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ON THE DEGENERATION OF SOLUTIONS OF THE ABSTRACT DIFFERENTIAL EQUATION $\epsilon u'' + Au = 0$

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ABSTRACT. Let Ω be an open subset of R^n, $\mathfrak{X}$ a locally convex linear topological space of functions defined on Ω, and $H^0 \subset \mathfrak{X}$ (as sets), where H^0 is the completion of $C_0^\infty(\Omega)$ with respect to the inner product $(f, g)_k = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha f \overline{D^\alpha g}\, dx$. Let $\mathfrak{X}_0$ be the set of limits of sequences of functions in $\mathfrak{X}$ which vanish on $\partial\Omega$. Under suitable conditions it is shown that the solution u_ϵ of the problem $\epsilon^2 u'' + Au = 0$, $u \in \mathfrak{X}_0$, $u(0) = f_1 + h$, $u'(0) = g$ with $h \in \mathfrak{X}$, f_1, $g \in H^0$, is such that $(t_2 - t_1)^{-1} \int_{t_1}^{t_2} u_\epsilon(\tau)\, d\tau \to V$ as $\epsilon \to 0$, where $AV = 0$ and $V - h \in H^0$. Applications to degeneration of the wave equation to Laplace's equation are made.

I. **Formulation of the problem.** Let Ω be an open subset of R^n with (sufficiently smooth) boundary $\partial\Omega$; we allow the possibility that $\partial\Omega$ is empty, thus including the Cauchy problem as well as the initial-boundary value problem in our analysis. Denote by $C_0^\infty(\Omega)$ the set of all infinitely differentiable functions having compact support in Ω. For $\alpha = (\alpha_1, \cdots, \alpha_n)$ with α_j nonnegative integers set $|\alpha| = \sum_{j=1}^n \alpha_j$ and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1{}^{\alpha_1} \cdots \partial x_n{}^{\alpha_n}.$$

For f, $g \in C_0^\infty(\Omega)$ and k a nonnegative integer define the inner product

$$(f, g)_k = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha f\, \overline{D^\alpha g}\, dx;$$

denote the corresponding norm by $\|\cdot\|_k$. Let $H_k{}^0$ be the completion of $C_0^\infty(\Omega)$ with respect to this norm; then $H_k{}^0$ is a Hilbert space. Throughout the following k will be a fixed integer (depending on the problem considered), and hence will be dropped from the notation. Let $\mathfrak{X}$ be a locally convex linear topological space of functions defined on $\bar{\Omega}$ and including (*as a set*) H^0; we suppose that the topology of $\mathfrak{X}$ is weaker than that of H^0. Let $\mathfrak{X}_0$ be a subspace of $\mathfrak{X}$ composed of limits of sequences of functions vanishing in a neighborhood (depending on the function) of $\partial\Omega$ and such that $\mathfrak{X}_0 \subset H^0$ (as sets). Let $h \in \mathfrak{X}$, and let A be an operator: $\mathfrak{X} \supset D(A) \to \mathfrak{X}$. We are con-

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cerned with the relation between solutions (in $\mathfrak{X}$) of the problems

$$AU = 0, \qquad U - h \in H^0 \tag{1}$$

(the degenerate problem) and

$$\begin{aligned} \epsilon^2 u_{tt} + Au &= 0, \qquad u - h \in \mathfrak{X}_0, \\ u|_{t=0} &= f_1 + h, \qquad u_t|_{t=0} = g \end{aligned} \tag{2}$$

(the perturbed problem); here f_1, $g \in H^0$ and the function $u = u_\epsilon$ is assumed to have two derivatives (in the topology of $\mathfrak{X}$) with respect to t. Concerning (1) we assume that (for the given $h \in \mathfrak{X}$) solutions exist (but are not necessarily unique). If we set $v = u - U$ where U is some solution of (1) we have, for v, the problem

$$\begin{aligned} \epsilon^2 v_{tt} + Av &= 0, \qquad v \in \mathfrak{X}_0, \\ v|_{t=0} &= f \equiv f_1 + (h - U) \in H^0, \qquad v_t|_{t=0} = g, \end{aligned} \tag{3}$$

where the derivatives of v are taken with respect to the topology of $\mathfrak{X}$. We shall impose conditions sufficient to guarantee, for every small $\epsilon > 0$ and suitable f, g, the existence of a unique solution to this problem *in* H^0 (i.e., $v \in H^0$ and derivatives taken with respect to the norm in H^0). We shall refer to this problem in H^0 as (3′). It follows then that $u = U + v$ is a solution of (2). We assume that solutions of (2) are unique, so every solution is obtained in this manner.

Our goal is to prove that $u_\epsilon(t) \to V$ in a suitable sense as $\epsilon \to 0$, where V is some solution of (1) (of course, $V = U$ if the solution of (1) is unique). If this happens, then we can use the degeneration of problem (2) to problem (1) to uniquely pick out solutions of (1) by obtaining them as limits of solutions of (2) for suitable initial data f_1, g. The precise sense in which we show that $u_\epsilon \to V$ is: Given any t_1, $t_2 > t_1$, then $(t_2 - t_1)^{-1} \int_{t_1}^{t_2} u_\epsilon(\tau)\, d\tau \to V$. Examples of this behavior are presented in the following section. In the final section we formulate and prove our theorems on degeneration (Theorems 1, 2) and show by example that $u_\epsilon \to V$ in the topology of $\mathfrak{X}$ does not occur in general.

Note that the change of the variable $t \to t/\epsilon$ removes the ϵ dependence from (2) and (3); hence our results concerning behavior as $\epsilon \to 0$ can be interpreted as holding when $t \to \infty$.

In 1950, N. Levinson [7] considered the Dirichlet problem

$$L_\epsilon[u] \equiv \epsilon(u_{xx} + u_{yy}) + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y),$$

$$u|_{\partial R} = \phi,$$

where R is a subset of the plane, and showed that $u(x, y, \epsilon)$ tends in the interior of R to a solution of $L_0[U] = D$ as $\epsilon \to 0+$. A number of authors have since studied singular perturbation problems for partial differential equations; an extensive bibliography can be found in [8]. Degeneration problems for abstract evolution equations have been studied in [2]-[6], [9], [10]. A classical problem related to the examples following may be found in [1].

II. Examples. Let Ω be a bounded domain in R^3 and $A = -\Delta$ ($\Delta =$ Laplacian in R^3); let $k = 1$ and $\mathfrak{X} = H^0$. Problem (1) asks for a harmonic function vanishing (in a generalized sense) on $\partial\Omega$. Problem (2) asks for the solution of the wave equation with wave speed $1/\epsilon$ which vanishes on $\partial\Omega$ and satisfies $u(0) = f_1 + h$, $u_t(0) = g$. Existence and uniqueness of solutions to both problems are known for suitable $\partial\Omega$, f_1, g. Our assertion is that the time average of the solution of (2) tends to the solution of (1) as $\epsilon \to 0$. This example can be extended to more general elliptic operators A.

We turn now to a second example. Let Ω be unbounded, take $k = 1$, and define a sequence of seminorms $\{\|\cdot\|_j\}$ on $C^\infty(\Omega)$ as follows. Let B_j be the intersection of Ω with the open ball of radius j and set

$$\|f\|_j^2 = \sum_{|\alpha| \leq 1} \int_{B_j} |D^\alpha f|^2 \, dx \qquad (j = 1, 2, \cdots).$$

Let $\mathfrak{X}$ be the completion of $C^\infty(\Omega)$ with respect to the family $\{\|\cdot\|_j\}$. Then $\mathfrak{X}$ is a Fréchet space, $H^0 \subset \mathfrak{X}$ as sets, and the topology of H^0 is stronger than the topology of $\mathfrak{X}$. Again let $A = -\Delta$. For $h = 0$, problem (1) asks for a harmonic function vanishing on $\partial\Omega$; as is well known, solutions of exterior Dirichlet problems are not unique. In physical applications it is customary to choose the desired solution by imposing some "condition at infinity" on the solution (such as boundedness, tending to zero, or $1/r$ behavior). Let h be some function in $\mathfrak{X}$ which vanishes in a neighborhood of $\partial\Omega$ and which has the desired behavior at infinity; we consider only the case where the condition at infinity can be expressed in the form $U - h \in H^0$. Only for certain $h \in \mathfrak{X}$ will (1) have solutions, of course. Our theorem below shows that if solutions of (1) exist, then some solution of (1) is the limit as $\epsilon \to 0$ of the time average of solutions of (2). Since the solutions of problem (2) for the wave equation are unique, the "condition at infinity" h receives an interpretation as an initial condition for the wave equation. Many physical problems involving Laplace's operator have natural interpretations as solutions of the wave equation in the limit as the speed of propagation tends to ∞; for such problems in unbounded domains the "initial state of the universe" (i.e., the initial data $f_1 + h$, g for the wave equation problem) thus dictates the "condition at infinity" which the desired solution of Laplace's equation must satisfy. And physically, it is always time averages of solutions that are observed.

A similar consideration applies to the "bent beam" operator $A = -(\partial/\partial x)^4$.

III. **Results.** We assume throughout the following that A is defined and selfadjoint on a dense subset of H^0 and that $(-\infty, 0) \subset \rho(A)$, the resolvent set of A. Since A is thus positive, it has a unique selfadjoint square root $S = A^{1/2}$ with $(-\infty, 0) \subset \rho(S)$. Since S, being selfadjoint, is closed, it generates a strongly continuous group e^{iSt} by

$$e^{iSt}x = \int_{0-}^{\infty} e^{i\lambda t}\, d(E_\lambda x)$$

for $x \in H^0$, where $\{E_\lambda\}$ is the resolution of the identity for S. From this it follows that $\sin(St) = (2i)^{-1}(e^{iSt} - e^{-iSt})$, $\cos(St) = \frac{1}{2}(e^{iSt} + e^{-iSt})$ are well-defined operators of norm 1. The usual identities of trigonometry and calculus follow at once from this definition and the continuity of the group e^{iSt}.

It is a straightforward matter to check that

$$v_\epsilon(t) = \cos\left(\frac{1}{\epsilon} St\right) f + \int_0^t \cos\left(\frac{1}{\epsilon} S\tau\right) g\, d\tau \tag{4}$$

is the solution of (3′) for $f \in D(A)$ and $g \in D(S)$, which we henceforth assume.

THEOREM 1. *If $0 \in \rho(A)$, then, for any t_1 and $t_2 > t_1$,*

$$\lim_{\epsilon \to 0+} \int_{t_1}^{t_2} v_\epsilon(\tau)\, d\tau = 0.$$

PROOF. The hypothesis guarantees that $0 \in \rho(S)$, so we have a generalization of the Riemann-Lebesgue Lemma:

$$\left\| \int_{t_1}^{t_2} \cos\left(\frac{1}{\epsilon} St\right) f\, d\tau \right\|$$

$$= \left\| \epsilon S^{-1} \left\{ \sin\left(\frac{1}{\epsilon} St_2\right) f - \sin\left(\frac{1}{\epsilon} St_1\right) f \right\} \right\|$$

$$\leqq 2\epsilon \|S^{-1} f\| \to 0$$

as $\epsilon \to 0$. Also

$$\left\| \int_0^{\tau} \cos\left(\frac{1}{\epsilon} S\sigma\right) g \, d\sigma \right\| \leqq 2\epsilon \|S^{-1}g\|,$$

which tends to zero uniformly in τ, whence its integral over $[t_1, t_2]$ also tends to zero. This proves the theorem.

Aside from the existence of S^{-1}, the proof above hinges only on the uniform boundedness of $\sin(St)$; hence it can be generalized to any locally convex linear topological space and any operator A possessing a square root S such that iS generates an equibounded group e^{iSt}. The more interesting situation $0 \notin \rho(S)$ is treated in the following theorem, whose proof depends on Hilbert-space methods.

THEOREM 2. *If* $0 \in \overline{\rho(A)}$, *then, for any* t_1 *and* $t_2 > t_1$,

$$V \equiv \lim_{\epsilon \to 0+} \int_{t_1}^{t_2} v_\epsilon(\tau) \, d\tau$$

exists and satisfies $AV = 0$. *Furthermore,* $V = 0$ *for all* $f \in D(A)$, $g \in D(S)$ *if and only if* $0 \notin \sigma_P(A)$, *the point spectrum of* A.

PROOF. Using the spectral representation for A it is easy to verify that $\sigma_P(A) = \{\sigma_P(S)\}^2 = \{z^2 : z \in \sigma_P(S)\}$; hence $0 \in \sigma_P(A)$ if and only if $0 \in \sigma_P(S)$. Again let $\{E_\lambda\}$ be the resolution of the identity for S. We have

$$\begin{aligned}
\int_{t_1}^{t_2} \cos\left(\frac{1}{\epsilon} S\tau\right) f \, d\tau &= \int_{t_1}^{t_2} \left\{\int_{0-}^{\infty} \cos\left(\frac{1}{\epsilon} \lambda\tau\right) d(E_\lambda f)\right\} d\tau \\
&= \int_{t_1}^{t_2} \left\{\int_{\delta}^{\infty} \cos\left(\frac{1}{\epsilon} \lambda\tau\right) d(E_\lambda f)\right\} d\tau \\
&\quad + \int_{t_1}^{t_2} \left\{\int_{0+}^{\delta} \cos\left(\frac{1}{\epsilon} \lambda\tau\right) d(E_\lambda f)\right\} d\tau \\
&\quad + (t_2 - t_1)[E_{0+}f - E_{0-}f] \\
&\equiv J_1(\epsilon, \delta) + J_2(\epsilon, \delta) + J_3.
\end{aligned}$$

$J_3 = (t_2 - t_1)E_{0+}f$ since $E_{0-} = 0$. J_3 is independent of ϵ, and zero if and only if $0 \notin \sigma_P(S)$; J_2 can be made small uniformly in ϵ by choosing $\delta > 0$ small because E_λ is continuous from the right. As to J_1, we have

$$\begin{aligned}
J_1 &= \int_{\delta}^{\infty} \left\{\int_{t_1}^{t_2} \cos\left(\frac{1}{\epsilon} \lambda\tau\right) d\tau\right\} d(E_\lambda f) \\
&= \epsilon \int_{\delta}^{\infty} \frac{1}{\lambda} \left[\sin\left(\frac{1}{\epsilon} \lambda t_2\right) - \sin\left(\frac{1}{\epsilon} \lambda t_1\right)\right] d(E_\lambda f)
\end{aligned}$$

whence

$$\|J_1(\epsilon, \delta)\| \leqq \frac{2\epsilon}{\delta} \|E_\infty f - E_\delta f\| \leqq \frac{2\epsilon}{\delta} \|f\|.$$

Thus

$$\lim_{\epsilon \to 0+} \int_{t_1}^{t_2} \cos\left(\frac{1}{\epsilon} S\tau\right) f\, d\tau = (t_2 - t_1) E_{0+} f.$$

Moreover, the convergence is uniform for $|t_2 - t_1| < M$.

Similarly,

$$\int_{t_1}^{t_2} \left\{ \int_0^\tau \cos\left(\frac{1}{\epsilon} S\sigma\right) g\, d\sigma \right\} d\tau$$

$$= \int_{t_1}^{t_2} \int_0^\tau \left\{ \int_{0-}^\infty \cos\left(\frac{1}{\epsilon} \lambda\sigma\right) d(E_\lambda g) \right\} d\sigma\, d\tau$$

$$\equiv I_1(\epsilon, \delta) + I_2(\epsilon, \delta) + I_3,$$

where I_1, I_2 are defined in the same manner as J_1, J_2, respectively, and

$$I_3 = \int_{t_2}^{t_1} \int_0^\tau (E_{0+} g)\, d\sigma\, d\tau = \frac{1}{2} (t_2{}^2 - t_1{}^2) E_{0+} g.$$

As before we can make I_2 arbitrarily small uniformly in ϵ small and in t_1, $t_2 \in [-T, T]$ for any $T > 0$ by choosing $\delta > 0$ small enough. For I_1 we have

$$I_1 = \int_\delta^\infty \left\{ \int_{t_1}^{t_2} \int_0^\tau \cos\left(\frac{1}{\epsilon} \lambda\sigma\right) d\sigma\, d\tau \right\} d(E_\lambda g),$$

and the innermost integral tends to zero as $\epsilon \to 0$ uniformly in τ for $t_1, t_2 \in [-T, T]$ as before.

We have thus shown that

$$\int_{t_1}^{t_2} v_\epsilon(\tau)\, d\tau \to (t_2 - t_1) E_{0+} f + \frac{1}{2} (t_2{}^2 - t_1{}^2) E_{0+} g$$

as $\epsilon \to 0$, uniformly in $t_1, t_2 \in [-T, T]$. If $E_{0+} = 0$, i.e., if 0 is not in the point spectrum $\sigma_P(S)$ of S, then $\int v_\epsilon(\tau)\, d\tau \to 0$; and conversely, if $\int v_\epsilon(\tau)\, d\tau \to 0$ for all $f \in D(A)$, $g \in D(S)$, then $E_{0+} = 0$ and $0 \notin \sigma_P(S)$. If $0 \in \sigma_P(S)$, then the eigenspace corresponding to the eigenvalue zero is precisely the range of E_{0+}, so that

$$A\left(\lim_{\epsilon \to 0} \int_{t_1}^{t_2} v_\epsilon(\tau)\, d\tau \right) = 0.$$

The example $\Omega = (-\infty, \infty)$, $H^0 = L_2(-\infty, \infty)$, $A = -(d/dx)^2$, $g \equiv 0$, and f smooth with compact support shows the necessity of the time average in our results. For the solution of (3′) is given by the classical result of d'Alembert

$$v_\epsilon(t, x) = \frac{1}{2}\,[f(x + t/\epsilon) + f(x - t/\epsilon)],$$

which certainly does not converge in L_2 as $\epsilon \to 0$.

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