

ON THE WEAK CONVERGENCE OF SIMILAR PROBABILITY LAWS

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The concept of types of random variables can be extended to random vectors in several ways. It is shown in this paper that with some natural extensions of this concept, convergence of laws is in reality convergence of types as in the case of random variables.

DEFINITION 1. We say two random variables X and Y (or their probability laws) are of the same (positive) *type* if $\exists b > 0$ and a such that $\mathcal{L}(Y) = \mathcal{L}(a + bX)$.

DEFINITION 2. We say a random variable X (or its law) is *degenerate* if $\exists c$ such that $P(X = c) = 1$.

DEFINITION 3. We say a distribution function $F(x)$ is *defective* if $F(+\infty) - F(-\infty) < 1$, or equivalently, if $F(+\infty) < 1$, or $F(-\infty) > 0$ or both.

In this paper we are primarily concerned with distribution functions of random variables which are necessarily nondefective.

DEFINITION 4. We say U_n *converges weakly* to U ($U_n \xrightarrow{w} U$) or $\mathcal{L}(U_n) \rightarrow \mathcal{L}(U)$ if for every continuity point z of $F(u) = P(U \leq u)$, $P(U_n \leq z) = F_n(z) \rightarrow F(z)$.

This definition holds both for random variables and random vectors.

We restate the familiar result about the weak convergence of types of random variables as follows. If, for sufficiently large n , X_n and Y_n are of the same type and $X_n \xrightarrow{w}$ nondegenerate X and $Y_n \xrightarrow{w}$ nondegenerate Y , then X and Y are of the same type [1, Theorem 1, p. 40]. It is this result that justifies the usage that a sequence of random variables Z_n has, asymptotically, a normal distribution (without specifying the norming constants). Since all (univariate) nondegenerate normal distributions are of the same type it follows that for any sequence of constants a_n and $b_n > 0$ the limit law of $a_n + b_n Z_n$ is also normal possibly with different mean and variance if it is nondefective and nondegenerate.

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The author introduced the following concepts of similar probability laws of random vectors in [3].

DEFINITION 5. We say two random vectors $U' = (U_1, U_2, \dots, U_p)$ and $V' = (V_1, V_2, \dots, V_p)$ or their probability laws are *similar* if \exists a nonrandom vector $A' = (a_1, a_2, \dots, a_p)$ and a nonrandom non-singular $p \times p$ matrix $B = (b_{ij})$ such that $\mathcal{L}(V) = \mathcal{L}(A + BU)$. We then write $U \sim V$. They are said to be *strongly similar* if \exists a scalar $b \neq 0$ such that $\mathcal{L}(V) = \mathcal{L}(A + bU)$. We write $U \sim^s V$ if they are strongly similar.

DEFINITION 6. We say a random vector U (or its law) is *nondegenerate* if there is no hyperplane in R_p containing the total mass of the probability distribution of U ; that is, if there exists no nonrandom $\ell' = (\ell_1, \ell_2, \dots, \ell_p) \neq (0, 0, \dots, 0)$ such that $P(\ell_1 U_1 + \ell_2 U_2 + \dots + \ell_p U_p = \ell) = 1$ for some scalar ℓ .

Some authors use 'nonsingular' instead of 'nondegenerate'; the latter is used here to avoid possible confusion with nonsingularity of matrices.

It is clear that U is nondegenerate if and only if $(U_{i_1}, U_{i_2}, \dots, U_{i_t})$ is nondegenerate for all distinct i_1, i_2, \dots, i_t such that $1 \leq i_j \leq p, j = 1, 2, \dots, t$.

LEMMA 1. If $P(X_n \leq x) = F_n(x) \xrightarrow{w} \text{nondefective } F(x)$ then, (a) $|a_n| \rightarrow \infty \Rightarrow (X_n/a_n) \xrightarrow{P} 0$, (b) $a_n \rightarrow \pm \infty \Rightarrow X_n + a_n$ converges weakly to a defective probability distribution.

PROOF. Given $\epsilon > 0$, let $A(\epsilon)$ be a continuity point of F such that $F(A) > 1 - \epsilon/4$. Since $F_n(A) \rightarrow F(A) \exists L(\epsilon)$ such that $n > L(\epsilon) \Rightarrow |F_n(A) - F(A)| < \epsilon/2$. Now for $n > L(\epsilon)$ and $x > A(\epsilon)$

$$\begin{aligned} |F_n(x) - F(x)| &= |\{1 - F_n(x)\} - \{1 - F(x)\}| \leq 1 - F_n(x) + 1 - F(x) \\ &\leq 1 - F_n(A) + 1 - F(A) \leq 1 - F(A) + \frac{\epsilon}{2} + 1 - F(A) \\ &\leq \frac{\epsilon}{2} + 2 \cdot \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

Similarly $\exists M(\epsilon)$ and $B(\epsilon)$ such that $F(B(\epsilon)) < \epsilon/4$, $x < B(\epsilon)$, and $n > M(\epsilon) \Rightarrow |F_n(x) - F(x)| < \epsilon$. If we let $N(\epsilon) = \max(L(\epsilon), M(\epsilon))$, then $n > N(\epsilon)$, $x < B(\epsilon)$ or $x > A(\epsilon) \Rightarrow |F_n(x) - F(x)| < \epsilon$. Now given $\delta > 0$, $\exists P(\delta, \epsilon)$ such that $n > P(\delta, \epsilon) \Rightarrow |a_n|\delta > A(\epsilon)$ and $-|a_n|\delta < B(\epsilon)$. For such n we have

$$P\left(\left|\frac{X_n}{a_n}\right| < \delta\right) = F_n(|a_n|\delta) - F_n(-|a_n|\delta)$$

$$\begin{aligned}
&\geq F(|a_n|\delta) - \epsilon - \{F(-|a_n|\delta) + \epsilon\} \\
&= F(|a_n|\delta) - F(-|a_n|\delta) - 2\epsilon \\
&\geq 1 - \frac{\epsilon}{4} - \frac{\epsilon}{4} - 2\epsilon = 1 - \frac{5\epsilon}{2}.
\end{aligned}$$

Since ϵ is arbitrary it follows that $P(|X_n/a_n| < \delta) \rightarrow 1$ as $n \rightarrow \infty$ for all $\delta > 0$. QED

Part (b) can be proved similarly.

LEMMA 2. Suppose \exists sequences of scalars $f_n, g_{n1}, \dots, g_{np}$ and sequences of random variables $U_{n1}, U_{n2}, \dots, U_{np}$ such that $(U_{n1}, U_{n2}, \dots, U_{np}) \xrightarrow{w}$ nondegenerate (U_1, U_2, \dots, U_p) and $f_n + \sum_{i=1}^p g_{ni} U_{ni} \xrightarrow{w}$ nondegenerate W_p . Then $\limsup |f_n| < \infty$, $\limsup |g_{ni}| < \infty$, $i = 1, 2, \dots, p$.

PROOF. Note that if $\limsup |g_{ni}| < \infty$, $i = 1, 2, \dots, p$, then $\limsup |f_n| < \infty$, for otherwise, \exists a subsequence $\{n_k\}$ such that $f_{n_k} \rightarrow \pm \infty$ and $g_{n_k i} \rightarrow g_i$. It follows from a theorem of Prokhorov [4, Theorem 1.10, p. 166] that $\mathcal{L}(\sum_{i=1}^p g_{n_k i} U_{n_k i}) \rightarrow \mathcal{L}(\sum_{i=1}^p g_i U_i)$ and from lemma 1(b) that $\mathcal{L}(f_{n_k} + \sum_{i=1}^p g_{n_k i} U_{n_k i}) \rightarrow$ a defective probability distribution which cannot be $\mathcal{L}(W_p)$. We shall show that $\limsup |g_{ni}| < \infty$, $i = 1, 2, \dots, p$ by induction on p . Suppose first that $p = 1$ and $\limsup |g_{n1}| = \infty$. Then \exists a subsequence $\{n_k\}$ such that $|g_{n_k 1}| \rightarrow \infty$, and from lemma 1(a) $f_{n_k}/g_{n_k 1} + U_{n_k 1} \xrightarrow{P} 0$. This contradicts $\mathcal{L}(U_{n_k 1}) \rightarrow \mathcal{L}(U_1)$ from lemma 1(b) if $f_{n_k}/g_{n_k 1}$ is unbounded. If $f_{n_k}/g_{n_k 1}$ is bounded, then \exists a subsequence $n_{k'}$ such that $f_{n_{k'}}/g_{n_{k'} 1}$ converges to some real c , and then, $\mathcal{L}(f_{n_{k'}}/g_{n_{k'} 1} + U_{n_{k'} 1}) \xrightarrow{w} \mathcal{L}(U_1 + c)$ nondegenerate, which contradicts $f_{n_{k'}}/g_{n_{k'} 1} + U_{n_{k'} 1} \xrightarrow{P} 0$. Now suppose the result is true for p , and $\mathcal{L}(f_n + \sum_{i=1}^{p+1} g_{ni} U_{ni}) \rightarrow$ nondegenerate $\mathcal{L}(W_{p+1})$. Suppose $\limsup |g_{ni_0}| = \infty$. Then \exists a subsequence $\{n_k\}$ such that

$$\frac{f_{n_k}}{g_{n_k i_0}} + \sum_{i \neq i_0} \frac{g_{n_k i}}{g_{n_k i_0}} U_{n_k i} + U_{n_k i_0} \xrightarrow{P} 0,$$

and

$$\mathcal{L}\left(\frac{f_{n_k}}{g_{n_k i_0}} + \sum_{i \neq i_0} \frac{g_{n_k i}}{g_{n_k i_0}} U_{n_k i}\right) \rightarrow \text{nondegenerate } \mathcal{L}(-U_{i_0}).$$

It follows from the induction hypothesis that $\exists \{n_\ell\} \subset \{n_k\}$ such that

$$\mathcal{L}\left(\frac{f_{n_\ell}}{g_{n_\ell i_0}} + \sum_{i \neq i_0} \frac{g_{n_\ell i}}{g_{n_\ell i_0}} U_{n_\ell i} + U_{n_\ell i_0}\right) \rightarrow \mathcal{L}\left(\alpha + \sum_{i \neq i_0} \beta_i U_i + U_{i_0}\right)$$

with α and β_i finite. Hence $P(\sum_{i \neq i_0} \beta_i U_i + U_{i_0} + \alpha = 0) = 1$, which

contradicts $(U_1, U_2, \dots, U_{p+1})$ is nondegenerate. Thus $\limsup |g_{ni}| < \infty$, $i = 1, 2, \dots, (p+1)$, and the proof is complete.

THEOREM. Let $U_n' = (U_{n1}, U_{n2}, \dots, U_{np})$, $V_n' = (V_{n1}, V_{n2}, \dots, V_{np})$, $U' = (U_1, U_2, \dots, U_p)$, and $V' = (V_1, V_2, \dots, V_p)$. If $\mathcal{L}(U_n) \rightarrow$ nondegenerate $\mathcal{L}(U)$ and $\mathcal{L}(V_n) \rightarrow$ nondegenerate $\mathcal{L}(V)$ and $U_n \sim V_n$ for sufficiently large n , then $U \sim V$.

PROOF. Let $\mathcal{L}(V_n) = \mathcal{L}(A_n + B_n U_n)$ for $n \geq N$ with $A_n' = (a_{n1}, a_{n2}, \dots, a_{np})$ and $B_n = (b_{nij})$. It follows from the Mann-Wald theorem [2, Theorem 5, p. 223] that $\mathcal{L}(U_{n1}) = \mathcal{L}(a_{n1} + \sum_{j=1}^p b_{n1j} U_{nj}) \rightarrow$ nondegenerate $\mathcal{L}(U_1)$ and from lemma 2 that $\limsup |a_{n1}| < \infty$, $\limsup |b_{n1j}| < \infty$, $j = 1, 2, \dots, p$. Similarly $\limsup |a_{ni}| < \infty$, $\limsup |b_{nij}| < \infty$, $i = 1, 2, \dots, p$; $j = 1, 2, \dots, p$. Hence $\exists \{n_k\}$ such that $A_{n_k} \rightarrow A$ and $B_{n_k} \rightarrow B$ with elements of A and B finite. Since $\mathcal{L}(V_{n_k}) = \mathcal{L}(A_{n_k} + B_{n_k} U_{n_k})$ for sufficiently large values of n_k it follows from the theorem of Prokhorov referred to earlier that $\mathcal{L}(V_{n_k}) \rightarrow \mathcal{L}(A + BU)$. Since $\mathcal{L}(V_{n_k})$ also converges to $\mathcal{L}(V)$ it follows that $\mathcal{L}(V) = \mathcal{L}(A + BU)$. Since V and U are both nondegenerate, B is nonsingular, and the proof is complete.

The following analogous result for strong similarity is proved in [5]. If $\mathcal{L}(U_n) \rightarrow$ nondegenerate $\mathcal{L}(U)$, $\mathcal{L}(V_n) \rightarrow$ nondegenerate $\mathcal{L}(V)$, and $U_n \stackrel{s}{\sim} V_n$ for sufficiently large n , then $U \stackrel{s}{\sim} V$. One can also define an intermediate type of similarity with diagonal matrices B , and the theorem remains true for this type of similarity also.

It is easy to see that any two (nondegenerate) multivariate normal distributions are similar, and it follows from the above theorem that if $\mathcal{L}(U_n) \rightarrow \mathcal{N}(\mu, \Sigma)$, then $\mathcal{L}(A_n + B_n U_n) \rightarrow \mathcal{N}(\nu, \psi)$ if B_n are nonsingular (for sufficiently large values of n) and the limit law of $A_n + B_n U_n$ exists and is nondegenerate. This justifies the usage that a sequence of random vectors U_n is asymptotically multivariate normal without specifying the "norming constants" A_n and B_n .

REFERENCES

1. B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*. Addison-Wesley Publishing Company, Inc., 1954.
2. H. B. Mann and A. Wald, *On stochastic limit and order relations*. Ann. Math. Statist. **14** (1943), 217-226.
3. R. Ponnappalli, *Types of laws of random vectors and best estimators*. Journal of Multivariate Analysis **5** (1975), 278-281.
4. Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability theory*. Theor. Prob. Appl. **1** (1956), 157-214.
5. K. Takano, *On the many-dimensional distribution functions*. Ann. Inst. Stat. Math., Tokyo **5** (1954), 40-58.