

CONTINUATION OF FUNCTIONS BEYOND  
 NATURAL BOUNDARIES\*

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One encounters series of the form

$$(1) \quad f(z) = \sum_n A_n/(z - z_n)$$

in many researches concerning analytic continuation. A family of functions defined by some restriction on the  $A_n$ , such as

$$(2) \quad \sum \frac{\log \log(1/|A_n|)}{\log(1/|A_n|)} \text{ converges,}$$

is quasi-analytic provided that two functions belonging to the class and coinciding on an arc of curve on which both series (1) converge uniformly coincide everywhere.

Carleman [1] has shown that the class defined by (2) is quasi-analytic. Carleman says that Denjoy has shown by example that the class defined by

$$(3) \quad |A_\nu| < \exp(-\nu^{1/2-\epsilon}),$$

is not quasi-analytic. Since (2) is satisfied by

$$(4) \quad |A_\nu| < \exp(-\nu^{1+\epsilon}),$$

there exists a gap in our knowledge: for example, is the class defined by

$$(5) \quad |A_\nu| < \exp(-\nu^{1/2})$$

quasi-analytic or not? I do not know whether or not this question has been answered since Carleman wrote his book in 1926.

I was led by these facts to study the convergence of the  $[N/N + 1]$  Padé approximants to

$$(6) \quad f(z) = \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} e^{-n} / \left( z - \exp(2\pi i \frac{m}{n}) \right),$$

where  $m$  and  $n$  are relatively prime. I also studied the example

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$$(7) \quad f(z) = \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} e^{-\sqrt{n}} / \left( z - \exp(2\pi i \frac{m}{n}) \right),$$

where  $m$  and  $n$  are relatively prime. Since on average there are about  $0.64 n m$ 's which are relatively prime to  $n$ , for the example of (6),

$$(8) \quad A_n \sim \exp(-\sqrt{\nu/0.32})$$

and for the example of (7),

$$(9) \quad A_n \sim \exp(-4\sqrt{\nu/0.32})$$

The unit circle is a natural boundary of both functions, and I am particularly interested in the convergence of the  $[N/N + 1]$  Padé approximants beyond the natural boundary, since, as is well known, Borel [2] has shown that there exists a kind of analytic continuation which differs from the usual kind (the theory of the usual kind is due to Weierstrass), and Borel made use of examples such as the ones studied here in showing that in some cases it is possible to continue functions beyond what Weierstrass called natural boundaries. I am interested in these examples because they seem to me suggestive of the direction in which comprehensive theorems about the domains in which Padé approximants converge and theorems about to what they converge are to be sought.

I do not think that it is possible to overemphasize the uniqueness question for Padé approximants. When it is established that the diagonal files in the Padé table converge (for series of Stieltjes say), it is still necessary to answer the question: To what? That is, it is still necessary to answer the question: in what sense is the function to which the Padé approximants have converged unique? I believe the theory of quasi-analytic functions answers this question.

I would expect that the Padé approximants in the case of the example of (7) do not converge beyond the natural boundary because it would be awkward to say to what they converge if they do. Denjoy's example does not exclude the possibility that the Padé approximants for the example of (6) converge.

In figure 1, I show the zeros and poles of the  $[N/N + 1]$  Padé approximant to (6). In Table I, I illustrate the convergence of the  $[N/N + 1]$  Padé approximants outside the unit circle.

In figure 2, I show the zeros and poles of the  $[N/N + 1]$  Padé approximant to (7). In Table II, I illustrate the divergence of the  $[N/N + 1]$  Padé approximants outside the unit circle.

These examples support the supposition that there is a connection between convergence of Padé approximants and quasi-analyticity.

## REFERENCES

1. T. Carleman, *Quasi-Analytic Functions*, translated by J. L. Gammel, LA-4702-TR, available from the translator upon request.
2. E. Borel, *Leçons sur les fonctions monogènes d'une variable complexe*, Gauthier-Villars, Paris, 1917.

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TABLE 1. Padé approximants for example of (6)

$z = 0.8$	$z = 2.0$
exact $f(z) = 0.164363$	exact $f(z) = 0.111937$
[1/2] = 0.163618	[1/2] = 0.135211
[2/3] = 0.170791	[2/3] = 0.151985
[3/4] = 0.165940	[3/4] = 0.125214
[4/5] = 0.162826	[4/5] = 0.206372
[5/6] = 0.164369	[5/6] = 0.103400
[6/7] = 0.164292	[6/7] = 0.093791
[7/8] = 0.164356	[7/8] = 0.108262
[8/9] = 0.164353	[8/9] = 0.107434
[9/10] = 0.164379	[9/10] = 0.101249
[10/11] = 0.164339	[10/11] = 0.104148
[11/12] = 0.164363	[11/12] = 0.114326
[12/13] = 0.164363	[12/13] = 0.304062
[13/14] = 0.164363	[13/14] = 0.114013
[14/15] = 0.164363	[14/15] = 0.112906
[15/16] = 0.164363	[15/16] = 0.113062
[16/17] = 0.164363	[16/17] = 0.113467
[17/18] = 0.164363	[17/18] = 0.112369
— —	— —
— —	— —
— —	— —
[34/35] = 0.164362	[34/35] = 0.112039

Table 2. Padé approximants to example of (7)

$z = 0.8$		$z = 2.0$	
exact $f(z)$	= 1.76988	exact $f(z)$	= + 3.07561
[1/2]	= 3.05682	[1/2]	= -0.465555
[2/3]	= 1.84244	[2/3]	= -0.832242
[3/4]	= 1.90467	[3/4]	= -0.782236
[4/5]	= 1.77184	[4/5]	= -1.21615
[5/5]	= 1.77219	[5/6]	= -1.20624
[6/7]	= 1.76994	[6/7]	= -1.36719
[7/8]	= 1.77041	[7/8]	= -1.22948
[8/9]	= 1.77020	[8/9]	= -1.32492
[9/10]	= 1.77020	[9/10]	= -1.32502
[10/11]	= 1.77020	[10/11]	= -1.32351
[11/12]	= 1.77022	[11/12]	= -1.31725
[12/13]	= 1.77047	[12/13]	= -1.24809
[13/14]	= 1.77043	[13/14]	= -1.25196
[14/15]	= 1.77011	[14/15]	= -1.35062
[15/16]	= 1.76976	[15/16]	= -1.21332
[16/17]	= 1.76986	[16/17]	= -1.14878
[17/18]	= 1.76983	[17/18]	= -1.16796
[18/19]	= 1.76988	[18/19]	= -1.06946
[19/20]	= 1.76988	[19/20]	= -1.10654
[20/21]	= 1.76988	[20/21]	= -1.10347
[21/22]	= 1.76988	[21/22]	= -1.13153
[22/23]	= 1.76988	[22/23]	= -1.17954
[23/24]	= 1.76988	[23/24]	= -1.17904
[24/25]	= 1.76988	[24/25]	= -1.08602
[25/26]	= 1.76988	[25/26]	= -1.10254
[26/27]	= 1.76988	[26/27]	= -1.07632
[27/28]	= 1.76988	[27/28]	= -1.29311