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INCLUSION THEOREMS AND SEMICONSERVATIVE *FK* SPACES

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Dedicated to the memory of Leo Moser

1. **Introduction.** The first half of this article gives a unified inclusion theorem for *FK* spaces. These spaces arose in the theory of summability (see [**11**]). In that context it was natural to assume that every such space includes c, the set of convergent (complex) sequences. Throwing off the connection with matrices, the theory became more general and more satisfactory, [**2**], [**6**], with topological methods replacing classical matrix arguments. However, it now appears that these methods depend crucially on a weaker property than inclusion of c, the property which we call *semiconservative.* With this hypothesis the classical results are obtained, often in a more natural and simple way.

2. **Acknowledgement.** The results of this article were obtained in the course of three years of seminars held at Lehigh University. Many of them are due to or are inspired by Grahame Bennett and Nigel Kalton; in particular the methods of §8 were suggested by them. Theorem 7 was also obtained by W. H. Ruckle (using different methods). Part of Theorem 10 was announced in [5]. Professors G. W. Goes, F. P. A. Cass, and Mr. J. H. Hampson made useful suggestions.

3. ***FK* spaces.** An *FK* space X is a vector space of complex sequences which is also a Fréchet space (linear, complete metric) with continuous coordinates. We shall also assume that $X \supset E^\infty$ the set of all finitely nonzero sequences. A *BK* space is a normed *FK* space. An introduction to *FK* spaces is given in [8, §§11.3 and 12.4]. Our results can easily be extended to function spaces which are *FH* spaces [8, §11.3], replacing E^∞ by the set of functions with compact support or any other dense subspace.

We shall call X *o-conservative* if $X \supset c_0$, the space of null sequences, and *semiconservative* if $\sum \delta^k$ is weakly Cauchy, i.e. $\sum f(\delta^k)$ is convergent for all $f \in X'$. Here δ^k is the sequence x with $x_k = 1, x_n = 0$ for $n \neq k$. An o-conservative space is semiconservative by the general

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theorem [8, §11.3, Corollary 1], that is, if $Y \subset X$ the inclusion map must be continuous. The difference between these two properties is shown by Corollary 5. If $X \subset Y$ and X is semiconservative, so is Y; if X is a closed subspace of Y and Y is semiconservative and locally convex, so is X. "Locally convex" and "closed" can be omitted if X has finite codimension, or more generally, is complemented in Y. (See [9, p. 78].)

4. **Inclusion theorems.** We give generic inclusion theorems which unify many known results and have some new special cases. By the general theorem mentioned in §3, the sufficient conditions given are also necessary. This remains true for all our inclusion theorems and we shall state them all as sufficient conditions for inclusion, omitting the trivial converse in some cases.

THEOREM 1 *Let X, Y be FK spaces with E^∞ dense in Y. Suppose that on E^∞ the topology induced by Y is larger than that induced by X. Then $X \supset Y$.*

Let $y \in Y$. There exists a sequence $\{u^n\}$ of points in E^∞ with $u^n \to y$. Then $\{u^n\}$ is Cauchy in X, say $u^n \to x$ in X. Then $u^n \to x$ and y in s (the space of all sequences), i.e., $u_k{}^n \to x_k$ and y_k for each k. Thus $x = y$ and so $y \in X$.

More generally Theorem 1 holds if X is a sequentially complete K space and Y is a metrizable K space; or X is a complete K space and Y is a K space. (A K space is a topological vector sequence space with continuous coordinates.) More generally Y may be closure sequential (see [10, p. 30] and [3, p. 19]).

THEOREM 2. *Let X, Y be FK spaces with E^∞ dense in Y. Suppose that every subset of E^∞ which is a bounded subset of Y is also a bounded subset of X. Then $X \supset Y$.*

Let A, B be E^∞ with, respectively, the relative topology of X, Y. The identity map from A to B is bounded, hence continuous by [8, §10.5, Example 4 and Theorem 4]. The result follows from Theorem 1.

Theorem 2 is extended in the same way as Theorem 1; in the second case add the assumption that every bornivore in X is a neighborhood of 0 (bornological, in the locally convex case).

THEOREM 3. *Let Y be a BK space with E^∞ dense. Let D be the intersection of E^∞ with the unit disc of Y. If X is an FK space and D is bounded in X, then $X \supset Y$.*

This is immediate from Theorem 2.

Now let l^p be the familiar FK space $\{x : \sum |x_n|^p < \infty\}$, $p > 0$. For l^1 we write l.

COROLLARY 1. *An FK space X includes l if and only if the absolutely convex hull of* $\{\delta^n\}$ *is bounded in X. A locally convex FK space X includes l if and only if* $\{\delta^n\}$ *is bounded in X.*

The first part follows from Theorem 3 since D is precisely the set mentioned. Local convexity cannot be omitted in the second half as is shown by $l^{1/2}$.

Let $\sigma^n = \sum_{k=1}^n \delta^k$, $bv = \{x : \sum |x_n - x_{n+1}| < \infty\}$, $bv_0 = bv \cap c_0$. Then $bv \subset c$ and bv is a BK space with $\|x\|_{bv} = |\lim x| + \sum |x_n - x_{n+1}|$.

COROLLARY 2. *An FK space X includes* bv_0 *if and only if the absolutely convex hull of* $\{\sigma^n\}$ *is bounded in X. A locally convex FK space X includes* bv_0 *if and only if* $\{\sigma^n\}$ *is bounded in X.*

Again D, in Theorem 3, is precisely the set mentioned.

COROLLARY 3. *Every locally convex semiconservative FK space includes* bv_0.

The hypothesis implies that $\{\sigma^n\}$ is weakly bounded, hence bounded [8, §12.3, Theorem 1].

Note however that bv_0 is not semiconservative; consider $f(x) = \sum (-1)^n x_n$.

The above arguments yield the following result. Let Z be the set of sequences in E^∞ all of whose nonzero terms are ± 1.

COROLLARY 4. *An FK space X is o-conservative if and only if the absolutely convex hull of Z is bounded in X. A locally convex FK space X is o-conservative if and only if Z is bounded in X.*

5. **Inclusion for locally convex spaces.** With each FK space X we associate the sequence space X^f consisting of all sequences of the form $\{f(\delta^k)\}$, $f \in X'$. Note that X is semiconservative if and only if $X^f \subset cs$, the set of convergent series. From $X \supset Y$ it follows that $X^f \subset Y^f$.

THEOREM 4. *Let Y be an FK space with* E^∞ *dense. Let X be a locally convex FK space with* $X^f \subset Y^f$. *Then* $X \supset Y$.

Let $S \subset E^\infty$ be a bounded subset of Y. Let $f \in X'$. By hypothesis there exists $g \in Y'$ with $f = g$ on E^∞. Then $g[S]$, and so $f[S]$ too, is bounded. Thus S is a weakly bounded subset of X, hence bounded. The result follows from Theorem 2.

If Y is an AK space (i.e. $\{\delta^n\}$ is a basis), Y^f may be replaced by $Y^\beta = \{u : \sum u_n y_n$ is convergent for all $y \in Y\}$, for these sets are equal if (and only if) Y is an AK space.

From Theorem 4 we obtain for a locally convex FK space the following results (the second and third of which are equivalent to the second part of Corollaries 1 and 2): $X \supset cs$, $X \supset l$, $X \supset bv_0$, $X \supset l^p$, $X = s$ if and only if $X^f \subset bv_0$, $X^f \subset m$ (the set of bounded sequences), $X^f \subset bs$ (the set of bounded series), $X^f \subset l^q$ $(p + q = pq)$, $X^f = E^\infty$. (The last result may be new.) We also obtain the following new result which is actually equivalent to the second part of Corollary 4.

COROLLARY 5. *A locally convex space X is o-conservative if and only if* $\sum |f(\delta^k)| < \infty$ *for all* $f \in X'$.

For the condition is precisely $X^f \subset l = c_0{}^f$.

The Silverman-Toeplitz conditions can be deduced from Corollary 5.

In Theorem 4 it is not sufficient to assume $X^\beta \subset Y^f$. To see this, let q be the closure in m of the periodic sequences. It is shown in [1, p. 367, Corollary 2], that $q^\beta = l$. Now let X be any FK space including q and E^∞. Then $X^\beta \subset q^\beta = l$ but X need not include c_0, for example we may take $X = l + q$.

COROLLARY 6. *Let X be a locally convex FK space with* $\{\delta^n\}$ *as basis. Then X is the largest subset of* $X^{\beta\beta}$ *which is an FK space in which* E^∞ *is dense.*

Let Y be any such space. Then $Y^f \supset Y^\beta \supset X^{\beta\beta\beta} = X^\beta = X^f$. By Theorem 4, $X \supset Y$.

For example c_0 is the largest subset of m which is an FK space in which E^∞ is dense.

For completeness we mention two inclusion results not related to the above theorems. (a) Any FK space which contains every sequence of zeros and ones must include m. (b) Any o-conservative FK space which is weakly sequentially complete (for example, semireflexive) must include m.

The proof of (a) is given for BK spaces in [4, Theorem 3.3c]. It can be extended to FK spaces. (This was pointed out to us by Nigel Kalton.) To prove (b), if $x \in m$, $\sum x_k \delta^k$ is weakly Cauchy in c_0 by Corollary 5, hence weakly convergent in X. It can only converge to x.

6. **Semiconservative matrices.** Given an infinite matrix A, let $c_A = \{x : Ax \in c\}$. In [8, §12.4, Theorem 2], it is shown that c_A is a locally convex FK space. We call A semiconservative, o-conservative, or conservative if c_A has the corresponding property. The following theorem is well known.

THEOREM 5. *For a matrix A,* $c_A \supset bv_0$ *if and only if*

(i) $\lim_n a_{nk} = a_k$ *exists for each k,*

(ii) $\sup_{m,n} | \sum_{k=1}^m a_{nk} | < \infty$.

Condition (i) states that $c_A \supset E^\infty$. Using [8, §12.1, Theorem 2 and §12.4, Theorem 2], it is easy to check that (ii) is equivalent to boundedness of $\{\sigma^n\}$ and Corollary 2 applies.

Consider the following conditions on a matrix A which obeys (i) (Theorem 5):

(iii) $\sum_k a_{nk}$ converges for each n,
(iv) $\sum a_k$ converges.

Theorem 6. *A matrix A is semiconservative if and only if conditions* (i), (ii), (iii) *and* (iv) *hold, that is, if and only if* $c_A \supset bv_0$ *and conditions* (iii) *and* (iv) *hold.*

If A is semiconservative, $c_A \supset bv_0$ by Corollary 3. Conditions (iii) and (iv) follow by consideration of the functions $x \to \lim Ax$ and $x \to (Ax)_n = \sum a_{nk} x_k$ and the definition of semiconservative. Conversely, if the conditions hold, let $f \in c_A{}'$. The facts now to be used are contained in [8, §12.4, Theorem 6]. We have $f(\delta^k) = \alpha a_k + \sum_r t_r a_{rk} + \beta_k$ where $\sum |t_r| < \infty$ and $\sum \beta_k x_k$ converges for all x such that Ax exists. Condition (iii) implies that $A1$ exists, hence $\sum \beta_k$ converges. Since $\sum a_k$ converges, the proof is concluded by showing the convergence of $\sum_k \sum_r t_r a_{rk}$. Now $\sum_{k=1}^{m-1} \sum_r t_r a_{rk} = \sum_r t_r \sum_k a_{rk} - \sum_r t_r b_{rm}$ where $b_{rm} = \sum_{k=m}^{\infty} a_{rk}$. The two series on the right converge by (ii); the second series is uniformly convergent by the Weierstrass M-test and so we may let $m \to \infty$ inside the summation.

It is easy to see that the intersection of countably many locally convex semiconservative spaces is semiconservative. This is false for intersections in general.

Theorem 7. *The intersection of all locally convex semiconservative FK spaces is* bv_0.

It is sufficient to show that bv is the intersection of all those spaces of the specified type which include it, for $bv_0 = bv \cap c_0$ and c_0 is semiconservative. Let $x \notin bv$. There exists y with $\sum y_i$ convergent, $\sum x_i y_i$ divergent. Then $\{z : \sum y_i z_i$ is convergent$\}$ is semiconservative, excludes x, and includes bv. (See [8, §12.4, Lemma 1].)

7. **Coregular and conull spaces.** We shall now use the word *space* as an abbreviation for locally convex semiconservative *FK* space. A *conull* (*coregular*) space is a space in which $\sum \delta^k$ is (not) weakly convergent. Thus every conull space contains 1 and so includes bv. The proof of Theorem 7 shows that the intersection of all conull spaces is bv. If $1 \notin X$, X is automatically coregular. Any space which includes a conull space is conull and any closed subspace of a conull space is conull. The intersection of countably many conull spaces is conull.

Consideration of Theorem 7 and spaces of the form $X \cap m$ shows that the intersection of all coregular spaces is bv_0. Since m is coregular, a conull space must contain unbounded sequences.

A classical application of these remarks is the fact that if $\sum_k a_{nk}$ is convergent for all n, there exists an unbounded x with $\sum_k a_{nk} x_k$ convergent for all n. To see this let $E_n = \{x : \sum_k a_{nk} x_k \text{ is convergent}\}$. Each E_n, hence $\bigcap E_n$, is conull, thus contains an unbounded sequence.

If a semiconservative space X is weakly sequentially complete it must be conull. (Compare §5, Remark (b).) This shows again that bv_0 is not semiconservative since norm and weak sequential convergence coincide. Also bv, having bv_0 as a closed subspace, is not semiconservative either.

We now extend some results known for conservative spaces. Let X be a space and assume that $1 \in X$. For $f \in X'$ let $\psi(f) = f(1) - \sum f(\delta^k)$. For a matrix A with $E^\infty \subset c_A$ and $1 \in c_A$, let $\chi(A) = \psi(\lim_A)$ where $\lim_A x = \lim Ax$. In the following theorem α is the number given in the proof of Theorem 6.

THEOREM 8. *Let A be a semiconservative matrix with $1 \in c_A$, and let $f \in c_A'$. Then $\psi(f) = \alpha\chi(A)$.*

The proof is similar to that given in Theorem 6 and [8, §12.4, Problem 25].

COROLLARY 7. *Let A be a semiconservative matrix with $1 \in c_A$. Then c_A is conull if and only if $\chi(A) = 0$.*

If c_A is conull, $\chi(A) = \lim_A 1 - \sum \lim_A \delta^k = 0$ since $\sum \delta^k$ converges weakly to 1. Conversely, if $\chi(A) = 0$, Theorem 8 implies that $\psi(f) = 0$ for all $f \in c_A'$; that is, $\sum \delta^k$ converges weakly to 1.

The condition $\chi(A) = 0$ is not sufficient that c_A be conull; for example, let $a_{nn} = n$, $a_{n,n-1} = -n$, $a_{nk} = 0$ otherwise.

8. **Two-norm criteria.** We now generalize, with a more elementary proof, a result of the first author [6, Theorem 1]. We use the notation $\|x\|_\infty = \sup |x_n|$, and $\|x\|_{bv} = \sum |x_k - x_{k+1}| + |\lim x|$. For a space X we say x is in the two-norm$_\infty$ closure of $S \subset X$ if there is a sequence $\{s^n\}$ in S with $s^n \to x$ in X and $\|s^n\|_\infty$ bounded. The two-norm$_{bv}$ closure is defined similarly. It is possible, in a conservative coregular space, to have 1 in the closure of E^∞. This shows the role of the extra conditions in the next two theorems.

THEOREM 9. *Let X be conservative. Then X is conull if and only if 1 is in the two-norm$_\infty$ closure of E^∞.*

THEOREM 10. *Let X be semiconservative. Then X is conull if and only if $1 \in X$ and 1 is in the two-norm$_{bv}$ closure of E^∞.*

To prove sufficiency in Theorem 9, let $f \in X'$, and $s^n \in E^\infty$ with $s^n \to 1$ in X, $\|s^n\|_\infty < M$. Then

$$f(s^n) = \sum s_k{}^n f(\delta^k) \to \sum f(\delta^k) \tag{1}$$

since the series is uniformly convergent by comparison with $\sum M|f(\delta^k)|$, and $s_k{}^n \to 1$ for each k by definition of *FK* spaces. But also $f(s^n) \to f(1)$ so X is conull. Necessity in Theorem 9 follows from Theorem 10 since $\|x\|_\infty \leqq \|x\|_{bv}$. To prove sufficiency in Theorem 10, let $f \in X'$ and $s^n \in E^\infty$ with $s^n \to 1$ in X, $\|s^n\|_{bv} < M$. Again (1) holds implying that X is conull; this is seen from the fact that the matrix $(s_k{}^n)$ is series-sequence regular. See for example [12, p. 58], or [7, Theorem 3.6]. Finally let X be conull. By definition, $\sigma^n = \sum_{k=1}^n \delta^k \to 1$ weakly so that 1 lies in the weak closure of Σ, the absolutely convex hull of $\{\sigma^n\}$. Hence 1 lies in the closure of Σ; that is, there exists $s^n \in \Sigma$ with $s^n \to 1$, $s^n = \sum t_k{}^n \sigma^k$, $\sum_k |t_k{}^n| \leqq 1$ for each n. It follows that $\|s^n\|_{bv} \leqq \sum |t_k{}^n| \, \|\sigma^k\|_{bv} = \sum_k |t_k{}^n| \leqq 1$.

Theorems 9 and 10 are best possible in the sense that conservative cannot be replaced by semiconservative in Theorem 9 and semiconservative cannot be omitted in Theorem 10. The relevant examples follow.

Example 1. Let $\sum b_k$ be a conditionally convergent series of nonzero numbers and $A = (a_{nk})$ where $a_{n,2k-1} = b_k$ for $2k - 1 \leqq n$; $a_{n,2k} = 1$ for $n = 2k$ and $n = 2k + 1$; $a_{nk} = 0$ otherwise. By Theorem 6 and Corollary 7, c_A is coregular. Let $\epsilon > 0$. Choose N and $P > N$ such that $|\sum_{k=n}^m b_k| < \epsilon$ for $m > n > N$, $|\sum_{k=N}^P |b_k| - 1| < \epsilon$. Let $\alpha_k = \operatorname{sgn} b_k$ and define $x_k = 0$ for $k < 2N - 1$, and for $k = 2N - 1, 2N, 2N + 1, \cdots$ define x_k to be, respectively, $-\alpha_N$, $|b_N|$, $-\alpha_{N+1}$, $|b_N| + |b_{N+1}|$, $-\alpha_{N+2}$, $|b_N| + |b_{N+1}| + |b_{N+2}|$, $-\alpha_{N+3}$, $\cdots$, continuing this until $-\alpha_P$ is reached. For all larger k let $x_k = 1$. It may now be checked that $\|x\|_\infty \leqq 1 + \epsilon$ and $\|Ax\|_\infty \leqq 2\epsilon$. Now c_A is a Banach space with $\|x\| = \|Ax\|_\infty$ [7, §5.2, Problem 9] so if we set $s = 1 - x$ we shall have $s \in E^\infty$, $\|s\|_\infty \leqq 2 + \epsilon$, $\|1 - s\| \leqq 2\epsilon$. Thus 1 is in the two-norm$_\infty$ closure of E^∞.

Example 2. Let $u(x) = Ax$ where $A = (a_{nk})$ is defined by $a_{n,2n} = -a_{n,2n+1} = 1$, $a_{nk} = 0$ otherwise. Then $X = u^{-1}[l]$ is a locally convex *FK* space [8, §12.4, Theorem 1] which is not semiconservative as is shown by $f(x) = \sum (x_{2k} - x_{2k+1})$. However, with $s^n = \sum_{k=1}^{2n-1} \delta^k$, $s^n \to 1$ in X and $\|s^n\|_{bv} = 1$.

For the space $\Omega(r)$ we refer to [6].

Theorem 11. *Let X be a locally convex FK space such that* 1 *is in the two-norm$_\infty$ closure of E^∞. Then $X + c_0 \supset \Omega(r)$ for some r.*

We make $X + c_0$ into an FK space in the standard way. See, for example, [8, p. 39, Theorem 7.3]. It is conservative; moreover, by Theorem 9 it is conull. The result follows from [6, Theorem 2].

For example, with A as in Example 1, $c_A + c_0 \supset \Omega(r)$ for some r.

COROLLARY 8. *Let X be a locally convex conull FK space. Then $X \cap m$ is a nonseparable subspace of m.*

More generally "conull" may be replaced by the assumption that 1 is in the two-norm$_\infty$ closure of E^∞. By Theorem 11, $(X + c_0) \cap m$ is not separable. If D is dense in $X \cap m$ and D_1 is dense in c_0, then $D + D_1$ is dense in $(X + c_0) \cap m$.

9. **Remarks.** A matrix A is conservative if and only if it maps c into itself. It is an attractive conjecture that there exists a space X such that a matrix of the above type is semiconservative if and only if it maps X into itself. That no such X exists is an immediate consequence of the fact that there exists a matrix A with $1 \in c_A$ such that A is, and A^2 is not semiconservative. Such a matrix is (a_{nk}) with $a_{nk} = (-1)^{n-k} n^{-1/4}$ for $k \leqq n$, $a_{nk} = 0$ for $k > n$. The first column of A^2 is unbounded.

10. **Questions.** 1. The referee has suggested that there might exist a space E such that $\{x : Ax \in E\}$ is semiconservative if and only if it includes E. Such a space E would have to be semiconservative as the identity matrix shows.

2. Does Corollary 5 hold without the assumption of local convexity?

3. Is there a smallest FK space in which $\{\delta^n\}$ is bounded? (Corollary 1 shows that there is a smallest locally convex one.)

4. Must the following be semiconservative? (a) A closed subspace of a semiconservative space? (b) The intersection of two semiconservative spaces?

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