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A MULTIPLE-ZERO LEMMA FOR LINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. The lemma gives conditions on n real-valued functions sufficient for some linear combination of these functions to have a zero of order n. The result is specialized to linear ordinary differential equations of order k and the method of application is considered.

1. **Introduction.** The purpose of this note is to communicate a special lemma from the theory of real functions. This lemma is potentially useful for the study of boundary value problems for linear ordinary differential equations of order k.

The spirit of the lemma is to assert under endpoint and differentiability conditions that a linear combination of n functions $u_1, \cdots, u_n$ has an n-th order zero at some point of the open interval.

In the case of two functions, this lemma has played a central role in existence and nonexistence arguments for boundary value problems associated with the k-th order linear differential equation

$$u^{(k)} + \sum_{j=0}^{k-1} p_j(t)u^{(j)} = 0.$$

The first use of the two-function lemma appears in the fundamental paper of Leighton and Nehari [4] on fourth order linear differential equations. Sherman [9] reformulated the Leighton-Nehari lemma for use in the study of the conjugate point function $\eta_1(t)$ associated with a k-th order linear differential equation. The Sherman lemma has played an important role in existence-nonexistence arguments of Bogar [1], Dolan [2], Peterson [6, 7], Ridenhour and Sherman [8] and the author [3].

The main result for real functions appears in Lemma 2.4; the novelty here is the precise information. A model lemma suitable for differential equations is given in § 3; a discussion of the method of application follows.

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Throughout this paper, $f = 0(h^p)$ means that $|f(h)| \leqq K|h^p|$ as $h \to 0$ for some constant $K > 0$.

Given n functions $u_1, \cdots, u_n$ of class $C^{n-1}[a, b]$, $W(u_1(x), \cdots, u_n(x))$ shall denote the *Wronskian Determinant*: $\det[u_j^{(i-1)}]$ $(1 \leqq i \leqq n, 1 \leqq j \leqq n)$.

A real-valued function $u \in C^n[a, b]$ shall be said to have a *zero of order* r $(r \leqq n)$ at $c \in [a, b]$ iff $u^{(i)}(c) = 0$, $0 \leqq i \leqq r - 1$. The zero shall be called of *order exactly* r iff $u^{(i)}(c) = 0$, $0 \leqq i \leqq r-1$, $u^{(r)}(c) \neq 0$.

2. **The multiple-zero lemma for real functions.** We first establish some technical lemmas on Wronskian determinants.

Lemma 2.1. *Let $0 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_n$ be integers satisfying $p_i \geqq i - 1$ $(1 \leqq i \leqq n)$, and suppose $A = [a_{ij}]$ is an $n \times n$ matrix of functions satisfying $a_{ij}(h) = 0(h^{\alpha_{ij}})$, $\alpha_{ij} \equiv \max\{p_j - i + 1, 0\}$, then*

$$\det[A(h)] = 0(h^T),$$

where

$$T = \sum_{i=1}^{n} p_i - \frac{1}{2}n(n-1).$$

Proof. Let's write $\det A = \sum \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$, the sum being extended over all σ in the symmetric group of order n. It suffices to show that $\prod_{i=1}^n a_{i,\sigma(i)} = 0(h^T)$.

To establish this, observe that $\prod_{i=1}^n a_{i,\sigma(i)} = 0(h^K)$, where $K = \sum_{i=1}^n \max\{(p_{\sigma(i)} - i + 1), 0\}$. Further, since σ is a permutation,

$$K \geqq \sum_{i=1}^{n} (p_{\sigma(i)} - i + 1) = \sum_{i=1}^{n} p_{\sigma(i)} - \frac{1}{2} n(n-1) = T.$$

This proves the result.

The result is best possible, because the product of the diagonal elements of A has order h^T.

Lemma 2.2. *Let $0 \leqq r_1 < r_2 < \cdots < r_n$ be integers, put $m_i = 1 + r_i (1 \leqq i \leqq n)$. Assume $u_i \in C^{m_i}[a, b] \cap C^{n-1}[a, b]$ $(1 \leqq i \leqq n)$ are given functions, u_i has a zero of order exactly r_i at $x = c \in [a, b]$, $1 \leqq i \leqq n$, and put $v_i(x) = u_i^{(r_i)}(c)(x-c)^{r_i}/r_i!$, $1 \leqq i \leqq n$. Then*

$$W(u_1(x), \cdots, u_n(x)) = W(v_1(x), \cdots, v_n(x)) + 0[(x-c)^{R+1}],$$

where

$$R = \sum_{i=1}^{n} r_i - \frac{1}{2} n(n-1).$$

PROOF. Define $h \equiv x - c$. Then $u_i(x) = u_i^{(r_i)}(c)h^{r_i}/r_i! + 0(h^{r_i+1})$, so

$$(2.1) \qquad \begin{cases} u_i^{(j)}(x) = v_i^{(j)}(x) + 0(h^{r_i+1-j}) & (0 \leqq j \leqq r_i) \\ u_i^{(j)}(x) = 0(1) & (j > r_i) \end{cases}$$

for $1 \leqq i \leqq n$.

Let us use the sum rule for determinants on the columns of $W(u_1, \cdots, u_n)$ together with relation (2.1), then

$$(2.2) \qquad W(u_1, \cdots, u_n) = W(v_1, \cdots, v_n) + \sum_{i=1}^{2^n-1} \det[A_i]$$

where each A_i is an $n \times n$ matrix. The preceding Lemma 2.1 applies to prove $\det[A_i] = 0(h^{R+1})$, where $R = \sum_{i=1}^{n} r_i - (1/2)n(n-1)$; indeed, each A_i has a column which starts with order one higher than the corresponding column of $W(v_1, \cdots, v_n)$. This completes the proof.

LEMMA 2.3. *Let $0 \leqq r_1 < r_2 < \cdots < r_n$ be integers, $a_1, \cdots, a_n$ nonzero constants and put $v_i(h) = a_i h^{r_i}/r_i!$. Then*

$$W(v_1, \cdots, v_n) = \alpha \left(\prod_{i=1}^{n} a_i \right) \frac{h^R}{R!} + 0(h^{R+1})$$

where $R = \sum_{i=1}^{n} r_i - (1/2)n(n-1)$ and α is a positive integer which depends only on n and the integers $r_1, \cdots, r_n$.

PROOF. Let $V(h) = [v_1(h), \cdots, v_n(h)]$. Define for each integer $k \geqq 0$ the set S_k to be the set of n-tuples $\sigma = (k_1, \cdots, k_n)$ of nonnegative integers such that $|\sigma| \equiv \sum_{i=1}^{n} k_i = k$. Define the operator T^σ to act on $W(v_1, \cdots, v_n)$ as follows: $T^\sigma W(v_1, \cdots, v_n)$ is $W(v_1, \cdots, v_n)$ with row i replaced by its k_i-th derivative $(1 \leqq i \leqq n)$.

The rule for differentiation of determinants gives

$$(2.3) \qquad \left(\frac{d}{dh} \right)^k W(v_1, \cdots, v_n) = \sum_{\sigma \in S_k} c_\sigma T^\sigma W(v_1, \cdots, v_n),$$

where each c_σ is a positive integer; this is easily proved by induction on k. Further, it is shown by induction that a term will occur on the right side of (2.3) iff $k_1 < 1 + k_2 < 2 + k_3 < \cdots < n - 1 + k_n$; here one appeals to the determinant rule that two equal rows yield zero determinant.

Let's show that $(d/dh)^k W(v_1, \cdots, v_n) = 0$ at $h = 0$ for $0 \leqq k < R$.

Let $\sigma \in S_k$, and assume that $T^\sigma W(v_1, \cdots, v_n) \neq 0$ at $h = 0$, then $k_1 < 1 + k_2 < \cdots < n - 1 + k_n$, and $|\sigma| = k$.

Define e_i to be the i-th unit vector of R^n $(1 \leqq i \leqq n)$. At $h = 0$, the rows of $T^\sigma W(v_1, \cdots, v_n)$ are multiples of the vectors e_i $(1 \leqq i \leqq n)$, hence

$$V^{(i-1+k_i)}(0) = a_{\ell_i} e_{\ell_i} \qquad (1 \leqq i \leqq n) \tag{2.4}$$

where $\ell_1, \cdots, \ell_n$ is a permutation of the integers $1, 2, \cdots, n$.

It follow that $r_{\ell_i} = k_i + i - 1$, hence

$$|\sigma| = \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} r_{\ell_i} - (1/2)n(n-1) = R,$$

a contradiction.

This proves that $(d/dh)^k W(v_1, \cdots, v_n) = 0$ at $h = 0$ for $0 \leqq k < R$.

Now consider the case $k = R$ in relation (2.3). Define $\sigma_0 \equiv (r_1, r_2 - 1, r_3 - 2, \cdots, r_n - n + 1)$. The claim is that the only term on the right side of (2.3) at $h = 0$ is $c_{\sigma_0} T^{\sigma_0} W(v_1, \cdots, v_n)$.

To prove this, let $|\sigma| = R$, $\sigma = (k_1, \cdots, k_n)$. If $T^\sigma W(v_1, \cdots, v_n) \neq 0$ at $h = 0$, then relation (2.4) holds and $r_{\ell_i} = k_i + i - 1$, $1 \leqq i \leqq n$. However, $k_1 < 1 + k_2 < \cdots < n - 1 + k_n$ implies $r_{\ell_1} < r_{\ell_2} < \cdots < r_{\ell_n}$, therefore $\ell_1 < \ell_2 < \cdots < \ell_n$, and we have $\ell_1 = 1$, $\ell_2 = 2$, $\cdots, \ell_n = n$; thus $\sigma \equiv \sigma_0$.

Relation (2.4) makes it easy to compute $T^{\sigma_0}(v_1, \cdots, v_n)$ at $h = 0$, the value being $\prod_{i=1}^n a_i$.

Put $\alpha = c_{\sigma_0}$. Then α is a positive integer, and $(d/dh)^R W(v_1, \cdots, v_n)|_{h=0} = \alpha \prod_{i=1}^n a_i$. By Taylor's theorem,

$$W(v_1, \cdots, v_n) = \sum_{k=0}^{R} [(d/dh)^k W(v_1, \cdots, v_n)|_{h=0}] h^k/k! + 0(h^{R+1}),$$

and this completes the proof.

Combining these lemmas, we obtain the following lemma about real functions:

Multiple-Zero Lemma

Lemma 2.4. *Let $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$ be two sets of distinct nonnegative integers and put*

$$m_i \equiv 1 + \max\{\alpha_i, \beta_i\} \qquad (1 \leqq i \leqq n).$$

Assume that functions $u_1, \cdots, u_n$ are given with

(2.5) $\qquad u_i \in C^{m_i}[a, b] \cap C^{n-1}[a, b] \qquad (1 \leqq i \leqq n)$

(2.6) *u_i has a zero of order exactly α_i at $x = a$ and a zero of order exactly β_i at $x = b$ $\quad (1 \leqq i \leqq n)$.*

(2.7) *The permutations π_1 and π_2 which send $(\alpha_1, \cdots, \alpha_n)$ and $(\beta_1, \cdots, \beta_n)$, respectively, into natural order, satisfy $\operatorname{sign}(\pi_1) \cdot \operatorname{sign}(\pi_2) \prod_{i=1}^{n} u_i^{(\alpha_i)}(a) u_i^{(\beta_i)}(b)(-1)^{\beta_i - i + 1} < 0$, or equivalently, for all $\epsilon > 0$ sufficiently small, $(-1)^{(1/2)n(n-1)} \operatorname{sign}(\pi_1) \operatorname{sign}(\pi_2) \prod_{i=1}^{n} u_i(a+\epsilon) u_i(b-\epsilon) < 0$.*

Then there exists constants $c_1, \cdots, c_n$ not all zero such that $u(x) \equiv \sum_{i=1}^{n} c_i u_i(x)$ has a zero of order at least n at some point $x_0 \in (a, b)$.

Proof. Let $W_i(x) = (\operatorname{sign} \pi_i) W(u_1(x), \cdots, u_n(x))$, $i = 1, 2$. Then $W_i(x) \equiv W(u_{\pi_i(1)}(x), \cdots, u_{\pi_i(n)}(x))$ $(i = 1, 2)$, (here, $\pi_1(k) = \pi_1(\alpha_k)$, $\pi_2(k) = \pi_2(\beta_k)$, for brevity) so lemmas 2.2, 2.3 apply to give

$$W_1(a+h) W_2(b-h) = \alpha\beta \prod_{i=1}^{n} (a_i b_i) \frac{h^{R_1}(-h)^{R_2}}{R_1! R_2!} + 0(h^{R_1 + R_2 + 1}).$$

Here, $R_1 = \sum_{i=1}^{n} \alpha_i - (1/2)n(n-1)$, $R_2 = \sum_{i=1}^{n} \beta_i - (1/2)n(n-1)$, $a_i = u_i^{(\alpha_i)}(a)$, $b_i = u_i^{(\beta_i)}(b)$ $(1 \leqq i \leqq n)$, and α and β are positive integers. A rearrangement of this relation gives

$$W(u_1(a+h), \cdots, u_n(a+h)) W(u_1(b-h), \cdots, u_n(b-h)) = kh^{R_1 + R_2} + 0(h^{R_1 + R_2 + 1})$$

$k \equiv \alpha\beta[\operatorname{sign} \pi_1]\ [\operatorname{sign} \pi_2]\ [\prod_{i=1}^{n} a_i b_i (-1)^{\beta_i - i + 1}]\ [R_1! R_2!]^{-1}$. Further, $k < 0$ by relation (2.7). Therefore, $W(u_1(x), \cdots, u_n(x))$ changes sign at some point $x_0 \in (a, b)$. The conclusion follows by solving the system $\sum_{j=1}^{n} u_j^{(i)}(x_0) c_j = 0$ $(0 \leqq i \leqq n-1)$ for nontrivial $c_1, \cdots, c_n$.

Remark. The most common kind of application is when each $u_i(x)$ is one-signed on (a, b), then relation (2.7) reduces to the requirement that

(2.8) $\qquad (-1)^{(1/2)n(n-1)} [\operatorname{sign} \pi_1]\ [\operatorname{sign} \pi_2] < 0$

3. **Boundary value problems.** Let $Lu \equiv u^{(k)} + \sum_{j=0}^{k-1} p_j(t) u^{(j)}$ be a linear ordinary differential operator with continuous coefficients. The following restatement of lemma 2.4 is suitable for applications to boundary value problems for $Lu = 0$.

Multiple-Zero Lemma

Lemma 3.1. *Let $u_1, \cdots, u_n$ be solutions of $Lu = 0$ such that*

(3.1) *u_i has a zero of order exactly α_i at a $(1 \leqq i \leqq n)$, $\alpha_1, \cdots, \alpha_n$ distinct,*

(3.2) *u_i has a zero of order exactly β_i at b $(1 \leqq i \leqq n)$, $\beta_1, \cdots, \beta_n$ distinct,*

(3.3) *u_i is one-signed in (a, b) $(1 \leqq i \leqq n)$.*

If π_1 and π_2 carry $(\alpha_1, \cdots, \alpha_n)$ and $\beta_1, \cdots, \beta_n)$, respectively, into natural order, and

$$(-1)^{(1/2)n(n-1)} \operatorname{sign} \pi_1 \operatorname{sign} \pi_2 < 0,$$

then for some $x_0 \in (a, b)$ there exists a nontrivial solution $u = c_1 u_1 + \cdots + c_n u_n$ of $Lu = 0$ with a zero of order at least n at x_0.

Application to Differential Equations

Consider a k-th order linear ordinary differential equation $Lu = 0$ and the boundary conditions

$$u^{(i)}(s_j) = 0 \quad (0 \leqq i \leqq n_j - 1,\ 0 \leqq j \leqq \nu) \tag{3.4}$$

which will hereafter be abbreviated to: u has a zero of order $(n_0, \cdots, n_\nu)$ at $\{s_0 < \cdots < s_\nu\}$. It is always assumed that $n_0 + \cdots + n_\nu = k$, and $a < s_0 < \cdots < s_\nu < b$.

Suppose $\alpha_p = (n_{0\,p}, \cdots, n_{\nu_p, p})$ $(1 \leqq p \leqq \ell)$ is a finite set of boundary data and the following *uniqueness condition* holds: for every choice of $\nu_p + 1$ points $s_0 < \cdots < s_{\nu_p}$ in (a, b) the only solution of $Lu = 0$ with a zero of order α_p at $\{s_0 < \cdots < s_{\nu_p}\}$ is $u \equiv 0$ $(1 \leqq p \leqq \ell)$.

Under this uniqueness assumption, certain kinds of other boundary value problems (3.4) also have a unique solution. For example, it is well-known that if the only solution with k distinct zeros in (a, b) is $u \equiv 0$, then all problems (3.4) have the unique solution $u \equiv 0$ (an elegant proof of this has been given by Z. Opial [5]).

A common use of this kind of uniqueness result is to obtain the existence of a Green's function $G(t, s)$ for boundary conditions (3.4), hence converting the problem $Lu = f$ with boundary conditions (3.4) into an integral equation

$$u(t) = \int_{s_0}^{s_\nu} G(t, s) f(s)\, ds.$$

The role of the multiple-zero lemma is to convert this question into possibly more tractable questions.

To illustrate the method for $n \geqq 3$, assume the uniqueness condition holds for the family $\{\alpha_1, \cdots, \alpha_9\}$ specified by

$$\alpha_1 = (h+1, m-1, \ell), \quad \alpha_2 = (h, m, \ell),$$

$$\alpha_3 = (h, m-2, 3, \ell-1), \quad \alpha_4 = (h+1, m-2, \ell+1),$$

$$\alpha_5 = (h+1, m-2, 1, \ell), \quad \alpha_6 = (h, m+1, \ell-1),$$

$$\alpha_7 = (h, m, 1, \ell-1), \quad \alpha_8 = (h, m-1, \ell+2),$$

$$\alpha_9 = (h, m-1, 1, \ell+1).$$

It will be shown that the only solution of the equation $Lu = 0$ with a zero of order $\alpha = (h, m-1, \ell+1)$ is the trivial solution $u \equiv 0$.

Suppose not, and let $u_1 \not\equiv 0$ be a solution of $Lu = 0$ with a zero of order α at $\{s_0 < s_1 < s_2\}$.

Construct solutions $u_2 \not\equiv 0$, $u_3 \not\equiv 0$ of $Lu = 0$, with zeros of order $(h+1, m-2, \ell)$, $(h, m, \ell-1)$ at $\{s_0 < s_1 < s_2\}$, respectively.

The uniqueness condition implies that u_1, u_2, u_3 have no other zeros on $[s_i, s_2]$, counting multiplicities. Hence, we may assume that $u_i(t) > 0$ on $s_1 < t < s_2$, $1 \leqq i \leqq 3$. The permutations π_1 and π_2 of the multiple-zero lemma are given by π_1: $(m-1, m-2, m) \to (m-2, m-1, m)$ and π_2: $(\ell+1, \ell, \ell-1) \to (\ell-1, \ell, \ell+1)$, therefore (2.8) holds:

$$(-1)^{(1/2)(3)(3-1)}[\operatorname{sign} \pi_1] \; [\operatorname{sign} \pi_2] < 0.$$

The multiple-zero lemma applies to give a solution $u = c_1u_1 + c_2u_2 + c_3u_3 \not\equiv 0$ of $Lu = 0$ with a triple zero at $t_0 \in (s_1, s_2)$. However, this implies u has a zero of order α_3 at $\{s_0 < s_1 < t_0 < s_2\}$, a contradiction to the uniqueness condition. Therefore, the only solution of $Lu = 0$ with a zero of order α in (a, b) is $u \equiv 0$.

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