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CONVERGENCE OF PADÉ APPROXIMANTS IN THE GENERAL CASE

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ABSTRACT. We shall present here a new method for proving the convergence of Padé approximants in the non-Stieltjes case. This method allows us to derive results which generalize Nuttall's theorem [2] on the convergence in measure of Padé approximants for meromorphic functions. We can prove convergence in measure for a class of functions meromorphic in a circle. We show that the same kind of results can be established for generalized Padé approximants, constructed from equations given by their values at certain points instead of their Taylor series.

I. **Outline of the results.** The convergence of $[N \pm j, N]$ Padé approximants has been known for a very long time in the case of Stieltjes functions [1]. Very precise results have been obtained in this case (See G. Baker review talk at this Conference). Little was known about the general case until recent years. The reason for this seems to be that convergence in the ordinary sense is not to be expected in general. The property which plays the essential role in the proof of convergence in the Stieltjes case is that the zeros of the denominators of Padé approximants can easily be located. In the general case, it is very likely that the poles of Padé approximants can even become dense in the complex plane. So probably only weaker forms of convergence can be shown. A result of this kind has been presented at this Conference by Nuttall [2]:

Let $f(z)$ be any given analytic function, meromorphic in the complex plane. Let R, ϵ, δ be three arbitrary positive numbers. It is always possible to find an integer N such that for $n \geqq N$ the $[n, n]$ Padé approximants satisfy

$$|f(z) - P_n(z)/Q_n(z)| < \epsilon, \text{ for } |z| \leqq R,$$

except on an open set D_n of measure smaller than δ. The diagonal Padé approximants converge in measure towards $f(z)$.

We shall present here a new derivation and a generalization of this result.

[1]Talk presented at the "International Conference on Padé Approximants, Continued Fractions and related Topics" June 21, 1972, Boulder, Colorado.

THEOREM 1. *The* $[M, N]$ *Padé approximants of an analytic function* $f(z)$ *meromorphic in the entire complex plane converge in measure in the complex plane when* M *tends towards infinity with* $N \leqq \lambda M$, *where* λ *is a fixed positive number. If* $f(z)$ *is an entire function of order smaller than* $2/\lambda$ *the* $[M, N]$ *Padé approximants converge except on an open set of measure zero.*

THEOREM 2. *The* $[M, N]$ *Padé approximants of a function meromorphic in the circle* $|z| \leqq 1$ *converge in measure towards the function, in the circle*

$$|z| \left[\frac{2|z|}{(1-|z|^2)} \right]^{\lambda} < 1,$$

when M *tends towards infinity with,* $N \leqq \lambda M$, *if the following condition is satisfied*

$$q_{N,M}/M \to 0,$$

where $q_{N,M}$ *is the number of poles of the* $[M, N]$ *Padé approximant inside the circle of radius* 1. *When* $q_{N,M}$ *satisfies*

$$q_{N,M}(\log M)/M \to 0, \text{ where } M \to \infty,$$

then we have convergence in the same circle except on a set of measure zero.

GENERALIZED PADÉ APPROXIMANTS [4]. Let $f(z)$ be an analytic function, $\{z_i\}$ a set of points belonging to the domain of analyticity of $f(z)$.

A generalized $[M, N]$ Padé approximants is defined by the ratio of two polynomials $P_M(z)$ and $Q_N(z)$, of degree M and N respectively, satisfying the condition

$$P_M(z_i)/Q_N(z_i) - f(z_i) = 0, \ \forall z_i, \ 1 \leqq i \leqq N + M + 1.$$

Analogous results can be derived for this kind of Padé approximants provided that the points $\{z_i\}$ stay at a finite given distance from all the singularities of the function $f(z)$. If the set $\{z_i\}$ has a singularity of $f(z)$ as limiting point, the results are largely modified, and depend critically on the way the subset of points tends towards the singularity.

II. **The Method.** We shall not go through all the details of the proof, which can be found in [5], but only give the general idea of the method.

A. GENERAL FORMULA. Let $f(z)$ be an analytic function which has a Taylor series expansion at the origin:

(A.1) $$f(z) = \sum_0^\infty a_n z^n.$$

The $[M, N]$ Padé approximant of $f(z)$, $P_M(z)/Q_N(z)$ satisfies

(A.2) $$f(z) - P_M(z)/Q_N(z) = O(z^{N+M+1}),$$

where $P_M(z)$ and $Q_N(z)$ are polynomials of degree M and N respectively. The elementary properties of Padé approximants can be found in [1], [4]. We can also write the definition (A.2)

(A.3) $$Q_N(z)f(z) - P_M(z) = O(z^{N+M+1}).$$

Now let us assume that the Taylor series has a nonzero radius of convergence. Then

(A.4) $$Q_N(z)f(z) - P_M(z) = \frac{z^{N+M+1}}{2i\pi}\int_C f(z')Q_N(z')\frac{dz'}{(z'-z)z'^{M+N+1}}$$

where C is a contour, around the origin, inside of which $f(z)$ is holomorphic. We can generalize (4) in the following way: let $R_N(z)$ be a polynomial of degree less than or equal to N. Then

(A.5) $$R_N(z)Q_N(z)f(z) - R_N(z)P_M(z) = O(z^{N+M+1}).$$

The polynomial $R_N(z)P_M(z)$ being at most of degree $N + M$, we have for every $R_n(z)$:

(A.6) $$\begin{aligned} R_N(z)(Q_N(z)f(z) - P_M(z)) &= \frac{z^{N+M+1}}{2i\pi} \\ &\cdot \int_C R_N(z')Q_N(z')f(z')\frac{dz'}{z'^{N+M+1}(z-z')}. \end{aligned}$$

We see reciprocally that the fact that (A.6) holds for any polynomial $R_N(z)$ shows that $P_M(z)$ is at most of degree M. Equation (A.6) will be the basic formula for the proofs of convergence.

Following Nuttall [2] we shall also use the following lemma of Szego [3].

Let δ be the measure of the set D on which the following inequality holds

(A.7) $$\prod_{i=1}^n |(z - z_i)| < \alpha^n,\ \alpha > 0,\ z \in D.$$

Then δ is maximum when all the z_i are equal and $\delta_{\max} = \pi\alpha^2$.

This lemma will be used in a systematic way in order to set an upper bound on $1/(Q_N(z)R_N(z))$ outside a set of given measure.

B. Meromorphic functions in the complex plane. Let $f(z)$ be a meromorphic function and let $\{z_n\}$ be the set of poles of the function with $|z_n| \leqq |z_{n+1}|$. We shall use identity (A.6)

$$\begin{aligned} f(z) - \frac{P_M(z)}{Q_N(z)} &= \frac{z^{N+M+1}}{2i\pi Q_N(z)R_N(z)} \\ &\cdot \int_{C(r_N)} f(z')Q_N(z')R_N(z') \frac{dz'}{(z'-z)z'^{N+M+1}} . \end{aligned} \tag{B.1}$$

We shall take for $R_N(z)$

$$R_N(z) = \prod_{i=1}^{p_N} (z - z_i) \text{ with } p_N \leqq N, \tag{B.2}$$

and for C a circle of radius r_N

$$|z_{p_N}| \leqq r_N < |z_{p_N+1}|. \tag{B.3}$$

The integer p_N is chosen such that

$$\frac{p_N}{M} \to 0, \text{ when } M \to \infty, \tag{B.4}$$

with $N \leqq \lambda M$, and $r_N \to \infty$ with $\operatorname{Sup}_\theta |f(r_N e^{i\theta})| < A^M$. One can then find a bound for $|f(z) - P_M(z)/Q_N(z)|$ which leads to theorem 1.

C. Functions meromorphic in a circle of radius 1. Let $f(z)$ be meromorphic in the circle $|z| \leqq 1$. Let $\{z_i\}$ be the finite set of the p poles of $f(z)$ in the circle. In order to find a bound on $|f(z) - P_M(z)/Q_N(z)|$ where P_M/Q_N is as before the $[M, N]$ *Padé approximant of* $f(z)$, one uses (A.6) with the following choice for $R_N(z)$:

$$R_N(z) = \prod_{i=1}^{p} (z - z_i)\bar{R}_{N-p}(z),$$

where $\bar{R}_{N-p}(z)$ has for zeros $N-p$ zeros of $Q_N(z)$. Binding $|Q_N(z)R_N(z)|^{-1}$ outside a set of measure δ_N by using Szego lemma yields to theorem 2.

D. Generalized Padé approximants. Let $f(z)$ be an analytic function, $\{z_i\}$ a set of points belonging to the domain of analyticity of $f(z)$. The generalized Padé approximant $[M, N]$ of $f(z)$ is, as explained in I, a ratio of two polynomials satisfying

$$f(z_i) - P_M(z_i)/Q_N(z_i) = 0, \ 1 \leqq i \leqq N + M + 1.$$

It is then easy to derive the analogue of formula (A.6)

$$R_N(z)Q_N(z)f(z) - R_N(z)P_N(z) = \prod_{i=1}^{N+M+1} (z - z_i)\frac{1}{2i\pi} \cdot \int_C \frac{R_N(z')Q_N(z')f(z')\,dz'}{\left[\prod_{i=1}^{N+M+1}(z' - z_i)\right](z' - z)},$$

when C is a contour inside the domain of analyticity of $f(z)$ which surrounds the points z_i, and $R_N(z)$ is as before an arbitrary polynomial of degree not higher than N. This yields to the inequality

$$\left| f(z) - \frac{P_M(z)}{Q_N(z)} \right| \leqq \frac{1}{2\pi} \left| \frac{\prod_{i=1}^{N+M+1}(z - z_i)}{Q_N(z)R_N(z)} \right| \cdot \left| \int_C \frac{R_N(z')Q_N(z')f(z')\,dz'}{\left[\prod_{i=1}^{N+M+1}(z' - z_i)\right](z' - z)} \right|;$$

which allows us to prove various analogues of theorems 1 and 2 depending on the conditions imposed to $f(z)$ and the set $\{z_i\}$.

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