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ON GENERATING SUBGROUPS OF THE AFFINE GROUP ON THE PLANE BY PAIRS OF INFINITESIMAL TRANSFORMATIONS*

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ABSTRACT. Let G be a Lie group with Lie algebra g and let X and Y be elements of g. If every element of G can be written as a product of elements taken alternately from $\exp(tX)$ and $\exp(tY)$, X and Y are said to generate G. A classification will be obtained here of all Lie subgroups of the affine group acting on the plane; for each such group, necessary and sufficient conditions will be found that a pair of elements in the Lie algebra generate the group. All but three subgroups of the affine group can be so generated.

I. **Introduction.** The real affine group $A(2)$ acting on the plane is the set of all transformations from R^2 to R^2 of the form $v \rightarrow Av + \ell$, where $A \in GL(2, R)$ and $\ell \in R^2$. From now on denote such a transformation by $\langle A, \ell \rangle$. Then $\langle A, \ell \rangle \circ \langle B, m \rangle = \langle AB, \ell + Am \rangle$. The Lie algebra $a(2)$ of $A(2)$ consists of all $\langle A, \ell \rangle$ with $A \in M_2(R)$ and $\ell \in R^2$; $[\langle A, \ell \rangle, \langle B, m \rangle] = \langle AB - BA, Am - B\ell \rangle$. We shall determine all Lie subalgebras of $a(2)$ up to conjugacy, and thereby all connected Lie subgroups of $A(2)$ up to conjugacy.

A connected Lie group G is generated by a pair of one-parameter subgroups if every element of G can be written as a finite product of elements chosen alternately from the two one-parameter subgroups. This happens just in case the Lie algebra of G is generated by the corresponding pair of infinitesimal transformations, because the set of all such finite products is an arcwise connected subgroup of G and so a Lie subgroup by Yamabe's theorem [4]. It is known that all connected subgroups of the Moebius group $w = (\alpha z + \beta)/(\nu z + \zeta)$, α, β, ν, and ζ complex, can be generated by an appropriate pair of infinitesimal transformations with the exception of the group $w = \alpha z + \beta$, $\alpha > 0$ [2]. This group is also a subgroup of $A(2)$; we will show that all subgroups of $A(2)$, with the exception of this group and two others, can be generated by a suitable pair of infinitesimal transformations.

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II. Subalgebras of $M_2(R)$.

THEOREM 1. *Let g be a subalgebra of $M_2(R)$. Then g is conjugate to precisely one of the following:*

(1) $\{0\}$

(2) $R\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ where $|\lambda| \leqq 1$

(3) $R\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$ where $0 \leqq \lambda$

(4) $R\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(5) $R\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(6) $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}$

(7) $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right\}$

(8) $\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right\}$

(9) $\left\{\begin{pmatrix} \lambda a & b \\ 0 & (\lambda + 1)a \end{pmatrix}\right\}$ where $\lambda \in R$

(10) $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right\}$

(11) $s\ell(2, R)$

(12) $M_2(R)$.

PROOF. The rational canonical form theorem implies that each one-dimensional g is conjugate to an algebra listed in (2) through (5).

LEMMA. *If $g \subset M_2(R)$ is isomorphic to $R \oplus R$, g contains the identity matrix I.*

PROOF. Suppose not and let e and f generate g. Choose $h \in M_2(R)$ so $\{e, f, I, h\}$ is a basis for $M_2(R)$. Then $s\ell(2, R) = [M_2(R), M_2(R)]$ is generated by $[e, h]$ and $[f, h]$, although it is three dimensional.

Assume that g is isomorphic to $R \oplus R$ and choose e so e and I generate g; after suitable conjugation we may suppose that e is one of the matrices listed in (2) through (5); after subtracting a suitable multiple of I, we may suppose that e is one of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus g is conjugate to 6, 7, or 8; no two of these algebras are conjugate because every matrix in 6 is diagonalizable and every matrix in 7 has at least one real eigenvalue.

If g is two dimensional and non-abelian, g has a basis $\{e, f\}$ so $[e, f] = e$. Notice that $\operatorname{tr} e = 0$; after suitable conjugation, then, $e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $[e, f]$ is then $\begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix}$, $\begin{pmatrix} -b-c & a-d \\ a-d & b+c \end{pmatrix}$, or $\begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$; this can equal e only if $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} a & b \\ 0 & a+1 \end{pmatrix}$, so g is conjugate to an algebra listed in (9). Distinct λ's give distinct conjugacy classes of algebras, for if g is conjugate to $\left\{\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix}\right\}$, g contains an element with two distinct eigenvalues one unit apart, and these eigenvalues must be λ and $\lambda + 1$.

Suppose $\dim g = 3$ and $I \notin g$. Then $g \oplus RI = M_2(R)$ and $s\ell(2, R) = [M_2(R), M_2(R)] \subset [g, g] \subset g$, so $s\ell(2, R) = g$.

Finally suppose $\dim g = 3$ and $I \in g$. Then $s\ell(2, R) \cap g$ is a two dimensional Lie algebra and so conjugate to one of 6, 7, 8, 9; since any algebra conjugate to $s\ell(2, R) \cap g$ is contained in $s\ell(2, R)$, $s\ell(2, R) \cap g$ is conjugate to $\begin{pmatrix} -a/2 & b \\ 0 & a/2 \end{pmatrix}$ and g is conjugate to 10.

THEOREM 2. *Every connected Lie subgroup of $GL(2, R)$ is conjugate to precisely one of the following:*

(1) $\{I\}$

(2) $\left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{\lambda t} \end{pmatrix}\right\}$ where $|\lambda| \leqq 1$

(3) $\left\{e^{\lambda t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\right\}$ where $0 \leqq \lambda$

(4) $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$

(5) $\left\{e^t\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$

(6) $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a > 0, b > 0\right\}$

(7) $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a > 0\right\}$

(8) $\left\{a\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \,\middle|\, a > 0\right\}$

(9) $\left\{\begin{pmatrix} a^\lambda & b \\ 0 & a^{\lambda+1} \end{pmatrix} \,\middle|\, a > 0\right\}$ where $\lambda \in R$

(10) $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a > 0, d > 0\right\}$

(11) $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1\right\}$

(12) $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc > 0\right\}$.

PROOF. An immediate consequence of Theorem 1.

III. **Subalgebras of $a(2)$.** Let g be a subalgebra of $a(2)$; define $g_0 = \{A \in M_2(R) \mid \text{there exists } \ell \in R^2 \text{ such that } \langle A, \ell\rangle \in g\}$ and $V = \{\ell \in R^2 \mid \langle 0, \ell\rangle \in g\}$. Then g_0 is a subalgebra of $M_2(R)$ and V is a subspace of R^2; $g_0(V) \subset V$ because $[\langle A, \ell\rangle, \langle 0, m\rangle] = \langle 0, Am\rangle$. The following sequence is exact:

$$0 \to V \to g \to g_0 \to 0$$

Notice that conjugation of $\langle A, \ell\rangle \in a(2)$ by $\langle B, m\rangle \in A(2)$ yields $\langle BAB^{-1}, B\ell - BAB^{-1}m\rangle$. In particular g_0 becomes Bg_0B^{-1} and V becomes BV.

THEOREM 3. *The pair* $\{\mathfrak{g}_0, V\}$ *is conjugate to precisely one of the following:*

(a) $\{\mathfrak{g}_0, \{0\}\}$ *where* $\mathfrak{g}_0$ *is one of the algebras listed in Theorem* 1

(b) $\{\mathfrak{g}_0, R^2\}$ *where* $\mathfrak{g}_0$ *is one of the algebras listed in Theorem* 1

(c) $\{\mathfrak{g}_0, R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$ *where* $\mathfrak{g}_0$ *is one of* $\{0\}$, $R\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $R\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $R\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $R\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\{\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix}\lambda a&b\\0&(\lambda+1)a\end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix}a&b\\0&a\end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\}$

(d) $\{R\left(\begin{smallmatrix}1&0\\0&\lambda\end{smallmatrix}\right), R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$ or $\{R\left(\begin{smallmatrix}1&0\\0&\lambda\end{smallmatrix}\right), R\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ for $|\lambda| < 1$.

PROOF. We can assume that $\mathfrak{g}_0$ is one of the algebras listed in Theorem 1. The condition $\mathfrak{g}_0(V) \subset V$ puts no restriction on V if $\dim V = 0$ or 2; when $\dim V = 1$, V can be arbitrary if $g = \{0\}$ or $R\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $V = R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ or $R\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ if $g = R\left(\begin{smallmatrix}1&0\\0&\lambda\end{smallmatrix}\right)$ $-1 \leqq \lambda < 1$ or $\{\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)\}$, $V = R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ if $g = R\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $R\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\{\left(\begin{smallmatrix}\lambda a&b\\0&(\lambda+1)b\end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix}a&b\\0&a\end{smallmatrix}\right)\}$, or $\{\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\}$, and otherwise no one-dimensional V will work. When $\mathfrak{g}_0 = \{0\}$ or $R\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $B\mathfrak{g}_0B^{-1} = \mathfrak{g}_0$ for all B; applying an appropriate B to V we can assume $V = R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. If $\mathfrak{g}_0 = \{\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)\}$ or $R\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathfrak{g}_0\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)^{-1} = \mathfrak{g}_0$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = R\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$; thus V can be taken to be $R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. If $\mathfrak{g}_0 = R\left(\begin{smallmatrix}1&0\\0&\lambda\end{smallmatrix}\right)$ for $|\lambda| < 1$ and $B\mathfrak{g}_0B^{-1} = \mathfrak{g}_0$, $B = \left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)$ so $R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $R\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ are not conjugate.

Choose a subspace V_1 of R^2 so $V \oplus V_1 = R^2$. Whenever $A \in \mathfrak{g}_0$, there is a unique $\varphi(A) \in V_1$ so $\langle A, \varphi(A)\rangle \in g$. Clearly $g = \{\langle A, \varphi(A) + \ell\rangle \mid A \in \mathfrak{g}_0, \ell \in V\}$.

Notice that $\varphi : \mathfrak{g}_0 \to V_1$ is linear. Let $P : R^2 \to V_1$ be the obvious projection map. Then $\varphi([A, B]) = P\{A\varphi(B) - B\varphi(A)\}$ since $[\langle A, \varphi(A)\rangle, \langle B, \varphi(B)\rangle] = \langle [A, B], A\varphi(B) - B\varphi(A)\rangle$.

Conjugation of g by $\langle I, m\rangle$ leaves $\mathfrak{g}_0$ and V fixed and converts $\langle A, \varphi(A)\rangle$ to $\langle A, \varphi(A) - Am\rangle$. Hence we may replace $\varphi(A)$ by $P\{\varphi(A) - Am\}$ and obtain a conjugate algebra.

If $\mathfrak{g}_0 = \{0\}$, $\varphi = 0$. If $\mathfrak{g}_0 = Re$ where $e = \left(\begin{smallmatrix}1&0\\0&\lambda\end{smallmatrix}\right)$ $0 < |\lambda| \leqq 1$, $\left(\begin{smallmatrix}\lambda&-1\\1&\lambda\end{smallmatrix}\right)$ $0 \leqq \lambda$ or $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, m can be found so $\varphi(e) = em$ since $\det e \neq 0$. Thus after conjugation $\varphi = 0$. If $I \in \mathfrak{g}_0$ we can suppose $\varphi(I) = 0$ since I is nonsingular; whenever $A \in \mathfrak{g}_0$, $0 = \varphi([I, A]) = P\{\varphi(A)\} = \varphi(A)$, so $\varphi = 0$. If $\mathfrak{g}_0 = \mathfrak{sl}(2, R)$, $\varphi = 0$ unless $V = \{0\}$ and $V_1 = R^2$. In this case let $e = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $g = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$; as usual we may assume $\varphi(e) = 0$. Then $2\varphi(f) = \varphi[e, f] = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\varphi(f)$, so $\varphi(f) = 0$. Similarly $-2\varphi(g) = \varphi[e, g] = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\varphi(g)$, so $\varphi(g) = 0$ and $\varphi = 0$. If $\mathfrak{g}_0 = \{\left(\begin{smallmatrix}\lambda a&b\\0&(\lambda+1)a\end{smallmatrix}\right)\}$, let $e = \left(\begin{smallmatrix}\lambda&0\\0&\lambda+1\end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. Suppose $\lambda \neq 0, -1$, or -2. Then $\det\left(\begin{smallmatrix}\lambda&0\\0&\lambda+1\end{smallmatrix}\right) \neq 0$ and we can suppose $\varphi(e) = 0$; $-\varphi(f) = \varphi[e, f] = P\{\left(\begin{smallmatrix}\lambda&0\\0&\lambda+1\end{smallmatrix}\right)\varphi(f)\}$. If $V = \{0\}$ and $\varphi(f) = \left(\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}(1+\lambda)v_1\\(2+\lambda)v_2\end{smallmatrix}\right) = 0$ so $v_1 = v_2 = 0$ and $\varphi = 0$. If $V = R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$

let $V_1 = R\binom{0}{1}$ and $\varphi(f) = \binom{0}{v}$; then $-\binom{0}{v} = \binom{0}{(\lambda+1)v}$ and $v = 0$, so $\varphi = 0$. In short, φ can be taken to be zero unless $g_0 = R\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $R\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\{\begin{pmatrix}0&b\\0&a\end{pmatrix}\}$, $\{\begin{pmatrix}a&b\\0&0\end{pmatrix}\}$, or $\{\begin{pmatrix}2a&b\\0&a\end{pmatrix}\}$. The following five lemmas complete the classification.

Lemma 1. *Let* $g_0 = R\begin{pmatrix}1&0\\0&0\end{pmatrix}$. *If* $V = R\binom{0}{1}$ *or* R^2, φ *can be taken to be zero. If* $V = \{0\}$ *or* $R\binom{1}{0}$, $\varphi\begin{pmatrix}1&0\\0&0\end{pmatrix}$ *can be taken to be* 0 *or* $\binom{0}{1}$.

Proof. If $Bg_0B^{-1} = g_0$, $B = \begin{pmatrix}a&0\\0&d\end{pmatrix}$; such a B preserves all possible V. Conjugation of $\langle \begin{pmatrix}1&0\\0&0\end{pmatrix}, \varphi\begin{pmatrix}1&0\\0&0\end{pmatrix}\rangle$ by $\langle \begin{pmatrix}a&0\\0&d\end{pmatrix}, m\rangle$ yields $\langle \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}a&0\\0&d\end{pmatrix}\varphi\begin{pmatrix}1&0\\0&0\end{pmatrix} - \begin{pmatrix}1&0\\0&0\end{pmatrix}m\rangle$. Clearly a, d, and m can be chosen to make the second part of this expression equal 0 or $\binom{0}{1}$. We are interested in the projection of this term on V_1; since $V_1 = R^2$, $R\binom{0}{1}$, $R\binom{1}{0}$, and $\{0\}$ when $V = \{0\}$, $R\binom{1}{0}$, $R\binom{0}{1}$, and R^2, the lemma follows.

Lemma 2. *Let* $g_0 = R\begin{pmatrix}0&1\\0&0\end{pmatrix}$. *If* $V = R^2$, $\varphi = 0$. *If* $V = \{0\}$ *or* $R\binom{1}{0}$, $\varphi\begin{pmatrix}0&1\\0&0\end{pmatrix}$ *can be taken to be* 0 *or* $\binom{0}{1}$.

Proof. If $Bg_0B^{-1} = g_0$, $B = \begin{pmatrix}a&b\\0&d\end{pmatrix}$; such a B preserves all possible V. Conjugation of $\langle \begin{pmatrix}0&1\\0&0\end{pmatrix}, \varphi\begin{pmatrix}0&1\\0&0\end{pmatrix}\rangle$ by $\langle \begin{pmatrix}a&b\\0&d\end{pmatrix}, m\rangle$ yields $\langle \begin{pmatrix}0&a/d\\0&0\end{pmatrix}, \begin{pmatrix}a&b\\0&d\end{pmatrix}\varphi\begin{pmatrix}0&1\\0&0\end{pmatrix} - \begin{pmatrix}0&a/d\\0&0\end{pmatrix}m\rangle$. Clearly $a = d$ and m can be chosen to make the second part of this expression equal 0 or $\binom{0}{1}$.

Lemma 3. *Let* $g_0 = \{\begin{pmatrix}0&b\\0&a\end{pmatrix}\}$. *If* $V = R\binom{1}{0}$ *or* R^2, φ *can be taken to be zero. If* $V = \{0\}$, $\varphi\begin{pmatrix}0&b\\0&a\end{pmatrix}$ *can be taken to be* 0 *or* $\binom{a}{0}$.

Proof. Let $e = \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f = \begin{pmatrix}0&0\\0&1\end{pmatrix}$, $\varphi(e) = \binom{v_1}{v_2}$, $\varphi(f) = \binom{w_1}{w_2}$. If $V = \{0\}$, $\binom{v_1}{v_2} = \varphi(e) = \varphi[e,f] = \begin{pmatrix}0&1\\0&0\end{pmatrix}\binom{w_1}{w_2} - \begin{pmatrix}0&0\\0&1\end{pmatrix}\binom{v_1}{v_2} = \binom{w_2}{-v_2}$, so $v_1 = w_2$ and $v_2 = 0$. If $V = R\binom{1}{0}$, let $V_1 = R\binom{0}{1}$; then $v_1 = w_1 = 0$ and $\binom{0}{-v_2} = \varphi(e) = \varphi[e,f] = P\{\binom{w_2}{-v_2}\} = \binom{0}{-v_2}$, so $v_2 = 0$.

If $Bg_0B^{-1} = g_0$, $B = \begin{pmatrix}a&b\\0&d\end{pmatrix}$; such a B preserves all possible V. Conjugation of $\langle e, \varphi(e)\rangle$ and $\langle f, \varphi(f)\rangle$ by $\langle \begin{pmatrix}a&b\\0&d\end{pmatrix}, m\rangle$ yields $\langle \begin{pmatrix}0&a/d\\0&0\end{pmatrix}, \begin{pmatrix}a&b\\0&d\end{pmatrix}\varphi(e) - \begin{pmatrix}0&a/d\\0&0\end{pmatrix}m\rangle$ and $\langle \begin{pmatrix}0&b/d\\0&0\end{pmatrix}, \begin{pmatrix}a&b\\0&d\end{pmatrix}\varphi(f) - \begin{pmatrix}0&b/d\\0&1\end{pmatrix}m\rangle$. If $V = R\binom{1}{0}$, we are only interested in the projection of the second parts of these expressions on $R\binom{0}{1}$; since $\varphi(e) = 0$, the projection of $\begin{pmatrix}a&b\\0&d\end{pmatrix}\varphi(e) - \begin{pmatrix}0&a/d\\0&0\end{pmatrix}m$ is automatically zero; clearly $a = d$, $b = 0$, and m can be chosen to make the projection of $\begin{pmatrix}a&b\\0&d\end{pmatrix}\varphi(f) - \begin{pmatrix}0&b/d\\0&1\end{pmatrix}m$ vanish. If $V = \{0\}$, let $\tilde{\varphi}$ be the conjugate of φ;

$$\tilde{\varphi}(e) = \frac{d}{a}\tilde{\varphi}\begin{pmatrix}0&a/d\\0&0\end{pmatrix} = \frac{d}{a}\left\{\begin{pmatrix}ab\\0d\end{pmatrix}\begin{pmatrix}v_1\\0\end{pmatrix} - \begin{pmatrix}0&a/d\\0&0\end{pmatrix}\begin{pmatrix}m_1\\m_2\end{pmatrix}\right\} = \begin{pmatrix}dv_1 - m_2\\0\end{pmatrix},$$

$$\tilde{\varphi}(f) = \tilde{\varphi}\begin{pmatrix} 0 & b/d \\ 0 & 1 \end{pmatrix} - \frac{b}{d}\tilde{\varphi}(e) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} w_1 \\ v_1 \end{pmatrix}$$
$$-\begin{pmatrix} 0 & b/d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} - \begin{pmatrix} bv_1 - b/d\, m_2 \\ 0 \end{pmatrix} = \begin{pmatrix} aw_1 \\ dv_1 - m_2 \end{pmatrix}.$$

Clearly a, d, and m_2 can be chosen to make $\tilde{\varphi}(e) = 0$ and $\tilde{\varphi}(f)$ either 0 or $\binom{1}{0}$.

Lemma 4. *Let* $g_0 = \{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)\}$. *If* $V = \{0\}$ *or* R^2, φ *can be taken to be zero. If* $V = R\binom{1}{0}$, $\varphi\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ *can be taken to be* 0 *or* $\binom{0}{a}$.

Proof. Let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\varphi(e) = \binom{v_1}{v_2}$, $\varphi(f) = \binom{w_1}{w_2}$. If $V = \{0\}$, $-\binom{v_1}{v_2} = -\varphi(e) = \varphi[e,f] = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \binom{w_1}{w_2} - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \binom{v_1}{v_2} = \binom{w_2 - v_1}{0}$, so $w_2 = v_2 = 0$. Conjugation by $\langle I, \binom{w_1}{v_1}\rangle$ converts $\langle e, \varphi(e)\rangle$ and $\langle f, \varphi(f)\rangle$ to $\langle e, 0\rangle$ and $\langle f, 0\rangle$, so φ can be taken to be zero. If $V = R\binom{1}{0}$, let $V_1 = R\binom{0}{1}$; then $v_1 = w_1 = 0$ and $-\binom{0}{v_2} = \varphi[e,f] = P\{\binom{w_2}{0}\} = 0$, so $v_2 = 0$. If $Bg_0B^{-1} = g_0$, $B = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Conjugation of $\langle e, 0\rangle$ and $\langle f, \varphi(f)\rangle$ by $\langle \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \binom{m_1}{m_2}\rangle$ yields $\langle \left(\begin{smallmatrix} 0 & a/d \\ 0 & 0 \end{smallmatrix}\right), -\left(\begin{smallmatrix} 0 & a/d \\ 0 & 0 \end{smallmatrix}\right)\binom{m_1}{m_2}\rangle$ and $\langle \left(\begin{smallmatrix} 1 & -b/a \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \binom{0}{w_2} - \left(\begin{smallmatrix} 1 & -bd \\ 0 & 0 \end{smallmatrix}\right) \binom{m_1}{m_2}\rangle$; we are only interested in the projection of the second parts of these expressions on $R\binom{0}{1}$, so $\tilde{\varphi}(e) = 0$, $\tilde{\varphi}(f) = \binom{0}{dw_2}$. Clearly d can be chosen so $\tilde{\varphi}(f) = 0$ or $\binom{0}{1}$.

Lemma 5. *Let* $g_0 = \{\left(\begin{smallmatrix} 2a & b \\ 0 & a \end{smallmatrix}\right)\}$. *If* $V = R^2$, φ *is zero. If* $V = \{0\}$ *or* $R\binom{1}{0}$, $\varphi\left(\begin{smallmatrix} 2a & b \\ 0 & a \end{smallmatrix}\right)$ *can be taken to be* 0 *or* $\binom{0}{b}$.

Proof. Let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since f is non-singular, $\varphi(f)$ can be taken to be zero; let $\varphi(e) = \binom{v_1}{v_2}$. If $V = \{0\}$, $-\binom{v_1}{v_2} = -\varphi(e) = \varphi[e,f] = -\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \binom{v_1}{v_2} = \binom{-2v_1}{-v_2}$, so $v_1 = 0$. If $V = R\binom{1}{0}$, let $V_1 = R\binom{0}{1}$; then $v_1 = 0$.

If $Bg_0B^{-1} = g_0$, $B = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Conjugation of $\langle e, \varphi(e)\rangle$ and $\langle f, 0\rangle$ by $\langle \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \binom{m_1}{m_2} \rangle$ yields respectively

$$\left\langle \begin{pmatrix} 0 & a/d \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & a/d \\ 0 & 0 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right\rangle,$$
$$\left\langle \begin{pmatrix} 2 & -b/d \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 2 & -b/d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right\rangle.$$

If $V = \{0\}$,

$$\tilde{\varphi}(e) = \begin{pmatrix} \dfrac{bd}{a} v_2 - m_2 \\ \dfrac{d^2}{a} v_2 \end{pmatrix}, \quad \tilde{\varphi}(f) = \begin{pmatrix} -2m_1 + \dfrac{b^2}{a} v_2 \\ \dfrac{bd}{a} v_2 - m_2 \end{pmatrix}.$$

Clearly a, b, d, m_1, and m_2 can be chosen so $\tilde{\varphi}(e) = 0$ or $\binom{0}{1}$ and $\tilde{\varphi}(f) = 0$. If $V = R\binom{1}{0}$, we are only interested in the projections of the above expressions on $R\binom{0}{1}$, so $\tilde{\varphi}(e) = \binom{0}{(d^2/a)v_2}$ and $\tilde{\varphi}(f) = \binom{0}{(bd/a)v_2 - m_2}$; clearly a, b, d, and m_2 can be chosen so $\tilde{\varphi}(e) = 0$ or $\binom{0}{1}$ and $\tilde{\varphi}(f) = 0$.

Combining the above results, we obtain:

THEOREM 4. *Let* g *be a subalgebra of* $a(2)$. *Then* g *is conjugate to precisely one of the following:*

A. $\{\langle A, 0\rangle \mid A \in g_0\}$ *or* $\{\langle A, \ell\rangle \mid A \in g_0, \ell \in R^2\}$ *where* g_0 *is one of the following*

1. $\{0\}$
2. $R\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}$ where $|\lambda| \leqq 1$
3. $R\begin{pmatrix}\lambda & -1\\ 1 & \lambda\end{pmatrix}$ where $0 \leqq \lambda$
4. $R\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$
5. $R\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$
6. $\left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\right\}$
7. $\left\{\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}\right\}$
8. $\left\{\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\right\}$
9. $\left\{\begin{pmatrix}\lambda a & b\\ 0 & (\lambda+1)a\end{pmatrix}\right\}$ where $\lambda \in R$
10. $\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\right\}$
11. $\mathfrak{sl}(2, R)$
12. $M_2(R)$.

B. $\{\langle A, \ell\rangle \mid A \in g_0, \ell \in V\}$ *where* $\{g_0, V\}$ *is one of*

13. $0, R\begin{pmatrix}1\\ 0\end{pmatrix}$
14. $R\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, R\begin{pmatrix}1\\ 0\end{pmatrix}$ where $|\lambda| \leqq 1$
15. $R\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, R\begin{pmatrix}0\\ 1\end{pmatrix}$ where $|\lambda| < 1$
16. $R\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, R\begin{pmatrix}1\\ 0\end{pmatrix}$
17. $R\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, R\begin{pmatrix}1\\ 0\end{pmatrix}$
18. $\left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\right\}, R\begin{pmatrix}1\\ 0\end{pmatrix}$
19. $\left\{\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}\right\}, R\begin{pmatrix}1\\ 0\end{pmatrix}$
20. $\left\{\begin{pmatrix}\lambda a & b\\ 0 & (\lambda+1)a\end{pmatrix}\right\}, R\begin{pmatrix}1\\ 0\end{pmatrix}$ where $\lambda \in R$
21. $\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\right\}, R\begin{pmatrix}1\\ 0\end{pmatrix}$

C.

22. $\left\{\left\langle \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ r \end{pmatrix} \right\rangle \middle| r \in R \right\}$

23. $\left\{\left\langle \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ r \end{pmatrix} \right\rangle \middle| r \in R \right\}$

24. $\left\{\left\langle \begin{pmatrix} 0 & r \\ 0 & s \end{pmatrix}, \begin{pmatrix} s \\ 0 \end{pmatrix} \right\rangle \middle| r, s \in R \right\}$

25. $\left\{\left\langle \begin{pmatrix} 2r & s \\ 0 & r \end{pmatrix}, \begin{pmatrix} 0 \\ s \end{pmatrix} \right\rangle \middle| r, s \in R \right\}$

26. $\left\{\left\langle \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} s \\ r \end{pmatrix} \right\rangle \middle| r, s \in R \right\}$

27. $\left\{\left\langle \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} s \\ r \end{pmatrix} \right\rangle \middle| r, s \in R \right\}$

28. $\left\{\left\langle \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} t \\ r \end{pmatrix} \right\rangle \middle| r, s, t \in R \right\}$

29. $\left\{\left\langle \begin{pmatrix} 2r & s \\ 0 & r \end{pmatrix}, \begin{pmatrix} t \\ s \end{pmatrix} \right\rangle \middle| r, s, t \in R \right\}$

THEOREM 5. *Let G be a connected Lie subgroup of $A(2)$. Then G is conjugate to precisely one of the following:*

A. $\{\langle A, 0\rangle \mid A \in G_0\}$ *or* $\{\langle A, \ell\rangle \mid A \in G_0, \ell \in R^2\}$ *where G_0 is one of the following,*

1. $\{I\}$

2. $\left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{\lambda t} \end{pmatrix}\right\}$ where $|\lambda| \leqq 1$

3. $\left\{e^{\lambda t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\right\}$ where $\lambda \geqq 0$

4. $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$

5. $\left\{e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$

6. $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a > 0, b > 0\right\}$

7. $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a > 0\right\}$

8. $\left\{a \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \middle| a > 0\right\}$

9. $\left\{\begin{pmatrix} a^\lambda & b \\ 0 & a^{\lambda+1} \end{pmatrix} \middle| a > 0\right\}$ where $\lambda \in R$

10. $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| a > 0, d > 0\right\}$

11. $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1\right\}$

12. $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc \neq 0\right\}$

B. $\{\langle A, \ell\rangle \mid A \in G_0, \ell \in V\}$ where $\{G_0, V\}$ is one of

13. $\{I\}, R\begin{pmatrix}1\\0\end{pmatrix}$

14. $\left\{\begin{pmatrix}e^t & 0\\0 & e^{\lambda t}\end{pmatrix}\right\}, R\begin{pmatrix}1\\0\end{pmatrix}$ where $|\lambda| \leqq 1$

15. $\left\{\begin{pmatrix}e^t & 0\\0 & e^{\lambda t}\end{pmatrix}\right\}, R\begin{pmatrix}0\\1\end{pmatrix}$ where $|\lambda| < 1$

16. $\left\{\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}\right\}, R\begin{pmatrix}1\\0\end{pmatrix}$

17. $\left\{e^t\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}\right\}$ $R\begin{pmatrix}1\\0\end{pmatrix}$

18. $\left\{\begin{pmatrix}a & 0\\0 & b\end{pmatrix} \;\middle|\; a > 0, b > 0\right\}, R\begin{pmatrix}1\\0\end{pmatrix}$

19. $\left\{\begin{pmatrix}a & b\\0 & a\end{pmatrix} \;\middle|\; a > 0\right\}, R\begin{pmatrix}1\\0\end{pmatrix}$

20. $\left\{\begin{pmatrix}a^\lambda & b\\0 & a^{\lambda+1}\end{pmatrix} \;\middle|\; a > 0\right\}, R\begin{pmatrix}1\\0\end{pmatrix}$ where $\lambda \in R$

21. $\left\{\begin{pmatrix}a & b\\0 & d\end{pmatrix} \;\middle|\; a > 0, d > 0\right\}, R\begin{pmatrix}1\\0\end{pmatrix}$

C.

22. $\left\{\left\langle\begin{pmatrix}e^t & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0\\t\end{pmatrix}\right\rangle \;\middle|\; t \in R\right\}$

23. $\left\{\left\langle\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}, \begin{pmatrix}t^2/2\\t\end{pmatrix}\right\rangle \;\middle|\; t \in R\right\}$

24. $\left\{\left\langle\begin{pmatrix}1 & t\\0 & e^u\end{pmatrix}, \begin{pmatrix}u\\0\end{pmatrix}\right\rangle \;\middle|\; t, u \in R\right\}$

25. $\left\{\left\langle\begin{pmatrix}t^2 & tu\\0 & t\end{pmatrix}, \begin{pmatrix}u^2/2\\u\end{pmatrix}\right\rangle \;\middle|\; t > 0, u \in R\right\}$

26. $\left\{\left\langle\begin{pmatrix}e^t & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}u\\t\end{pmatrix}\right\rangle \;\middle|\; t, u \in R\right\}$

27. $\left\{\left\langle\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}, \begin{pmatrix}u\\t\end{pmatrix}\right\rangle \;\middle|\; t, u \in R\right\}$

28. $\left\{\left\langle\begin{pmatrix}e^t & u\\0 & 1\end{pmatrix}, \begin{pmatrix}v\\t\end{pmatrix}\right\rangle \;\middle|\; t, u, v \in R\right\}$

29. $\left\{\left\langle\begin{pmatrix}t^2 & tu\\0 & t\end{pmatrix}, \begin{pmatrix}v\\u\end{pmatrix}\right\rangle \;\middle|\; t > 0, u, v \in R\right\}$

IV. Generation of subalgebras of $M_2(R)$ by pairs of infinitesimal transformations.

THEOREM 6. *Let* $X, Y \in M_2(R)$.

a. *If X and Y have no common complex eigenvector and* tr $X \neq 0$ *or* tr $Y \neq 0$, *X and Y generate* $M_2(R)$

b. *If X and Y have no common complex eigenvector and* $\operatorname{tr} X = \operatorname{tr} Y = 0$, *$X$ and Y generate $\mathfrak{sl}(2, R)$*

c. *If X and Y have a common complex eigenvector and X, Y, $[X, Y]$ are linearly independent, X and Y generate a subalgebra conjugate to* $\{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}\}$

d. *Otherwise X and Y generate a subalgebra of dimension $\leqq 2$*

e. *All subalgebras of $M_2(R)$ can be generated by appropriate X and Y.*

Proof. If g is an algebra listed in Theorem 1 other than $\mathfrak{sl}(2, R)$ and $M_2(R)$, there is a complex vector v such that $Av = \lambda v$ for all $A \in g$. Consequently if X and Y have no common complex eigenvector, they generate a subalgebra conjugate to $\mathfrak{sl}(2, R)$ or $M_2(R)$. But both of these subalgebras are self-conjugate. The rest of the theorem follows immediately.

Remark. If g is a subalgebra of $M_2(R)$ and X and Y belong to g, the above theorem gives a satisfactory necessary and sufficient condition that X and Y generate g, since the generation problem is trivial when $\dim g \leqq 2$.

V. **Generation of subalgebras of $a(2)$ by pairs of infinitesimal transformations.** Let g be a subalgebra of $a(2)$, X and Y elements of g. We seek a simple necessary and sufficient condition that X and Y generate g. The problem is trivial when $\dim g \leqq 2$; if $\dim g = 3$, X and Y generate g just in case X, Y, and $[X, Y]$ are linearly independent. When $g \to g_0$ is an isomorphism, the problem was solved in the previous section. Referring to Theorem 4, we are left with algebras 6 through 12 when $V = R^2$ and 21.

Theorem 7. *Let $g = \{\langle A, \ell \rangle \mid A \in g_0, \ell \in R^2\}$ where $g_0 = R\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$ $0 \leqq \lambda$, $\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\}$, $\mathfrak{sl}(2, R)$, or $M_2(R)$. Let $X = \langle A, \ell \rangle$ and $Y = \langle B, m \rangle$ belong to g. Then X and Y generate g if and only if*

(1) *A and B generate g_0*

(2) *The equations $Av = \ell$ and $Bv = m$ cannot be simultaneously solved for v; equivalently A is nonsingular and $m \neq BA^{-1}\ell$ or B is nonsingular and $\ell \neq AB^{-1}m$ or A and B are singular and one of $\ell \notin$ range A, $m \notin$ range B.*

Moreover, such a pair always exists.

Proof. The first condition is obviously necessary. If v satisfies the second condition, conjugation of X and Y by $\langle I, v \rangle$ produces $\langle A, 0 \rangle$ and $\langle B, 0 \rangle$, so X and Y generate a subalgebra conjugate to $\{\langle A, 0 \rangle \mid A \in g_0\}$.

Conversely, suppose 1 and 2 hold; then X and Y generate a subalgebra $\tilde{g}$ such that $\tilde{g}_0 = g_0$. If $\tilde{V} = R^2$, $\tilde{g} = g$; otherwise $\tilde{V} = \{0\}$ by Theorem 3. Then $\tilde{g}$ is conjugate to $\{\langle A, 0\rangle \mid A \in g_0\}$ by Theorem 4. Let $\langle C, w\rangle \in A(2)$ induce this conjugation; then X and Y become $\langle CAC^{-1}, C\ell - CAC^{-1}w\rangle$ and $\langle CBC^{-1}, Cm - CBC^{-1}w\rangle$, so $C\ell - CAC^{-1}w = 0$, $Cm - CBC^{-1}w = 0$; since C is nonsingular, $A(C^{-1}w) = \ell$, $B(C^{-1}w) = m$.

If A is nonsingular, the vector v obviously exists just in case $m = BA^{-1}v$. If A and B generate g_0 and both are singular, g_0 must be $\mathfrak{sl}(2, R)$ or $M_2(R)$, so A and B must have rank 1 and $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$. Suppose $\ell \in$ range A and $m \in$ range B. Let $Av = \ell$ and suppose that v_1 generates Ker A; then $A(v + \lambda v_1) = \ell$; since $v_1 \notin \operatorname{Ker} B$, Bv_1 generates the range of B and λ exists such that $B(v + \lambda v_1) = m$.

The existence of a generating pair is clear.

THEOREM 8. *Let* $g = \{\langle\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right), \ell\rangle \mid \ell \in R^2\}$. *Let* $X = \langle A, \ell\rangle$ *and* $Y = \langle B, m\rangle$ *belong to* g. *Then* X *and* Y *generate* g *if and only if*

(1) *A and B are linearly independent,*

(2) $Am - B\ell$ *belongs to neither* $R\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$ *nor* $R\left(\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right)$.

Moreover, g can always be generated by such a pair.

PROOF. These conditions are necessary. For example, suppose $Am - B\ell \in R\left(\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right)$. Then the subspace of g generated by X, Y, and $[X, Y] = \langle 0, Am - B\ell\rangle$ is a subalgebra, so X and Y generate an algebra of dimension at most 3.

Conversely suppose the above conditions hold. Then X and Y generate an algebra $\tilde{g}$ with $\tilde{g}_0 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right)\}$; it is enough to prove that $\tilde{V} = R^2$. At any rate $\tilde{V}$ is invariant under $\tilde{g}_0$ and so equal to $\{0\}$, $R\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$, $R\left(\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right)$, or R^2. But $[\langle A, \ell\rangle, \langle B, m\rangle] = \langle 0, Am - B\ell\rangle$ and $Am - B\ell \notin R\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$ or $R\left(\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right)$.

The above conditions are satisfied by $X = \langle\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right), 0\rangle$ and $Y = \langle\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1\end{smallmatrix}\right)\rangle$.

THEOREM 9. *Let* $g = \{\langle A, \ell\rangle \mid A \in g_0, \ell \in R^2\}$ *where* $g_0 = \{\left(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right)\}$ *or* $\{\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)\}$. *Let* $X = \langle A, \ell\rangle$ *and* $Y = \langle B, m\rangle$ *belong to* g. *Then* X *and* Y *generate* g *if and only if*

(1) *A and B generate g_0,*

(2) $Am - B\ell \notin R\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$.

Moreover, g can always be generated by such a pair.

PROOF. These conditions are necessary. Indeed suppose $Am - B\ell \in R\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$. If $g_0 = \{\left(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right)\}$, the subspace generated by X, Y, and $\langle 0, \left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)\rangle$ is a subalgebra of dimension at most 3. If $g_0 = \{\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)\}$, the subspace generated by X, Y, $\langle\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right), 0\rangle$ and $\langle 0, \left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)\rangle$ is a subalgebra of dimension at most 4.

Conversely suppose 1 and 2 hold; then X and Y generate an algebra $\tilde{g}$ with $\tilde{g}_0 = g_0$. It is enough to prove that $\tilde{V} = R^2$; at any rate, $\tilde{V}$ is invariant under g_0 and so $\{0\}$, $R\binom{1}{0}$, or R^2. If $g_0 = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\}$, $Am - B\ell \in \tilde{V}$, so $\tilde{V}$ is not $\{0\}$ or $R\binom{1}{0}$. If $g_0 = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\rangle$, $\tilde{g}$ contains $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \binom{s_1}{s_2}\rangle$ and $[X, Y] = \langle \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \binom{t_1}{t_2}\rangle$ for some s_1, s_2, t, t_1, and t_2; we are assuming $t_2 \neq 0$. Thus $\tilde{g}$ contains the bracket of these two elements, $\langle 0, \binom{t_1}{t_2} - \binom{ts_2}{0}\rangle$, and $\tilde{V}$ is not $\{0\}$ or $R\binom{1}{0}$.

If $g_0 = \langle \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\}$, the conditions are satisfied by $X = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0\rangle$ and $Y = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \binom{0}{1}\rangle$. If $g_0 = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\}$ the conditions are satisfied by $X = \langle \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, 0\rangle$ and $Y = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \binom{0}{1}\rangle$.

THEOREM 10. *Let* $g = \{\langle \begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix}, \ell\rangle \mid a, b \in R, \ell \in R^2\}$ *for* $\lambda \in R$. *Let* $X = \langle A, \ell\rangle$ *and* $Y = \langle B, m\rangle$ *belong to* g. *Then* X *and* Y *generate* g *if and only if*

(1) *A and B are linearly independent,*

(2) $(A - \beta)m - (B + \alpha)\ell \notin R\binom{1}{0}$ *where* $[A, B] = \alpha A + \beta B$.

Moreover g *can be generated by such a pair unless* $\lambda = -2$.

PROOF. If $(A - \beta)m - (B - \alpha)\ell \in R\binom{1}{0}$, the subspace generated by X, Y, and $\langle 0, \binom{1}{0}\rangle$ is a subalgebra of dimension at most 3.

Conversely suppose these conditions hold and let X and Y generate $\tilde{g}$; then $\tilde{g}_0 = \{\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix}\}$ and it is sufficient to show that $\tilde{V} = R^2$; since $\tilde{V}$ is invariant under $\tilde{g}_0$, $\tilde{V} = \{0\}$, $R\binom{1}{0}$, or R^2. But $\tilde{g}$ contains $[X, Y] - \alpha X - \beta Y = \{0, (A - \beta)m - (B + \alpha)\ell\rangle$, so $\tilde{V} = R^2$.

If $\lambda \neq -2$, let $X = \langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{pmatrix}, 0\rangle$, $Y = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \binom{0}{1}\rangle$ and notice that the above conditions hold. If $\lambda = -2$. notice that

$$\left[\begin{pmatrix} 2a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} 2c & d \\ 0 & c \end{pmatrix}\right] = \begin{pmatrix} 0 & ad - bc \\ 0 & 0 \end{pmatrix} = -c\begin{pmatrix} 2a & b \\ 0 & a \end{pmatrix} + a\begin{pmatrix} 2c & d \\ 0 & c \end{pmatrix}$$

and

$$\left\{\begin{pmatrix} 2a & b \\ 0 & a \end{pmatrix} - a\right\}\begin{pmatrix} m_x \\ m_y \end{pmatrix} - \left\{\begin{pmatrix} 2c & d \\ 0 & c \end{pmatrix} - c\right\}\begin{pmatrix} \ell_x \\ \ell_y \end{pmatrix}$$
$$= \begin{pmatrix} am_x + bm_y - c\ell_x - d\ell_y \\ 0 \end{pmatrix}.$$

THEOREM 11. *Let* $g = \{\langle \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \binom{e}{0}\rangle\}$. *Let* $X = \langle A, \ell\rangle$ *and* $Y = \langle B, m\rangle$ *belong to* g. *Then* X *and* Y *generate* g *if and only if*

(1) *A and B generate* $\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\}$,

(2) $Am - B\ell \neq 0$.

Moreover, g *can always be generated by such a pair.*

PROOF. If $Am - B\ell = 0$, the subspace generated by X, Y, and $\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\rangle$ is a subalgebra of dimension at most 3.

Conversely suppose these conditions hold and let X and Y generate $\tilde{g}$; then $\tilde{g}_0 = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\}$ and it is enough if $\tilde{V} \neq \{0\}$. But $\tilde{g}$ contains $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} s \\ 0 \end{pmatrix} \rangle$ and $[X, Y] = \langle \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \rangle$ for some s, t, u; we are assuming $u \neq 0$. Thus $\tilde{g}$ contains the bracket of these two elements, $\langle 0, \begin{pmatrix} u \\ 0 \end{pmatrix} \rangle$, and $\tilde{V} \neq \{0\}$.

The conditions are satisfied by $X = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \rangle$ and $Y = \langle \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$.

Theorem 12. *Every connected subgroup of* $A(2)$ *not conjugate to* $\{\langle \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ell \rangle \mid a > 0, \ \ell \in R^2\}$, $\{\langle \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c \\ 0 \end{pmatrix} \rangle \mid a > 0, \ b, \ c \in R\}$, *or* $\{\langle \begin{pmatrix} a^2 & b \\ 0 & a \end{pmatrix}, \ell \rangle \mid a > 0, \ b \in R, \ \ell \in R^2\}$ *can be generated by an appropriate pair of infinitesimal transformations.*

Proof. It suffices to consider the three dimensional g on the list in Theorem 4. Every non-abelian three dimensional Lie algebra can be generated by appropriate X and Y except the Lie algebra $\{\langle \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ell \rangle \mid a \in R, \ell \in R^2\}$ [1]; we should show that only two algebras on our list are isomorphic to this algebra. It is easier to proceed directly; we already know that 1 through 12 can be generated if $V = \{0\}$. If $V = R^2$ pick $X = \langle 0, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$; let $Y = \langle \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, 0 \rangle$ in case 2 if $\lambda \neq 1$, $\langle \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}, 0 \rangle$ in case 3 if $\lambda \neq 1$, $\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \rangle$ in case 4, and $\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 0 \rangle$ in case 5. In case 3 when $\lambda = 1$ let $X = \langle 0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$, $Y = \langle \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, 0 \rangle$. In case 18 let $X = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \rangle$, $Y = \langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$. In case 19 let $X = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \rangle$, $Y = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$. In case 20 when $\lambda \neq -1$ let $X = \langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{pmatrix}, 0 \rangle$, $Y = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$. In case 28 let $X = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \rangle$, $Y = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$; in case 29 let $X = \langle \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, 0 \rangle$, $Y = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$.

Remark. Notice that the only subgroup of $A(2)$ conjugate to $G = \{\langle \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ell \rangle \mid a > 0, \ell \in R^2\}$ is G itself.

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