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ON LIAPUNOV'S DIRECT METHOD

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We shall consider the system of ordinary differential equations

$$x' = f(t, x), \qquad t \in [0, \infty), \quad x \in D, \tag{1}$$

where D is an open connected subset of R^n containing the zero vector and f is a function from $[0, \infty) \times D$ to R^n such that solutions to (1) exist locally in the Carathéodory sense (cf. [4, p. 42]). *We denote by* $\mathcal{L}(1)$ *the class of real-valued functions* $V(t, x)$ *on* $[0, \infty) \times D$ *such that* $V(t, x(t))$ *is nonincreasing whenever* $x(t)$ *is a solution of* (1). A sufficient condition for $V \in \mathcal{L}(1)$ is that V be continuous in (t, x), locally Lipschitzian in x and satisfy

$$\limsup_{h \to 0+} [V(t + h, x + hf(t, x)) - V(t, x)]/h \leqq 0$$

for all (t, x), when f is continuous (cf. [14, p. 4]).

All of the applications of Liapunov's direct method with which we are here concerned are based on the observation that if $V \in \mathcal{L}(1)$ and (t_0, x_0), (t_1, x_1) are such that $t_0 < t_1$ and $V(t_0, x_0) < V(t_1, x_1)$ then there is no solution $x(t)$ of (1) such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

Notation. (i) A solution $x(t)$ such that $x(t_0) = x_0$ will often be denoted $x(t; t_0, x_0)$.

(ii) If $x_0 \in R^n$, $r \in (0, \infty)$, then $B(x_0, r) = \{x : |x - x_0| < r\}$, where $|\ |$ denotes any norm.

(iii) $2^D = \{X : X \subset D\}$.

(iv) Let $x, y \in R^n$:

$$\rho(x, y) = |x - y|, \quad \text{if } x \neq \infty,\ y \neq \infty;$$

$$\rho(x, y) = \frac{1}{|x|}, \quad \text{if } y = \infty;$$

$$\rho(x, X) = \inf\{\rho(x, y) : y \in X\}, \quad \text{if } X \subset R^n$$

and $x \to X$ means $\rho(x, X) \to 0$.

(v) If $X \subset R^n$ then $\bar{X}$ and ∂X denote the closure and boundary of X respectively.

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$$\text{(vi)}\qquad \begin{aligned} V_*(t, x) &= \inf \{V(\tau, x) : 0 \leqq \tau \leqq t\}, \\ V^*(t, x) &= \sup \{V(\tau, x) : 0 \leqq \tau \leqq t\}. \end{aligned}$$

Definitions. (i) A solution $x(t)$, $t \geqq t_0$, *exists in the future* if there is a (not necessarily unique) continuation of $x(t)$ throughout $[t_0, \omega)$ for each $\omega > t_0$.

(ii) A solution $x(t)$, $t \geqq t_0$, is *bounded in D* if there is a compact $\Delta \subset D$ such that $x(t) \in \Delta$ for all $t \geqq t_0$.

(iii) The solutions of (1) are *uniformly bounded in D* if, for each compact $E \subset D$, there is a compact $\Delta(E) \subset D$ such that $x_0 \in E$ implies $x(t; t_0, x_0) \in \Delta(E)$ for all $t \geqq t_0$ and all $t_0 \geqq 0$.

(iv) A solution $x(t)$, $t \geqq t_0$, is *unique in the future* if it has at most one continuation throughout $[t_0, \omega]$ for each $\omega > t_0$.

(v) A solution $x(t)$, $t \geqq 0$, which exists in the future, is *stable* if, for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $x_0 \in B(x(0), \delta(\epsilon))$ implies $x(t; 0, x_0)$ exists in the future and $x(t; 0, x_0) \in B(x(t), \epsilon)$ for all $t \geqq 0$.

(vi) A solution $x(t)$, $t \geqq 0$, which exists in the future, is *uniformly stable* if, for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $x_0 \in B(x(t_0), \delta(\epsilon))$ implies $x(t; t_0, x_0)$ exists in the future and $x(t; t_0, x_0) \in B(x(t), \epsilon)$ for each $t \geqq t_0$ and each $t_0 \geqq 0$.

Let $\mathcal{C}$ be a function from $[0, \infty)$ to 2^D.

(vii) If E, B are such that $E \subset D$, E compact, $B \subset \bar{D}$ then $\mathcal{C}$ *separates E from B*, $E|\mathcal{C}|B$, if for each $t \in [0, \infty)$ there exists a neighborhood $U(t)$ of B such that:

(a) $U(t) \cap E = \emptyset$.

(b) Every connected subset of D which intersects $U(t)$ and E also intersects $\mathcal{C}(t) - (U(t) \cup E)$.

If, furthermore, U above may be chosen independently of t then we say $\mathcal{C}$ *separates E from B uniformly* and write $E\|\mathcal{C}\|B$.

If $A \subset D$ and $B \subset \bar{D}$ then we write $A|\mathcal{C}|B$ $(A\|\mathcal{C}\|B)$ if $E|\mathcal{C}|B$ $(E\|\mathcal{C}\|B)$ *for each compact* $E \subset A$.

For example, consider $D = B(0, 1)$. Let

$$\mathcal{C}_1(t) = \bigcup_{n=1}^{\infty} \partial B\left(0, 1 - \frac{e^{-t}}{2n}\right),$$

$$\mathcal{C}_2(t) = \bigcup_{n=1}^{\infty} \partial B\left(0, 1 - \frac{e^{t}}{(e^t + 1)n}\right),$$

$$\mathcal{C}_3(t) = \bigcup_{n=1}^{\infty} \partial B\left(0, \frac{e^{-t}}{2n}\right),$$

$$\mathcal{C}_4(t) = \bigcup_{n=1}^{\infty} \partial B\left(0, \frac{1}{n}\right),$$

$$\mathcal{C}_5(t) = D;$$

then $D|\mathcal{C}_1|\partial D$, $D\|\mathcal{C}_2\|\partial D$, $D - \{0\}|\mathcal{C}_3|\{0\}$, $D - \{0\}\|\mathcal{C}_4\|\{0\}$, $D\|\mathcal{C}_5\|\partial D$ and $D - \{0\}\|\mathcal{C}_5\|\{0\}$.

THEOREM 0.1. *Let* $V \in \mathcal{L}(1)$.

(a) *Suppose* $V(t, x) \to +\infty$ *as* $x \to \partial D$ *uniformly on* $[0, T]$ *for each* $T \in (0, \infty)$.

If $V(t_0, x_0) < +\infty$ *then each solution* $x(t; t_0, x_0)$ *exists in the future* (*cf.* [**5**], *also* [**11**]).

(b) *Suppose there is a real-valued function* ω *on* D *such that* $\omega(x) \to +\infty$, *as* $x \to \partial D$, *and* $\omega(x) \leqq V(t, x)$, *for* $(t, x) \in [0, \infty) \times D$.

If $V(t_0, x_0) < +\infty$, *then each solution* $x(t; t_0, x_0)$ *is bounded in* D (*cf.* [**16**]).

(c) *If there exist real-valued functions* ω_1 *and* ω_2 *on* D *such that* ω_1 *is bounded above on compact subsets of* D, $\omega_2(x) \to +\infty$ *as* $x \to \partial D$ *and* $\omega_2(x) \leqq V(t, x) \leqq \omega_1(x)$ *for every* $(t, x) \in [0, \infty) \times D$, *then the solutions of* (1) *are uniformly bounded in* D (*cf.* [**16**]).

THEOREM 0.2. *Let* $V \in \mathcal{L}(1)$.

(a) *Suppose* $V(t, x) > 0$, *for* $x \neq 0$, $t \geqq 0$.

If $V(0, 0) = 0$, *then* $x(t) \equiv 0$ *is a solution of* (1) *which is unique in the future* (*cf.* [**2**]).

(b) *Suppose there is an increasing function* θ *on* $[0, \infty)$ *such that* $\theta(0) = 0$ *and* $\theta(|x|) \leqq V(t, x)$ *for all* $(t, x) \in [0, \infty) \times D$.

If $V(0, 0) = 0$ *and* $V(0, x)$ *is continuous at* $x = 0$, *then* $x(t) \equiv 0$ *is a solution of* (1) *which is stable* (*cf.* [**15**]).

(c) *If there exist real-valued functions* θ_1 *and* θ_2 *on* $[0, \infty)$ *such that* $\theta_i(0) = 0$, $i = 1, 2$, θ_2 *is increasing and* θ_1 *is continuous at* 0 *and* $\theta_2(|x|) \leqq V(t, x) \leqq \theta_1(|x|)$, *for all* $(t, x) \in [0, \infty) \times D$ *then* $x(t) \equiv 0$ *is a solution of* (1) *which is uniformly stable* (*cf.* [**15**]).

Although stated here only as sufficient conditions, the conditions of Theorems 0.1 and 0.2 are, under very general circumstances also necessary (e.g. cf. [**14**, Chapter V]). Nevertheless, because of the difficulty of finding functions $V \in \mathcal{L}(1)$ for specific equations of the type (1), it is of interest to relax these conditions. In particular, a number of authors (see [**3**] and [**14**, p. 18], for references) have profitably studied functions V satisfying less restrictive requirements than the assumption that $V(t, x(t))$ be nonincreasing. A more restrictive requirement is often used in other contexts, for example, asymptotic stability. In the present paper we devote our attention to relaxing the restrictions on the range of V; for example our generalization of Theorem 0.2 (c) allows us to conclude that $x(t) \equiv 0$ is uniformly stable from the existence of a function $V \in \mathcal{L}(1)$ which may be of indefinite sign or even unbounded above and below in every neighborhood of $x = 0$. We also show in Theorems 2 and 3 how an infinite collection of functions V may be used to obtain information about the stability and boundedness of solutions to a system of differential equations.

THEOREM 1.1. *Let* $V \in \mathcal{L}(1)$.

(a) *Suppose there exists a function* $\mathcal{C}$ *from* $[0, \infty)$ *to* 2^D *such that*:

(i) $V_*(t, x) \to +\infty$, *as* $x \to \partial D$, $x \in \mathcal{C}(t)$, *for each* $t > 0$.

(ii) $D|\mathcal{C}|\partial D$.

If $V(t_0, x_0) < +\infty$ *then any solution* $x(t;\ t_0, x_0)$ *exists in the future.*

(b) *Suppose there is a real-valued function* ω *on* D *such that*:

(i) $\omega(x) \to +\infty$, *as* $x \to \partial D$,

(ii) *if* $\mathcal{C}(t) = \{x : V_*(t, x) \geqq \omega(x)\}$ *then* $D\|\mathcal{C}\|\partial D$.

If $V(t_0, x_0) < +\infty$ *then any solution* $x(t;\ t_0, x_0)$ *is bounded in* D.

(c) *Suppose there exist real-valued functions* ω_1 *and* ω_2 *on* D *such that*:

(i) $\omega_2(x) \to +\infty$, *as* $x \to \partial D$, *while* ω_1 *is bounded above on each compact subset of* D:

(ii) *if* $\mathcal{C}_1(t) = \{x : V^*(t, x) \leqq \omega_1(x)\}$, $\mathcal{C}_2(t) = \{x : V_*(t, x) \geqq \omega_2(x)\}$ *then* $D\|\mathcal{C}_i\|\partial D$, $i = 1, 2$.

Then the solutions of (1) *are uniformly bounded in* D.

THEOREM 1.2. *Let* $V \in \mathcal{L}(1)$.

(a) *Suppose* $\mathcal{C}(t) = \{x : V_*(t, x) > 0\}$ *is such that* $D - \{0\}|\mathcal{C}|\{0\}$.

If $V(0, 0) = 0$ *then* $x(t) \equiv 0$ *is a solution of* (1) *which is unique in the future.*

(b) *Suppose there exists a function* θ *on* $[0, \infty)$ *such that*:

(i) θ *is increasing and* $\theta(0) = 0$,

(ii) *if* $\mathcal{C}(t) = \{x : V_*(t, x) \geqq \theta(|x|)\}$ *then* $D - \{0\}\|\mathcal{C}\|\{0\}$.

If $V(0, 0) = 0$ *and* $V(0, x)$ *is upper semicontinuous (u.s.c.) at* $x = 0$ *then* $x(t) \equiv 0$ *is a solution of* (1) *which is stable.*

(c) *Suppose there exist real-valued functions* θ_1 *and* θ_2 *on* $[0, \infty)$ *such that*:

(i) $\theta_i(0) = 0$, $i = 1, 2$, θ_1 *is continuous at* 0 *and* θ_2 *is increasing.*

(ii) *If* $\mathcal{C}_1(t) = \{x : V^*(t,x) \leqq \theta_2(|x|)\}$, $\mathcal{C}_2(t) = \{x : V_*(t,x) \geqq \theta_2(|x|)\}$ *then* $D - \{0\}\|\mathcal{C}_i\|\{0\}$, $i = 1, 2$.

Then $x(t) \equiv 0$ *is a solution of* (1) *which is uniformly stable.*

Theorems 0.1 and 0.2 are special cases of Theorems 1.1 and 1.2, respectively, with $D = \mathcal{C}(t) = \mathcal{C}_i(t)$, $i = 1, 2$.

PROOF OF THEOREM 1.1 (a). Suppose there is a solution $x(t) = x(t;\ t_0,\ x_0)$ which does not exist in the future, i.e., $x(t)$ exists on a right-maximal interval $[t_0,\ T)$, $T < +\infty$. Then $x(t) \to \partial D$ as $t \to T-$ (cf. [7, p. 12]). Condition (a) (i) implies that there is a compact subset E of D such that $x_0 \in E$ and

$$\text{if } x \in \mathcal{C}(T) - E, \quad \text{then } V_*(T, x) > V(t_0, x_0). \tag{2}$$

Since (a) (ii) holds, there is a neighborhood $U(T)$ of ∂D such that $E \cap U(T) = \emptyset$ and

$$\begin{gathered} \text{if } G \subset D,\ G \text{ connected},\ G \cap E \neq \emptyset, \\ G \cap U(T) \neq \emptyset, \text{ then } G \cap \{\mathcal{C}(T) - [U(T) \cup E]\} \neq \emptyset. \end{gathered} \tag{3}$$

Thus, by (3), there exists t_1, $t_0 < t_1 < T$, such that $x(t_1) \in \mathcal{C}(T) - [U(T) \cup E]$, and hence

$$\begin{aligned} V(t_1, x(t_1)) &\geqq V_*(t_1, x(t_1)) \\ &\geqq V_*(T, x(t_1)), \quad \text{since } t_1 < T \\ &> V(t_0, x_0), \quad \text{by (2), since } x(t_1) \in \mathcal{C}(T) - E. \end{aligned}$$

But this contradicts $V(t_1,\ x(t_1)) \leqq V(t_0,\ x_0)$ (i.e., $V \in \mathcal{L}(1)$) so that we must have $T = +\infty$; i.e., $x(t)$ exists in the future.

PROOF OF THEOREM 1.2 (b). If $\epsilon > 0$, (b)(ii) implies that there exists $\delta_1(\epsilon)$, $0 < \delta_1(\epsilon) < \epsilon$, such that:

(4) Every connected set which intersects $B(0,\ \delta_1(\epsilon))$ and $\partial B(0,\ \epsilon)$ also intersects $\mathcal{C}(t) - [B(0, \delta_1(\epsilon)) \cup \partial B(0,\ \epsilon)]$, for each $t \geqq 0$.

Since $V(0,\ x)$ is u.s.c. at $x = 0$ then, by (b) (i) there exists $\delta(\epsilon)$, $0 < \delta(\epsilon) < \delta_1(\epsilon)$, such that:

$$\text{If } x_0 \in B(0, \delta(\epsilon)) \quad \text{then } V(0, x_0) < \theta(\delta_1(\epsilon)). \tag{5}$$

Consider $x(t) = x(t;\ 0,\ x_0)$; if there exists $T > 0$ such that $x(t) \in \partial B(0, \epsilon)$ then, by (4) there exists t_1, $0 < t_1 < T$, such that

$$x(t_1) \in \mathcal{C}(T) - [B(0, \delta_1(\epsilon)) \cup \partial B(0, \epsilon)]$$

and hence

$$\begin{aligned} V(t_1, x(t_1)) &\geqq V_*(T, x(t_1)), \quad \text{since } t_1 < T \\ &\geqq \theta(|x(t_1)|), \quad \text{since } x(t_1) \in \mathcal{C}(T) \\ &\geqq \theta(\delta_1(\epsilon)), \quad \text{since } |x(t_1)| \geqq \delta_1(\epsilon) \\ &> V(0, x_0), \quad \text{by (5), since } |x_0| < \delta(\epsilon) \end{aligned}$$

contradicting $V \in \mathcal{L}(1)$. The conclusion of Theorem 1.2 (b) is now apparent.

PROOF OF THEOREM 1.2 (c). If $\epsilon > 0$, it follows from (c) (ii) that there exists $\delta_2(\epsilon)$, $0 < \delta_2(\epsilon) < \epsilon$, such that:

(6) Every connected set which intersects $B(0, \delta_2(\epsilon))$ and $\partial B(0, \epsilon)$ also intersects $\mathcal{C}_2(t) - B(0, \delta_2(\epsilon))$ for each $t \geqq 0$.

There exists $\delta_1(\epsilon)$, $0 < \delta_1(\epsilon) < \delta_2(\epsilon)$, such that:

$$\theta_1(\delta_1(\epsilon)) < \theta_2(\delta_2(\epsilon)). \tag{7}$$

There exists $\delta(\epsilon)$, $0 < \delta(\epsilon) < \delta_1(\epsilon)$, such that:

(8) Every connected set which intersects $B(0, \delta(\epsilon))$ and $\partial B(0, \delta_1(\epsilon))$ also intersects $\mathcal{C}_1(t) \cap B(0, \delta_1(\epsilon))$.

We assert that if $x_0 \in B(0, \delta(\epsilon))$, then $x(t;\, t_0, x_0) \in B(0, \epsilon)$ for all $t \geqq t_0$ and all $t_0 \geqq 0$. If there were a solution $x(t)$ such that $x(t_0) = x_0 \in B(0, \delta(\epsilon))$ and $x(T) \in \partial B(0, \epsilon)$ for some $T > t_0$ then, by (6) and (8), there exist t_1 and t_2 such that $t_0 < t_1 < t_2 < T$ and

$$x(t_1) \in \mathcal{C}_1(T) \cap B(0, \delta_1(\epsilon)), \qquad x(t_2) \in \mathcal{C}_2(T) - B(0, \delta_2(\epsilon)).$$

Therefore,

$$\begin{aligned} V(t_2, x(t_2)) &\geqq V_*(T, x(t_2)) \\ &\geqq \theta_2(|x(t_2)|), \quad \text{since } x(t_2) \in \mathcal{C}_2(T) \\ &\geqq \theta_2(\delta_2(\epsilon)), \quad \text{since } |x(t_2)| > \delta_2(\epsilon) \\ &> \theta_1(\delta_1(\epsilon)), \quad \text{by (7)} \\ &\geqq \theta_1(|x(t_1)|), \quad \text{since } |x(t_1)| < \delta_1(\epsilon) \\ &\geqq V^*(t, x(t_1)), \quad \text{since } x(t_1) \in \mathcal{C}_1(T) \\ &\geqq V(t_1, x(t_1)), \end{aligned}$$

i.e., $V(t_2, x(t_2)) > V(t_1, x(t_1))$, contradicting $V \in \mathcal{L}(1)$.

The proofs of the other sections of Theorems 1.1 and 1.2 follow a similar pattern to those above. These are not the most general results that can be obtained in this direction. For example, the conclusion of Theorem 1.1 (b) holds if $\lim_{x \to \partial D} \omega(x) = a > V(t_0, x_0)$ (cf. [9]) and, in fact, a solution $x(t;\, t_0, x_0)$ exists in the future (is bounded) if $\mathcal{C}(t) = \{x : V_*(t,x) > V(t_0, x_0)\}$ satisfies $\{x_0\} | \mathcal{C} | \partial D(\{x_0\} \| \mathcal{C} \| \partial D)$. Also we may have $\lim_{x \to \partial D} V(t, x) \leqq V(t_0, x_0)$ for all (t_0, x_0) and still conclude that the solutions are uniformly bounded; this is the case if the function $V(t, x) = (1/|x|) \sin^2 |x| \, (D = R^n)$ belongs to the class $\mathcal{L}(1)$, since $V_*(t, x) = V^*(t, x) = V(t, x)$ and if $\mathcal{C}_1(t) = \{x : V(t, x) = 0\}$ and $\mathcal{C}_2(t) = \{x : V(t, x) > 0\}$ then $D \| \mathcal{C}_i \| \partial D$, $i = 1, 2$.

EXAMPLE 1. Consider the system

$$x_1' = x_2, \qquad x_2' = (t - |x_1|^{-1/2})x_1 \tag{9}$$

which is equivalent to the scalar equation

$$x'' + (|x|^{-1/2} - t)x = 0.$$

Let $V(t, x_1, x_2) = |x_1|^{3/2}(4/3 - t|x_1|^{1/2}) + x_2{}^2$; then $(d/dt)V(t, x_1(t), x_2(t)) = -(x_1(t))^2 \leqq 0$ whenever $(x_1, x_2)(t)$ is a solution of (9). Also $V_*(t_1, x_1, x_2) = V(t, x_1, x_2) > 0$ whenever $(x_1, x_2) \in \mathcal{C}(t)$, where

$$\mathcal{C}(t) = \{(x_1, x_2) \neq (0, 0) : 0 \leqq |x_1| < 16/(9t^2)\}.$$

Then $R^2 - \{0\} | \mathcal{C} | \{0\}$ and the solution $(x_1, x_2)(t) \equiv (0, 0)$ is unique in the future, by Theorem 1.2 (a).

Notice that $V(t, x_1, x_2) \to -\infty$ as $t \to +\infty$ if $x_1 \neq 0$.

EXAMPLE 2. Consider the scalar equation

$$\begin{aligned} x' &= a(t)|x|^{1/2} \sin(1/x), && \text{if } x \neq 0, \\ &= 0, && \text{if } x = 0, \end{aligned} \tag{10}$$

where $a(t)$ is of constant sign ($a(t) \leqq 0$, say). Define $V(x) = \cos(1/x)$ if $x \neq 0$, $V(0)$ arbitrary. Then

$$\frac{d}{dt} V(x(t)) = a(t)|x(t)|^{-3/2} \sin^2 \frac{1}{x(t)} \leqq 0$$

whenever $x(t) \neq 0$ is a solution of (10). Let $\theta_1(r) = \theta_2(r) = r$, $r \geqq 0$. It can be seen that the functions $\mathcal{C}_i$, $i = 1, 2$, as defined in Theorem 1.2 (c) satisfy $R - \{0\} \| \mathcal{C}_i \| \{0\}$ and hence the solution $x(t) \equiv 0$ of (10) is uniformly stable.

EXAMPLE 3. The system

$$x_1' = x_2, \qquad x_2' = -\phi(x_1, x_2)x_2 - h(x_1), \qquad \phi(x_1, x_2) \geqq 0, \tag{11}$$

where ϕ and h are continuous, is equivalent to the Liénard equation

$$x'' + \phi(x, x')x' + h(x) = 0.$$

Theorem 1.1 (c) with

$$V(x_1, x_2) = 2H(x_1) + x_2{}^2, \text{ where } H(x) = \int_0^x h,$$

implies that the solutions are uniformly bounded if

$$\limsup_{|x|\to\infty} H(x) = +\infty.$$

In particular it is known that if $\lim_{|x|\to\infty} H(x) = +\infty$ then the solutions of (11) are uniformly bounded (cf. Utz [12]). The reader is also referred to the paper of Willett and Wong [13] where the role of the function $\phi(x_1, x_2)$ is investigated more thoroughly.

Theorem 1.2 (c), with V as above, implies that $(x_1, x_2)(t) \equiv (0, 0)$ is a solution of (11) which is uniformly stable if there exists a sequence $\{\alpha_n\}$ such that

$$(-1)^n \alpha_n > 0, \qquad H(\alpha_n) > 0, \qquad \lim_{n\to\infty} \alpha_n = 0.$$

In particular these conditions hold if $xh(x) > 0$, $x \neq 0$, in a neighborhood of $x = 0$.

For a study of some equations it may be convenient to use more than one (and possibly infinitely many) functions V. *For example, if, for each* $\epsilon > 0$, *there exists a* $\delta(\epsilon)$, $0 < \delta(\epsilon) < \epsilon$ *and* $V_\epsilon(t, x)$ *such that*:

(i) $V_\epsilon(t, x(t))$ *is nonincreasing when* $x(t)$ *is a solution of* (1) *and* $x(t) \in B(0, \epsilon)$.

(ii) $\mathcal{C}_\epsilon(t) = \{x : V_{\epsilon^*}(t, x) > \sup V_\epsilon(0, y),\ |y| < \delta(\epsilon)\}$ *satisfies* $\partial B(0, \epsilon) | \mathcal{C}_\epsilon | B(0, \delta(\epsilon))$.

Then $x(t) \equiv 0$ *is a solution of* (1) *which is stable.* We illustrate this by extending some known results for the systems

$$x_1' = x_2, \qquad x_2' = -\phi(x_1, x_2)x_2 - h(x_1) + e(t), \tag{12}$$

$$x_1' = \frac{1}{p(t)}\, x_2, \qquad x_2' = -q(t)f(x_1) \tag{13}$$

which are equivalent to the scalar equations

$$x'' + \phi(x, x')x' + h(x) = e(t), \qquad (p(t)x')' + q(t)f(x) = 0,$$

respectively.

THEOREM 2. *Suppose*

(i) ϕ *and* h *are continuous on* R^2 *and* R, *respectively, and* $\phi \geqq 0$.

(ii) *There exist* $\alpha_n \in (n, \infty)$, $\alpha_{-n} \in (-\infty, -n)$, $n = 1, 2, \cdots$, *such that*

$$\lim_{n\to\infty} \{H(\alpha_{\pm n}) + H(\beta_n)\} = +\infty,$$

where $H(x) = \int_0^x h$ *and* $H(\beta_n) = \inf\{H(x) : x \in [\alpha_{-n}, \alpha_n]\}$.

If e is measurable and $\int_0^t |e|$ exists and is finite for each $t \in [0, \infty)$ then the solutions of (12) exist in the future.

If, in addition, $\int_0^\infty |e| < +\infty$ then the solutions of (12) are uniformly bounded.

REMARKS. If $H(x) \geqq H_0 > -\infty$ for all x then the condition (ii) is simply

$$\limsup_{|x|\to\infty} H(x) = +\infty.$$

This result has been proved by Antosiewicz [1] for the case $H(x) \geqq 0$, $\lim_{|x|\to\infty} H(x) = +\infty$.

For the case $e(t) \equiv 0$ see Example 3 above; in this case there is no restriction on $H(\beta_n)$.

PROOF. Let $V_n(t, x_1, x_2) = (2H(x_1) - 2H(\beta_n) + x_2^2)^{1/2} - \int_0^t |e|$, $t \in [0, \infty)$, $x_1 \in (\alpha_{-n}, \alpha_n)$, $x_2 \in R$, $n = 1, 2, \cdots$. If $(x_1, x_2)(t)$ is a solution of (12) then

$$\frac{d}{dt} V_n(t, x_1(t), x_2(t)) \leqq 0, \quad \text{if } x_1(t) \in (\alpha_{-n}, \alpha_n). \tag{14}$$

Suppose there is a solution $(x_1, x_2)(t)$ which does not exist in the future, i.e., $(x_1, x_2)(t)$ exists on a right-maximal interval $[t_0, T)$, $T < +\infty$. Hence $x_1(t)$ and/or $x_2(t)$ must be unbounded on $[t_0, T)$. Condition (ii) implies that there exists a positive integer n such that $x_1(t_0) \in (\alpha_{-n}, \alpha_n)$ and

$$H(\alpha_{\pm n}) + H(\beta_n) > 2H(x_1(t_0)) + (x_2(t_0))^2 + \left(\int_{t_0}^T |e|\right)^2. \tag{15}$$

But (14) implies that if $x_1(s) \in (\alpha_{-n}, \alpha_n)$ for $s \in [t_0, t]$ then

$$V_n(t, x_1(t), x_2(t)) \leqq V_n(t_0, x_1(t_0), x_2(t_0))$$

from which it follows that

$$\begin{aligned} &2H(x_1(t)) + 2H(\beta_n) + (x_2(t))^2 \\ &\qquad\qquad \leqq 4H(x_1(t_0)) + 2\,(x_2\,(t_0))^2 + 2\left(\int_{t_0}^T |e|\right)^2. \end{aligned}$$

Therefore, from (15), $x_1(t) \in (\alpha_{-n}, \alpha_n)$ and $x_2(t)$ is bounded, $t \in [t_0, T)$, so that $[t_0, T)$ is not a maximal interval if $T < +\infty$.

The statement about boundedness follows analogously if we let $T = +\infty$ above.

LEMMA. *Suppose*:

(i) *f is continuous on R.*

(ii) *p and q are measurable and $1/p$, q are locally integrable on* $[0, \infty)$.

(iii) *There exists a function* ϕ *on* $[0, \infty)$ *such that* ϕ/p *and* ϕq *are nonincreasing.*

Let $F(x) = \int_0^x f$. *If* $F(x) \geqq F_0 > -\infty$ *for all* $x \in (a, b)$ *then*

$$V = 2\phi q(F(x_1) - F_0) + \frac{\phi}{p} x_2^2$$

is nonincreasing if $(x_1, x_2)(t)$ *is a solution of* (13) *and* $x_1(t) \in (a, b)$.

PROOF. We prove this lemma by means of an integration by parts technique used by Klokov [8]. (13) implies

$$\phi q f(x_1) x_1' + \frac{\phi}{p} x_2 x_2' = 0.$$

Therefore

$$\begin{aligned} 0 &= 2 \int_{t_1}^{t_2} \phi q \, d(F(x_1) - F_0) + \int_{t_1}^{t_2} \frac{\phi}{p} d(x_2^2) \\ &= \left[2\phi q(F(x_1) - F_0) + \frac{\phi}{p} x_2^2 \right]_{t_1}^{t_2} \\ &\quad - 2 \int_{t_1}^{t_2} (F(x_1) - F_0) d(\phi q) - \int_{t_1}^{t_2} x_2^2 d\left(\frac{\phi}{p}\right). \end{aligned}$$

Since $d(\phi/p) \leqq 0$ and $d(\phi q) \leqq 0$ it follows that

$$0 \geqq \left[2\phi q(F(x_1) - F_0) + \frac{\phi}{p} x_2^2 \right]_{t_1}^{t_2} = V(t_2) - V(t_1)$$

if $t_1 < t_2$ and $x(s) \in (a, b)$, $t_1 \leqq s \leqq t_2$.

REMARKS. If $pq > 0$ and $\log pq$ is locally of bounded variation on $[0, \infty)$ then

$$V = 2 \exp(-N)(F(x_1) - F_0) + \frac{1}{pq(0)} \exp(-P) x_2^2$$

satisfies the hypotheses of the lemma where $P(t)$ and $N(t)$ denote the positive and negative variation respectively of $\log pq$ on $[0, t]$. In this case $\phi(t) = (1/q(t)) \exp(-N(t))$. This result has been proved by Gollwitzer [**6**], using Stieltjes integral inequalities in the case that p and q are positive, locally of bounded variation and continuous. Here, however, it is not assumed that pq is continuous. If pq is locally absolutely continuous and positive then this is a special case of a result of Petty and Leitmann ([**10**, Lemma 1]) for a more general equation than (13).

Our main purpose in the following theorem is to extend known stability results for (13) to the case where F may change sign infinitely often in a neighborhood of an equilibrium and to extend boundedness theorems to include the possibility $\liminf_{|x|\to\infty} F(x) = -\infty$. We achieve this by the use of one-parameter families of Liapunov functions.

In Theorem 3, whenever $\lim_{t\to\infty} \lambda(t)$ exists for some function λ we denote this limit by $\lambda(\infty)$.

THEOREM 3. *Let conditions* (i), (ii), *and* (iii) *of the lemma hold.*

(a) *Suppose*:

(i) $\phi(t)/p > 0$, $\phi q(t) > 0$, *for each* $t \in [0, \infty)$.

(ii) *There exist* $\alpha_n \in (n, \infty)$, $\alpha_{-n} \in (-\infty, -n)$, $n = 1, 2, \cdots$, *such that*

$$\lim_{n\to\infty} F(\alpha_{\pm n}) + \left[\frac{\phi q(0)}{\phi q(t)} - 1\right] F(\beta_n) = +\infty$$

for each $t \in [0, \infty)$, *where*

$$F(\beta_n) = \inf\{F(x) : x \in [\alpha_{-n}, \alpha_n]\}.$$

Then the solutions of (13) *exist in the future.*

Furthermore if (a) (i) *and* (a) (ii) *hold at* $t = +\infty$ *then the solutions of* (13) *are uniformly bounded.*

(b) *Suppose*:

(i) $\phi(t)/p > 0$, $\phi q(t) > 0$, *for each* $t \in [0, \infty)$.

(ii) *For each* $\epsilon > 0$ *there exist* $\alpha_\epsilon \in (0, \epsilon)$, $\alpha_{-\epsilon} \in (-\epsilon, 0)$ *such that*

$$F(\alpha_{\pm\epsilon}) + \left[\frac{\phi q(0)}{\phi q(t)} - 1\right] F(\beta_\epsilon) > 0$$

for each $t \in [0, \infty)$, *where*

$$F(\beta_\epsilon) = \inf\{F(x) : x \in [\alpha_{-\epsilon}, \alpha_\epsilon]\}.$$

Then $(x_1, x_2)(t) \equiv (0, 0)$ *is a solution of* (13) *which is unique in the future.*

If (b) (i) *and* (b) (ii) *hold at* $t = +\infty$ *then* $(x_1, x_2)(t) \equiv (0, 0)$ *is also uniformly stable.*

REMARKS. If $F(x) \geqq F_0 > -\infty$ for all x, then condition (a) (ii) is $\limsup_{|x|\to\infty} F(x) = +\infty$ and is independent of ϕq.

If $F(x) > 0$, $x \neq 0$, in a neighborhood of $x = 0$ then condition (b)(ii) holds independently of ϕq, since $F(\beta_\epsilon) = 0$.

Also if pq is nondecreasing and positive we may choose $\phi = 1/q$ and then (a) (ii) and (b) (ii) are $\limsup_{|x|\to\infty} F(x) = +\infty$ and $F(\alpha_{\pm\epsilon}) > 0$, respectively, without any restriction on $F(\beta_n)$ or $F(\beta_\epsilon)$ since, in this case, $\phi q(0)/\phi q(t) - 1 = 0$.

PROOF OF THEOREM 3(b). By the lemma, the function $V_\epsilon = 2\phi q\{F(x_1) - F(\beta_\epsilon)\} + \phi x_2{}^2/p$, for each $\epsilon > 0$, is nonincreasing whenever $(x_1, x_2)(t)$ is a solution of (13) satisfying $x_1(t) \in (\alpha_{-\epsilon}, \alpha_\epsilon)$. Hence

$$2\phi q(t)\left\{F(x_1(t)) + \left[\frac{\phi q(t_0)}{\phi q(t)} - 1\right]F(\beta_\epsilon)\right\} + \frac{\phi}{p}(t)(x_2(t))^2 \leqq 2\phi q(t_0)F(x_1(t_0)) + \frac{\phi}{p}(t_0)(x_2(t_0))^2 \tag{16}$$

for each $t \geqq t_0$ such that $x(s) \in (\alpha_{-\epsilon}, \alpha_\epsilon)$, $t_0 \leqq s \leqq t$.

We first prove that $(x_1, x_2)(t) \equiv (0, 0)$ is a solution which exists and is unique in the future. Suppose $x_1(0) = x_2(0) = 0$ and $|x_1(T)| = \epsilon > 0$ for some $T > 0$; then $x_1(t_1) = \alpha_{\pm\epsilon}$ for some t_1, $0 < t_1 < T$. We assume that t_1 is the least such number and hence, from (16)

$$2\phi q(t_1)\left\{F(\alpha_{\pm\epsilon}) + \left[\frac{\phi q(0)}{\phi q(t_1)} - 1\right]F(\beta_\epsilon)\right\} + \frac{\phi}{p}(t_1)(x_2(t_1))^2 \leqq 0$$

which is obviously false if (b)(i) and (b)(ii) hold for each $t \in [0, \infty)$. Thus $x_1(t) \equiv 0$ and, for every $\epsilon > 0$,

$$\frac{\phi}{p}(t)(x_2(t))^2 \leqq 2[\phi q(t) - \phi q(t_0)]\, F(\beta_\epsilon).$$

But $F(\beta_\epsilon) = o(1)$ as $\epsilon \to 0$ so that $x_2(t) \equiv 0$.

The statement about uniform stability follows from the fact that (16) implies

$$2\phi q(\infty)\left\{F(x_1(t)) + \left[\frac{\phi q(0)}{\phi q(\infty)} - 1\right]F(\beta_\epsilon)\right\} + \frac{\phi}{p}(\infty)(x_2(t))^2 \leqq \phi q(0)F(x_1(t_0)) + \frac{\phi}{p}(0)(x_2(t_0))^2 \tag{17}$$

if $x(s) \in (\alpha_{-\epsilon}, \alpha_\epsilon)$, for $t_0 \leqq s \leqq t$. Now suppose (b) (i) and (b) (ii) hold at $t = +\infty$. Since F is continuous at 0 and $F(0) = 0$ there exists a $\delta(\epsilon) > 0$ such that if $|(x_1, x_2)(t_0)| < \delta(\epsilon)$ then the right-hand side of (17) is strictly less than

$$2\phi q(\infty)\left\{F(\alpha_{\pm\epsilon}) + \left[\frac{\phi q(0)}{\phi q(\infty)} - 1\right]F(\beta_\epsilon)\right\}.$$

Thus $x_1(t) \in (\alpha_{-\epsilon}, \alpha_\epsilon) \subset (-\epsilon, \epsilon)$ for all $t \geqq t_0$. Then $F(x_1(t)) \geqq F(\beta_\epsilon)$ for $t \in [t_0, \infty)$ and therefore

$$\frac{\phi}{p}(\infty)(x_2(t))^2 \leqq \phi q(\infty)\{F(\alpha_{\pm\epsilon}) - F(\beta_\epsilon)\} = o(1)$$

as $\epsilon \to 0$.

Part (a) may be proved similarly by considering the functions

$$V_n = 2\phi q\{F(x_1) - F(\beta_n)\} + \frac{\phi}{p}x_2{}^2, \qquad n = 1, 2, \cdots.$$

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