $0\mathscr{R}1$ but $2 \cdot 0 = 2\mathscr{R}0 = 2 \cdot 1$. Moreover since $Ga = G$ for any $a \in G$ and $bG = G$ for any $b \in G\setminus\{0, 1\}$ we have that $(xG)y = x(Gy)$ for any $x, y \in G$ (when $x \in \{0, 1\}$, $xG = \{0, 1\}$ and $\{0, 1\}y = \{0, 1\}$) so that G is weakly intra-consistent. Thus by (1.3) we can define $\mathscr{R}$ and $\mathscr{L}$ either equationally (1.1) or by using principal one-sided ideals. However as this example shows $\mathscr{R}$ need not be a left congruence on a weakly intra-consistent groupoid.

We now make the following observations when the given groupoid G is *commutative*, (i.e., when $ab = ba$ for all $a, b \in G$).

·	0	1	2	3	4	5	6	·	·	·
0	0	0	1	1	0	0	1	·	·	·
1	1	1	0	0	1	1	0	·	·	·
2	2	0	2	1	3	4	5	·	·	·
3	3	2	0	1	2	3	4	5	·	·
4	4	3	2	0	1	2	3	4		
5	5	4	3	2	0	1	2	3		
6	6	5	4	3	2	0	1	2		
·	·	·			·					
·	·		·			·				
·	·			·			·			

(1.6) PROPOSITION. *Let G be commutative groupoid. Then G is [weakly] left consistent if and only if G is [weakly] right consistent and therefore [weakly] consistent; in either case G is [weakly] intra-consistent.*

PROOF. Let $x, y \in G$ and H be any subgroupoid. Using the commutativity of G we have $(xy)H = H(yx)$ and $x(yH)\cdot = (yH)x = (Hy)x$. The two sets of equations are bridged if G is either [weakly] left or [weakly] right consistent and hence these conditions [with $H = G$] are equivalent.

Now, if G is [weakly] right (or left) consistent we have using the commutativity of G: $(xH)y = y(xH) = (yx)H = (xy)H = x(yH) = x(Hy)$ where H is any subgroupoid of G [with $H = G$]. It follows that G is [weakly] intra-consistent.

Consider the following examples:

(1.7) EXAMPLE. Let $G = \{a, b, c\}$ where multiplication is defined by

·	a	b	c
a	c	b	c
b	b	b	c
c	c	c	c

It is clear that G is commutative and one can readily check that G is weakly intra-consistent. However $a(aG) = \{b, c\} \neq \{c\} = (aa)G$ and hence G is not weakly (right or left) consistent so that the "weak" converse of (1.6) is false. Unfortunately G is not intra-consistent, for if we take $H = \{a, c\}$ then $(bH)a \neq b(Ha)$.

It can be checked by computer that all three-element commutative groupoids which are weakly intra-consistent but not weakly right consistent are isomorphic to this one.

(1.8) EXAMPLE. Let $G = \{0, 1, 2\}$ where multiplication is defined by

·	0	1	2
0	2	2	0
1	2	0	0
2	0	0	2

G is clearly commutative.

One can readily check that G is both weakly intra-consistent and weakly consistent. Here the only proper subgroupoids are $\{2\}$ and $\{0, 2\}$. But $(11)2 = 02 = 0$ while $1(12) = 10 = 2$ so that G is not consistent. Now with the commutativity $(xH)y = y(xH)$ and $x(Hy) = x(yH)$ and the reader may check to see that $x(yH) = y(xH)$ for any subgroupoid H. Thus it follows that G is intra-consistent but not consistent so that the other converse of (1.6) is also invalid.

(1.9) EXAMPLE. Let $G = \{a, b, c, d\}$ where multiplication is defined by the Cayley table below. Then G is weakly left consistent and weakly intra-consistent but not weakly right consistent. We leave the verification to the reader with the hint that $Gx = G$ for any $x \in G$ can be used to simplify the task.

·	a	b	c	d
a	a	a	b	b
b	b	b	a	a
c	c	c	d	d
d	d	d	c	c

Taking the dual of G by interchanging rows and columns in the Cayley table of G would give us an example of groupoid which is both weakly right and weakly intra-consistent but not weakly left consistent. This leads us to the following ((1.6) without the overall assumption of commutativity):

PROBLEM 1. Find an example of a groupoid which is both weakly left and right consistent but not weakly intra-consistent or prove that a weakly consistent groupoid is weakly intra-consistent.

One can check by computer that in Problem 1 any groupoid that is weakly consistent but not weakly intra-consistent must have at least four elements.

In order to prove the first part of Green's Lemma we will need the following result, whose proof is straightforward.

(1.10) LEMMA. *If G is a weakly right consistent groupoid then gG is a subgroupoid for any $g \in G$.*

(1.11) COROLLARY. *If G is a weakly right consistent groupoid then $\{g\} \cup gG$ is a subgroupoid for any $g \in G$.*

(1.12) THEOREM. *Let G be a consistent groupoid and suppose $c\mathcal{R}b$ for some $c \neq b$. Then there are s, $s' \in G$ such that $cs = b$, $bs' = c$ and the right translations ρ_s, $\rho_{s'}$, are, respectively, mappings from L_c into L_b and L_b into L_c, which are $\mathcal{R}$-class preserving, i.e., for $x \in L_c$, $x\mathcal{R}x\rho_s$ and for $y \in L_b$, $y\mathcal{R}y\rho_{s'}$.*

PROOF. Since $c \neq b$ the existence of s, s' follows from the Definition (1.1). Now let $a\mathcal{L}c$ and $d\mathcal{L}b$. By (1.4) $\mathcal{L}$ is a right congruence so that $as\mathcal{L}cs = b$ and $ds'\mathcal{L}bs' = c$. Thus $L_c\rho_s \subseteq L_b$ and $L_b\rho_{s'} \subseteq L_c$.

Now if $a \neq c$ then $aG \supseteq a(sG) = (tc)(sG)$ where $tc = a$ for some $t \in G$, by (1.1). Since sG is a subgroupoid by (1.10), we have $(tc)(sG) = t[c(sG)] = t[(cs)G] = t(bG)$. Thus $a = tc = t(bs') \in t(bG) \subseteq aG$. Continuing we have $aG \supseteq (as)G = t(bG) \supseteq t[b(s'G)] = t[(bs')G] = t(cG) = (tc)G = aG$, whence $aG = (as)G$. But $a \in aG$ and $as \in aG = (as)G$ so that we can conclude by (1.3) that $a\mathcal{R}as$. If $a = c$ the preceding argument can be simplified to show that $c\mathcal{R}cs$. In a similar manner $d\mathcal{R}ds'$.

(1.13) COROLLARY. *If G is a consistent groupoid then $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ on G.*

PROOF. If $a\mathcal{L}c\mathcal{R}b$ then the above Theorem yields an $s \in G$ such that $a\mathcal{R}as\mathcal{L}b$ and so $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. The reverse inclusion is a direct dualization.

We conclude this section with the following definition.

(1.14) DEFINITION. A groupoid G is said to be $\mathcal{D}$-defined or a $\mathcal{D}$-groupoid if $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ on G.

In such cases we define $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and $\mathcal{D}$ is then clearly an equivalence relation.

We observe that consistent groupoids are $\mathcal{D}$-defined though the converse is not necessarily true.

(1.15) EXAMPLE. Let $G = \{a, b\}$ where multiplication is defined by the table below.

·	a	b
a	b	a
b	b	a

Here we have that $\mathcal{L} = \mathcal{R} = \omega$, the universal relation on G, so that G is certainly $\mathcal{D}$-defined while $G(bb) = \{b\} \neq \{a\} = (Gb)b$.

2. **Green's Lemma and examples.** In (1.12) we saw that for a consistent groupoid G certain right translations define maps between two $\mathcal{L}$-classes which are $\mathcal{R}$-class preserving. It is not known, in general, if ρ_s and $\rho_{s'}$ are mutually inverse maps between L_c and L_b, as is the case for semigroups. In this section we find two additional conditions which suffice to guarantee this result.

(2.1) DEFINITION. Let G be a $\mathcal{D}$-defined groupoid. A $\mathcal{D}$-class, D, of G is said to be *regular* if there is an idempotent element $(x^2 = x)$ in each $\mathcal{L}$ and $\mathcal{R}$-class of D.

(2.2) LEMMA. *Let G be a consistent groupoid. If $e^2 = e$ then e is a right identity on its $\mathcal{L}$-class, L_e, and a left identity on its $\mathcal{R}$-class R_e.*

PROOF. Let $x \in L_e$. Then $x = te$ for some $t \in G$. Now $xe = (te)e = t(ee) = te = x$ since $\{e\}$ is a subgroupoid of G. The other result is dual.

(2.3) PROPOSITION. *Let D be a regular $\mathcal{D}$-class of a consistent groupoid G. Then for any $a \in D$ there exist $t, t' \in G$ such that $a = a(ta) = (at')a$.*

PROOF. Let $a \in D$. Since D is regular there is an idempotent $e \in L_a = L_e$. By (2.2) $ae = a$. Since $e\mathcal{L}a$ there is a $t \in G$ such that $ta = e$ (if $a = e$ the result is immediate). Then $a = ae = a(ta)$. Dually one obtains $a = (at')a$.

(2.4) REMARK. A converse of (2.3) is false: A groupoid G may be consistent, and for every $a \in G$ we may have a $t, t' \in G$ such that $a = a(ta) = (at')a$, and yet G may have no idempotents. Thus in the groupoid (cf. [2, p. 9]) $G = \{a, b, c\}$, where multiplication is defined by the table below, there are no idempotents and yet, e.g., $a = a(aa) = (aa)a$. Moreover, here we have $\mathcal{L} = \mathcal{R} = \omega$.

·	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

(2.5) DEFINITION. A groupoid G is said to be *almost associative* if whenever H is a subgroupoid of G and $a, b, c \in G$ we have $H[(ab)c] = H[a(bc)]$ and $[a(bc)]H = [(ab)c]H$.

(2.6) THEOREM. *A regular, consistent, almost associative groupoid is associative.*

PROOF. Let $a, b, c \in G$. Then $(ab)c = (ab)(ce)$ for some $e^2 = e \in L_c$. But $(ab)(ce) = [(ab)c]e = [a(bc)]e = a[(bc)e] = a[b(ce)] = a(bc)$ since G is consistent and almost associative and $\{e\}$ is a subgroupoid. We thus have that $(ab)c = a(bc)$ for any $a, b, c \in G$, i.e., G is associative.

In light of the above the following is not at all surprising.

(2.7) COROLLARY (GREEN'S LEMMA). *Let G be a consistent almost associative groupoid. If D is a regular $\mathcal{D}$-class of G and $c\mathcal{R}b$ for $c, b \in D$ then there exist $s, s' \in G$ such that $cs = b$ and $bs' = c$. Moreover the right translations $\rho_s, \rho_{s'}$, are mutually inverse bijections between L_c and L_b and are $\mathcal{R}$-class preserving.*

PROOF. We need now only show that ρ_s and $\rho_{s'}$ are mutually inverse bijections (cf. (1.12)). Let $f^2 = f \in R_c$ and $fu = c$ for some $u \in G$. Then $c = bs' = (cs)s' = ((fu)s)s' = (f(us))s' = f((us)s') = f(u(ss'))$ $= (fu)(ss') = c(ss')$.

Now let $a \in L_c$ and $e^2 = e \in L_c$. There exists a $t \in G$ such that $tc = a$. Then $(a)\rho_s\rho_{s'} = (as)s'$. Since $(as)s' \in L_a \cap R_a$ by (1.12) $(as)s' = [(as)s']e = [a(ss')]e = [(tc)(ss')]e = [t(c(ss'))]e = [tc]e$ $= ae = a$ since $c = c(ss')$ from above. Thus $\rho_s\rho_{s'}$ is the identity map on L_c. Similarly $\rho_{s'}\rho_s$ is the identity map on L_b. The result now follows.

(2.8) OBSERVATION. We see that in example (1.5) we have a weakly intra-consistent groupoid, G, which consists of one $\mathcal{D} = \mathcal{L}$-class and two $\mathcal{R} = \mathcal{H}$-classes: $\{0, 1\}$ and $\{2, 3, 4, \cdots\}$, each of which is regular (since $0^2 = 0$, $2^2 = 2$). (We define $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.) However the right translations between $L_m = L_n$ for $m, n \geqq 2$ defined by ρ_s, $s \geqq 2$, will not be 1-1 or $\mathcal{R}$-class preserving. For example: if $m = 3$, $n = 4$ and $s = 6$ then $3 \cdot 6 = 4$ and $4 \cdot 6 = 3$ yet $6 \cdot 6 = 6\rho_6 = 1 \notin R_3$. Indeed, here we see that an $\mathcal{H}$-class ($\{0, 1\}$) may have more than one idempotent, which is quite unlike the case for semigroups (cf. [3, Lemma 2.15]).

(2.9) EXAMPLE. Let G be defined by the Cayley table below. It is clear that if $k > 0$ then $k^2 < k$. Thus after a finite number of steps one obtains 0, 1 in the subgroupoid generated by k. It is now clear $(k \cdot (k + 1) = k + 2)$ that there are no proper subgroupoids of G. Because of the expanding diagonal it is also clear that $xG = Gx = G$ for any $x \in G$. Thus $\mathcal{D} = \mathcal{L} = \mathcal{R} = \mathcal{H}$ in this example. Moreover, G is both consistent and intra-consistent. However each right translation ρ_s is far from being 1-1 so that Green's Lemma need not be valid for infinite consistent groupoids. Furthermore it is apparent that no two right translations can be mutually inverse.

·	0	1	2	3	4	5	6	7	8	·	·	·
0	1	2	0	3	1	4	0	5	1	·	·	
1	2	0	3	1	4	0	5	1	6	·		
2	0	3	1	4	0	5	1	6	2			
3	3	1	4	0	5	1	6	2	7			
4	1	4	0	5	1	6	2	7	0			
5	4	0	5	1	6	2	7	0	8			
6	0	5	1	6	2	7	0	8	1			
7	5	1	6	2	7	0	8	1	9			
8	1	6	2	7	0	8	1	9	2			
·	·	·								3		
·	·										0	
·												1

One can now pose the following problems:

PROBLEM 2. Just how necessary is the regularity in (2.7)?

PROBLEM 3. Is there any natural condition for groupoids weaker than that of almost associativity which can replace it leaving (2.7) true but (2.6) false?

PROBLEM 4. Is there a regular consistent groupoid in which the right translations of (1.12) are not mutually inverse? We remark that the regular and almost associative conditions were added since there seemed to be no simple example for this problem.

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