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CONVERGENCE LATTICES[1]

ARMANDO R. GINGRAS

ABSTRACT. A complete lattice in which order convergence coincides with topological convergence with respect to the order topology Θ is termed a *convergence lattice.* Each complete chain, each arbitrary product of convergence lattices, and each lattice in which each chain is finite is a convergence lattice. We say that a complete lattice is *locally-intervaled* if every Θ-neighborhood of each point in the lattice contains an interval that is also a Θ-neighborhood of the point.

THEOREM. *A complete lattice is a convergence lattice if and only if it is locally-intervaled.*

Introduction. When considering simultaneous topological structures and lattice-order relations on a set, there are three basic approaches possible. The first point of view superimposes a topology on a lattice that is required to interact (i.e., be compatible) with the order relation in a prescribed way. For instance, the topology may be required to render the lattice a Hausdorff space in which the lattice operations are continuous. This inaugurates the study of topological lattices.

The second approach examines topologies on a lattice arising naturally from the lattice structure itself. Examples of such intrinsic topologies are those defined by Frink [7], Rennie [15], Birkhoff [3], Insel [9], Wolk [18], and the topology generated by order convergence (i.e., the order topology) [2].

The third viewpoint introduces a lattice-order on a topological space in such a way that a specific intrinsic topology derived from the lattice-order agrees with the original topology of the space. For example, the classic theorem of Eilenberg [5, Theorem 1] gives a sufficient condition on a connected Hausdorff space so that it may be totally ordered with the interval topology and the original topology coinciding.

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Of the three approaches to a study of topology on lattices, this last one remains the most intractable and least developed today.

This paper is a study of an important class of complete lattices for which the first two points of view merge, namely, those complete lattices in which order convergence coincides with the topological convergence derived from the order topology. We call such lattices convergence lattices. Here we initiate a systematic investigation of the mutual dependence of topological and order properties of convergence lattices.

Section 1 details those definitions and conventions basic to this work. It contains preliminary results concerning order convergence and the order topology Θ and sets the background for our study.

General convergence lattices are introduced in Section 2 and several characterization theorems are presented. There it is shown that every neighborhood of a point of a convergence lattice contains an interval that is also a neighborhood of the point; this condition is also sufficient to characterize convergence lattices. While this study bears mainly on fundamental properties in the topology of convergence lattices, we find that a corollary (Corollary 2.10) to the preceding result opens a way for a pure lattice-theoretic treatment of convergence lattices. In Section 2 it is also shown that each complete chain and each product of convergence lattices is a convergence lattice.

Various examples of convergence lattices (important in other areas of mathematics) and complete lattices failing to be convergence lattices are collected in Section 3.

Finally, a set of four unsolved problems is presented in Section 4.

§ 1. **Preliminaries.** The lattice-theoretic terminology used herein is consistent with that of Birkhoff [2″]; the topological terms can be found in Kelley [**10**]. The exceptions are noted below. In usage of words common to topology and lattice theory, the topological meaning will take precedence (e.g., compact and closed).

For typographical convenience, the following deviations from more or less standard notation are observed here. We let X be a complete lattice with a topology, then if A is a subset of X we set

O $\equiv$ generic symbol for a lattice's zero element,
I $\equiv$ generic symbol for a lattice's unit element,
A' $\equiv$ set-theoretic and lattice-theoretic complement,
$A \nabla B$ $\equiv$ set-theoretic difference $\equiv A \cap B'$,
A° $\equiv$ topological interior of A,
A^{c} $\equiv$ topological closure of A,
$\eta(x)$ $\equiv$ local neighborhood system at the point x in X.

A *convergence scheme* C for a set X is a class of pairs (Φ, x) where Φ

is a net in X and x belongs to X. If $(\Phi, x) \in C$, then we say that Φ C-converges to x and write $C\text{-}\lim \Phi = x$. If the domain of Φ is of interest, we may write $C\text{-}\lim \Phi_\alpha = x$ or even $C\text{-}\lim_\alpha \Phi_\alpha = x$. The C-closure, denoted by $C\text{-cl}(A)$, of a subset A of X is defined to be A together with all C-limits of every C-convergent net in A. A subset A is called C-closed if $A = C\text{-cl}(A)$. We say that a net Φ $C*$-converges (or simply $*$-converges if the convergence scheme C is understood) to x if an arbitrary subnet of Φ contains a further subnet that C-converges to x.

We now enter upon our subject proper.

Let X be a complete lattice. A net x_α is said to *order converge* (or *o-converge*) to a point x in X if

(1) there exist subsets A and B of X such that A is up-directed and B is down-directed,

(2) $\sup A = x = \inf B$,

(3) for each a in A and b in B, there exists β in the domain of x_α such that for $\alpha \geqq \beta$, $a \leqq x_\alpha \leqq b$.

A direct verification shows that the foregoing definition is equivalent to the following statement: x_α o-converges to x if and only if lim-inf x_α $= x =$ lim-sup x_α, where

$$\text{lim-inf}\, x_\alpha = \bigvee_\beta \bigwedge_{\alpha \geqq \beta} x_\alpha,$$

and

$$\text{lim-sup}\, x_\alpha = \bigwedge_\beta \bigvee_{\alpha \geqq \beta} x_\alpha.$$

It is equally easy to see that o-convergence is a convergence scheme in which every net that is eventually constant and every subnet of an o-convergent net is also o-convergent.

PROPOSITION 1.1. *The family C of o-closed sets of a complete lattice satisfies the following axioms for closed sets* [**10**, Theorem 4, pg. 40]:

(1) $X \in C$,

(2) If $F_1, F_2 \in C$, then $F_1 \cap F_2 \in C$,

(3) If $F_\lambda \in C$ for $\lambda \in \Lambda$, $\Lambda \neq \emptyset$, then $\bigcap_\lambda F_\lambda \in C$.

PROOF. See, for example, Vulikh [**17**, pg. 33]. //

We use the modified Halmos symbol, //, to indicate the end of a proof.

The *order topology* on a complete lattice is defined to be that (unique) topology having for its closed sets the family C of o-closed sets. Hereafter, we always let Θ denote the order topology of a complete lattice. Note that Θ is always T_1.

In general, o-closure is not a (Kuratowski) closure operator only in that o-closure need not be idempotent. Example 3.7 displays calculations showing the failure of idempotency of o-closure in a particular complete lattice.

Kelly [10] approaches convergence in the following way. Let C be a convergence scheme for a set X. He calls C a convergence class if C satisfies the following four conditions:

C1. If $\Phi(\alpha) = x$ for each α, then $(\Phi, x) \in C$.

C2. If Φ C-converges to x, so does each subnet of Φ.

C3. If ΦC $*$-converges to x, then Φ C-converges to x.

C4. (Birkhoff's Law of Iterated Limits) Let D be a directed set and E_α a directed set for each $\alpha \in D$. Let $E = D \times \prod\{E_\alpha \mid \alpha \in D\}$ and direct E coordinate-wise. Let $B = \{(\alpha, \gamma) \mid \alpha \in D, \gamma \in E_\alpha\}$. Define a net ψ on E in B by $\psi(\alpha, f) = (\alpha, f(\alpha))$. If Φ is a net on B in X, then $\Phi \circ \psi$ (composition) is a net on E in X. Finally, if C-$\lim_\alpha$ C-$\lim_{\gamma \in E_\alpha} \Phi(\alpha, \gamma) = x$, then $\Phi \circ \psi$ C-converges to x.

Kelley then proceeds to prove the following interesting theorem [10, Theorem 9, pg. 74] that characterizes a closure operator, hence a topology, in terms of convergence via convergence classes.

THEOREM 1.2. *Let C be a convergence class for a set X. Then C-cl is a closure operator. Furthermore, letting T denote the associated topology, a net Φ C-converges to x in X if and only if Φ converges to x with respect to T (i.e., T-converges).*

To see that o-convergence is not in general a convergence class, the reader is directed to the discussion of Example 3.6. The last part of the discussion under Example 3.6 reveals a curious fact. It is shown there that o-closure is indeed a closure operator for that particular lattice even though C3 and C4 are not satisfied. Thus, while condition C4 itself is sufficient for o-closure to be a closure operator, it is not necessary.

PROPOSITION 1.3. *Let x_α be a net in a complete lattice X. Then o-$\lim x_\alpha = x$ implies that $*$-$\lim x_\alpha = x$, which, in turn, implies that Θ-$\lim x_\alpha = x$.*

PROOF. As every subnet of an o-convergent net also o-converges, we have the first implication. Now assume that $*$-$\lim x_\alpha = x$. If x_α fails to Θ-converge to x, there exists some open neighborhood N of x such that x_α is frequently outside of N. From this we may construct a subnet x_γ of x_α which is always outside of N. Since x_α $*$-converges to x, x_γ contains a subnet that o-converges to x. Since N' is closed, $x \in N'$.

This is the required contradiction. Therefore, x_α o-converges to x. //

A net Φ in a lattice is called isotone if $\alpha \leqq \beta$ implies that $\Phi_\alpha \leqq \Phi_\beta$. An antitone net is defined dually and a monotone net is a net that is either isotone or antitone.

PROPOSITION 1.4. *If x_α is an isotone net in a complete lattice X, then the following statements are equivalent:* o-lim $x_\alpha = x$, $\sup_\alpha x_\alpha = x$, *and* Θ-lim $x_\alpha = x$. *Dual statements hold for antitone nets.*

PROOF. Clearly, o-lim $x_\alpha = x$ and sup $x_\alpha = x$ are equivalent and either statement implies that x_α Θ-converges to x. Now assume that x_α Θ-converges to x, but that sup $x_\alpha = y \neq x$. Then o-lim $x_\alpha = y$ implies that x_α Θ-converges to y. If $x \nleqq y$, then $x \notin [O, y]$. Since $[O, y]$ is Θ-closed, $G = X \nabla [O, y]$ is open and $x \in G$; Θ-lim $x_\alpha = x$ implies that x_α is eventually in G. But this contradicts the assumption that for each α, $x_\alpha \in [O, y]$. Therefore, $x \leqq y$. Since x_α is isotone, for any β, $\alpha \geqq \beta$ means that $x_\alpha \in [x_\beta, y]$. Because $[x_\beta, y]$ is Θ-closed, $x \in [x_\beta, y]$. Therefore, $x_\beta \leqq x$ for each β, hence $y \leqq x$. Thus, $x = y$ and o-lim $x_\alpha = x$. //

We shall have occasion to consider one other topology on a complete lattice which is also derived from the order structure of the lattice. Frink's interval topology I [7] is that topology defined by choosing the closed intervals $[a, b]$, $a \leqq b$, as a subbase for the closed sets. We let I denote the interval topology on a lattice.

Since we may be considering more than one topology on a set X, we mention that if T is a topology on X that possesses a particular attribute, say, that of being compact or Hausdorff, then we write that X is T-compact or T-Hausdorff.

As an example of the interplay between two topologies on a lattice, consider the result that follows. (We mention in passing that Example 3.3 shows that the converse of this result is false.)

PROPOSITION 1.5. *Let X be a complete lattice. If X is* I-*Hausdorff, then Θ-convergence coinsides with $*$-convergence (i.e., $o*$-convergence).*

PROOF. By Proposition 1.3, all that remains to be shown is that if Φ is a net Θ-converging to x in X, then Φ $*$-converges to x. Atsumi [1, Theorem 3] has shown that every net in a I-Hausdorff complete lattice contains an o-convergent subnet. From this it follows directly that X is Θ-Hausdorff (since I is weaker than Θ) and Θ-compact. Therefore, if Φ Θ-converges to x, then Φ contains an o-convergent subnet ψ. Since Θ is Hausdorff and ψ also Θ-converges to x, ψ o-converges to x. Because every subnet of Φ Θ-converges to x, we see that Φ $*$-converges to x. //

§ 2. **General Convergence Lattices.** In the second edition of his monumental treatise, *Lattice Theory*, Garrett Birkhoff [2′, Theorem IV.13] erroneously assumed that o-convergence and Θ-convergence invariably agree in lattices. Evidently, Northam [14] was the first to publish a counterexample and Birkhoff acknowledged the error (in an exercise) in the next edition of his book [2″]. Nevertheless, we take as our starting point Birkhoff's assumption and initiate a study of those complete lattices satisfying this condition. Accordingly, we begin with the following definition.

A *convergence lattice* is a complete lattice within which o-convergence and Θ-convergence coincide.

THEOREM 2.1. *Every convergence lattice is a regular Hausdorff space with respect to its order topology* Θ.

PROOF. Since limits are unique with respect to o-convergence, a convergence lattice is a Hausdorff space. Also, it is known that the order topology on a complete lattice is regular if o-convergence coincides with Θ-convergence (e.g., [4]). //

We now present various necessary and sufficient conditions for a complete lattice to be a convergence lattice. The first is a direct corollary to Proposition 1.3.

SCHOLIUM 2.2. *A complete lattice X is a convergence lattice if and only if for any net Φ and x in X both of the following two conditions are satisfied:*

(1) *If Φ Θ-converges to x, then Φ $*$-converges to x.*

(2) *If Φ $*$-converges to x, then Φ o-converges to x.*

Notice that condition (2) above is the same as condition C3 of Section 1.

COROLLARY 2.3. *If a complete lattice is* I-*Hausdorff, then it is a convergence lattice if and only if it also satisfies condition* C3.

PROOF. Apply Proposition 1.5 and Scholium 2.2. //

We consider other properties on a complete lattice satisfying C3 that make it a convergence lattice. The following theorem contains several equivalents of Birkhoff's rather complicated Law of Iterated Limits (i.e., condition C4 of Section 1). Statement (4) is a simple observation due to Gaina [8].

THEOREM 2.4. *Let X be a complete lattice satisfying condition* C3. *Then the following statements are equivalent:*

(1) *X is a convergence lattice.*

(2) *If* Φ Θ*-converges to* x, *then* Φ $*$*-converges to* x.

(3) X *satisfies* C4.

(4) *If* Φ Θ*-converges to* x, *then* Φ *contains a subnet o-converging to* x.

(5) *If* Φ *is a net and* $x \in \bigcap_\beta \Theta\text{-cl}\{\Phi(\alpha) \mid \alpha \geqq \beta\}$, *then* Φ *contains a subnet o-converging to* x.

PROOF. Theorem 1.2 and Scholium 2.2 join to show that statements (1), (2), and (3) are equivalent. Clearly, (1) implies (4) and an examination of the proof of Proposition 1.5 reveals a proof of the converse. To show that (5) implies (1), suppose Φ Θ-converges to x, but that Φ fails to o-converge to x. By C3, we may further suppose that no subnet of Φ o-converges to x. Since x clearly belongs to $\bigcap_\beta \Theta\text{-cl}\{\Phi(\alpha) \mid \alpha \geqq \beta\}$, by (5), we have that Φ contains a subnet o-converging to x and a contradiction. Conversely, suppose X is a convergence lattice and that $x \in \bigcap_\beta \Theta\text{-cl}\{\Phi(\alpha) \mid \alpha \geqq \beta\}$. Then, for any β in D (the domain of Φ), there exists a net $(\psi(\beta,\gamma); \gamma \in E_\beta)$ in $\{\Phi(\alpha) \mid \alpha \geqq \beta\}$ such that o-lim ψ (β,γ) $= x$. Let $E = D \times \prod \{E_\beta \mid \beta \in D\}$ and define a net Υ on E by $\Upsilon(\beta,f) = \psi(\beta,f(\beta))$. Since C4 holds in X for o-convergence, Υ o-converges to x. To show that Υ is a subnet of Φ, let M be a well-defined mapping of E into D defined by $M(\beta,f) = \hat{\alpha}$ where $\beta \leqq \hat{\alpha}$ and $\psi(\beta,f(\beta)) = \Phi(\hat{\alpha})$. Then $\Upsilon = \Phi \circ M$ since $\Upsilon(\beta,f) = \psi(\beta,f(\beta)) = \Phi(\hat{\alpha}) = \Phi \circ M(\beta,f)$. Finally, if β is an arbitrary element of D, then for (β,f) in E where f is arbitrary, we have $M(\beta,f) = \hat{\alpha} \geqq \beta$. Thus, Υ is indeed a subnet of Φ that o-converges to x. Therefore, (1) implies (5). //

We now present two theorems each giving a single condition on a complete lattice equivalent to having the lattice be a convergence lattice.

THEOREM 2.5. *A complete lattice* X *is a convergence lattice if and only if for each element* x *in* X, $\inf\{\sup N \mid N \in \eta(x)\} = x = \sup\{\inf N \mid N \in \eta(x)\}$, *where* $\eta(x)$ *is the* Θ*-neighborhood system at* x.

PROOF. Suppose X is a convergence lattice. Set $D = \{(n, N) \mid n \in N \in \eta(x)\}$ and direct D by $(n, N) \leqq (m, M)$ if $M \subset N$. Define a *net* Φ on D in X by $\Phi(n, N) = n$. Then Φ Θ-converges to x. For if $N \in \eta(x)$, then $(x, N) \in D$. If $(m, M) \geqq (x, N)$, then $\Phi(m, M) = m \in M \subset N$. Thus, Φ is eventually in every N in $\eta(x)$. Since X is a convergence lattice, Φ o-converges to x, i.e.,

$$\bigvee_{(n,N)} \bigwedge_{(m,M) \geqq (n,N)} \Phi(m, M) = x = \bigwedge_{(n,N)} \bigvee_{(m,M) \geqq (n,N)} \Phi(m, M).$$

Since $N = \{\Phi(m, M) \mid (m, M) \geqq (n, N)\}$, we have that $\inf N = \inf\{\Phi(m, M) \mid (m, M) \geqq (n, N)\}$ and $\sup N = \sup\{\Phi(m, M) \mid (m, M) \geqq (n, N)\}$. Therefore, $x = \text{lim-inf}\,\Phi(m, M) = \sup\{\inf N \mid N \in \eta(x)\}$ and $x = \text{lim-sup}\,\Phi(m, M) = \inf\{\sup N \mid N \in \eta(x)\}$. Hence, the above condition holds for each x in X. Conversely, suppose that X satisfies the condition in the statement of the theorem. Let Φ Θ-converge to x. Thus for each N in $\eta(x)$, there exists β in the domain of Φ such that if $\alpha \geqq \beta$, then $\Phi(\alpha) \in N$. Direct $\eta(x)$ by $N \leqq M$ if $M \subset N$. Define the sets $A = \{\inf N \mid N \in \eta(x)\}$ and $B = \{\sup N \mid N \in \eta(x)\}$. Then A is up-directed and B is down-directed. To show that A is up-directed, for example, let p and q be in A. Then there exist N and M in $\eta(x)$ such that $\inf N = p$ and $\inf M = q$. Let $L = N \cap M$ and set $r = \inf L$. Then $r \in A$ and $r \geqq p$ and $r \geqq q$. Using the theorem's condition, $\sup A = \sup\{\inf N \mid N \in \eta(x)\} = x = \inf\{\sup N \mid N \in \eta(x)\}$. Now let a be in A and b be in B. Then there exist N_a and N_b in $\eta(x)$ such that $\inf N_a = a$ and $\sup N_b = b$. Let $N = N_a \cap N_b$ and β in the domain of Φ be such that if $\alpha \geqq \beta$, then $\Phi(\alpha) \in N$. Then for $\alpha \geqq \beta$, we have $a = \inf N_a \leqq \Phi(\alpha) \leqq \sup N_b = b$. Therefore, Φ o-converges to x. //

THEOREM 2.6. *A complete lattice X is a convergence lattice if and only if X is Θ-regular and satisfies the following condition: for any $N \in \eta(x)$, there exists $M_N \in \eta(x)$ such that $M_N \subset N$ and if $M \in \eta(x)$ with $M \subset M_N$, then* $\inf M$ *and* $\sup M$ *both belong to N.*

PROOF. Suppose X is a convergence lattice. To show that X satisfies that theorem's condition, we begin by directing $\eta(x)$ by saying that $N \leqq M$ if $M \subset N$. The nets $x(N) = \inf N$ and $y(N) = \sup N$, N in $\eta(x)$, are isotone and antitone, respectively. As shown in the proof of Theorem 2.5, we have $\sup x(N) = \sup\{\inf N \mid N \in \eta(x)\} = x = \inf\{\sup N \mid N \in \eta(x)\} = \inf y(N)$. Therefore, both nets o-converge, hence Θ-converge, to x. Then if $N \in \eta(x)$, there exists M_1 and M_2 in $\eta(x)$ such that if $M \geqq M_1$, then $\sup M = y(M) \in N$ and if $M \geqq M_2$, then $\inf M = x(M) \in N$. Let $M_N = M_1 \cap M_2 \cap N$. Then $M_N \subset N$. Furthermore, if $M \in \eta(x)$, with $M \subset M_N$, then $M \geqq M_N$ and $x(M) = \inf M$ and $y(M) = \sup M$ both belong to N. The regularity of Θ follows from Theorem 2.1.

Conversely, suppose that X satisfies the above condition. Define $\bar{\eta}(x)$ to be the set of all closed neighborhoods of x (i.e., Θ-closed neighborhoods). For the moment, fix N in $\bar{\eta}(x)$. By the above condition and the Axiom of Choice, we may associate with N a set M_N such that $M_N \in \eta(x)$, $M_N \subset N$, and such that if $M \subset M_N$, where $M \in \eta(x)$, then $\sup M$ and $\inf M$ both belong to N. Define D_N to be the set of all open neighborhoods of x contained in M_N. Then D_N is not empty. Direct D_N by saying that for $M_1 \leqq M_2$ if $M_1 \subset M_2$. Set $y(M) = \sup M$ for each M in D_N. Then $y(M)$ is an isotone net on D_N in N. If we let $y = \sup y(M)$, then $y(M)$ Θ-converges to y. Since for each $M \in D_N$, $y(M)$ belongs to N and because N is Θ-closed, y also belongs to N. Setting $U_N = \bigcup\{M \mid M \in D_N\}$, then U_N is an open neighborhood of x contained in M_N. Therefore, $U_N \in D_N$ and $\sup U_N = \sup y(M) = y \in N$.

We may now define $u(N) = \sup U_N$ for each N in $\bar{\eta}(x)$, where $\bar{\eta}(x)$ is directed by setting $N_1 \leqq N_2$ if $N_2 \subset N_1$. Then $u(N)$ is an antitone net defined on $\eta(x)$. By the regularity hypothesized for Θ, $u(N)$ is eventually in each M in $\bar{\eta}(x)$. For if $M \in \eta(x)$, then there exists $N \in \bar{\eta}(x)$ with $N \subset M$. Thus $U_N \subset N \subset M$. By regularity again, there exists $K \in \bar{\eta}(x)$ such that $K \subset U_N$. Now if $L \geqq K$ in $\bar{\eta}(x)$, then $U_L \subset L \subset K \subset U_N$, so that $\sup U_L = u(L) \in N$. As $N \subset M, u(L)$ is indeed eventually in M. Therefore $u(N)$ Θ-converges to x. Proposition 1.4 together with the fact that $u(N)$ is antitone show that $u(N)$ o-converges to x. Using dual concepts and arguments, we may define a comparable net $v(N)$ on $\bar{\eta}(x)$ so that $v(N)$ also o-converges to x.

To complete the proof, assume that $\Phi(\alpha)$ is a net that Θ-converges to x. By the previous paragraph, for any M in $\eta(x)$, there exists N in $\bar{\eta}(x)$ such that $N \subset M$ and $U_N \subset N$. As $U_N \in \eta(x)$, there exists β in the domain of Φ such that if $\alpha \geqq \beta$, then $\Phi(\alpha) \in U_N$. Thus, $v(N) = \inf U_N \leqq \Phi(\alpha) \leqq \sup U_N = u(N)$ for $\alpha \geqq \beta$. Therefore, $\Phi(\alpha)$ o-converges to x. //

Corollary 2.7. *Let X be a convergence lattice. Then for each N in $\eta(x)$, there exist y and z in N such that $[y, z]$ is a neighborhood of x.*

We also obtain the following immediate corollary to Theorem 2.6. This gives us our first family of convergence lattices.

Corollary 2.8. *Let X be a complete lattice. If every point in X is topologically isolated* (*i.e.*, $\{x\}$ *is open*), *then X is a convergence lattice.*

A lattice X is said to be *locally-intervaled* if for each point x in X and each neighborhood N of x, there exists an interval $[a, b]$ such that $[a, b]$ is a Θ-neighborhood of x and $[a, b]$ is contained in N.

Theorem 2.9. *Every convergence lattice is locally-intervaled.*

Proof. Let X be a convergence lattice. Define $\iota(x)$ to be the collection of all neighborhoods of x that are also intervals. Direct $\iota(x)$ by setting $I \leqq J$ if $J \subset I$. Define two nets on $\iota(x)$ by $w(I) = \sup I$ and $z(I) = \inf I$. Set $A = \{\inf M \mid M \in \eta(x)\}$ and $B = \{\sup M \mid M \in \eta(x)\}$. Then A is up-directed, B is down-directed, and by Theorem 2.5,

$\sup A = x = \inf B$. If $a \in A$ and $b \in B$, then there exist M_1 and M_2 in $\eta(x)$ such that $a = \inf M_1$ and $b = \sup M_2$. Letting $N = M_1 \cap M_2$, we have $a = \inf M_1 \leqq \inf N \leqq \sup N \leqq \sup M_2 = b$. By Corollary 2.7, there exists $I \in \iota(x)$ such that $\inf N \leqq \inf I = z(I) \leqq w(I) = \sup I \leqq \sup N$. If $J \geqq I$ in $\iota(x)$, then $z(I) \leqq z(J) \leqq w(J) \leqq w(I)$, so that $a \leqq z(J) \leqq w(J) \leqq b$. Thus, $z(J)$ and $w(J)$ both o-converge, hence Θ-converge, to x. Therefore, for any N in $\eta(x)$, there exists $I \in \iota(x)$ such that if $J \geqq I$, then $z(J)$ and $w(J)$ both belong to N.

Now set $C = \{z(J) \mid J \in \iota(x)\}$ and $D = \{w(J) \mid J \in \iota(x)\}$ and notice that C is up-directed and D is down-directed, and that $\sup C = x = \inf D$. Let $E = \{(n, I) \mid n \in I \text{ and } I \in \iota(x)\}$ and direct E by setting $(n, I) \leqq (m, J)$ if $J \subset I$. Define a net Φ on E by $\Phi(n, I) = n$. Then Φ Θ-converges to x. For if we let c be in C and d be in D, then there exist I_1 and I_2 in $\iota(x)$ such that $c = z(I_1)$ and $d = w(I_2)$. Let $I = I_1 \cap I_2$ and we have $c = z(I_1) \leqq z(I) \leqq w(I) \leqq w(I_2) = d$. Now if $(m, J) \geqq (x, I)$, then $J \subset I$ and from $\Phi(m, J) = m \in I$ we have that $z(I) \leqq \Phi(m, J) \leqq w(I)$. Hence $c \leqq \Phi(m, J) \leqq d$ for $(m, J) \leqq (x, I)$. Therefore, Φ o-converges and Θ-converges to x. Finally, if $N \in \eta(x)$, then there exists $(n, I) \in E$ such that if $(m, J) \geqq (n, I)$, then $\Phi(m, J) \in N$. Since $I = \bigcup \{\Phi(m, j) \mid (m, J) \geqq (n, I)\}$, we have that $I \subset N$. Therefore, X is locally-intervaled. //

Let X be a complete lattice and x an element in X. An element a is called an *unavoidable lower bound* for x if for each up-directed subset D of X with $\sup D = x$, there exists a member d of D such that $a \leqq d$. We let $\lambda(x)$ denote the set of all unavoidable lower bounds for x. An unavoidable upper bound for x and the set $v(x)$ are dually defined. Notice that for each x in X, $O \in \lambda(x)$ and $I \in v(x)$ and also that $\lambda(x)$ and $v(x)$ are up-directed and down-directed, respectively.

The foregoing definition receives motivation from two sources. First, in any complete chain C with the intrinsic neighborhood topology, if $x \in C$, then any element a, with $a < x$, is an unavoidable lower bound for x. (This is not true in an arbitrary lattice.) This explains the terminology. Second, by simultaneously weakening the definition of a compact element and partially strengthening the definition of a join-inaccessible element [2″, p. 186], we may arrive at the definition of an unavoidable lower bound.

With this new terminology, we may restate Theorem 2.9 as follows: Let X be a convergence lattice and x an element of X. Then $B_x = \{[a, b] \mid a \in \lambda(x), b \in v(x)\}$ constitutes a base for the local neighborhood system at x.

COROLLARY 2.10. *Let X be a complete lattice. Then X is a convergence lattice if and only if for each x in X, $\sup \lambda(x) = x = \inf v(x)$.*

Proof. By Proposition 1.4, if $N \in \eta(x)$, then $\inf N$ is an unavoidable lower bound for x and $\sup N$ is an unavoidable upper bound for x. The condition then becomes a reformulation of Theorem 2.5. //

The foregoing corollary suggests a pure lattice-theoretic characterization of convergence lattices ("pure" in the sense that only lattice operations are involved). Thus, it provides an alternate definition of a convergence lattice and creates an avenue to a lattice-theoretic (as opposed to a topological) study of such lattices. Since this work bears mainly on topological aspects of convergence lattices, this algebraic path will not be systematically explored at this time.

The following theorem shows that the concept of being locally-intervaled completely characterizes the subclass of convergence lattices among complete lattices.

Theorem 2.11. *A complete lattice X is a convergence lattice if and only if it is locally-intervaled.*

Proof. Necessity is precisely Theorem 2.9. As in the proof of Corollary 2.10, for any x in X and any N in $\eta(x)$, $\inf N$ is an unavoidable lower bound for x. Now suppose that there exists some element x in X with $\sup \lambda(x) < x$. Set $y = \sup \lambda(x)$ and $G = X \nabla [O, y]$. Then G is a neighborhood of x. Therefore, there exists an interval $[a, b] \subset G$ with $[a, b]$ a neighborhood of x. Since $a \notin [O, y]$, $a \nleqq y$. But $a \in \lambda(x)$ and $a \leqq y$, a contradiction. Therefore, $y \nless x$, and $\sup \lambda(x) = x$ for each x in X. A dual argument and Corollary 2.10 complete the proof. //

Corollary 2.10 also yields as an immediate corollary a result due to Kent and Atherton [**12**, Theorem 1]. First we recall the following definitions. A lattice X is called compactly-generated if each element in X is the supremum of a set of compact elements. A co-compactly-generated lattice is dually defined and, furthermore, is said to be bicompactly-generated if it is also compactly-generated.

Corollary 2.12. *A bicompactly-generated complete lattice is a convergence lattice.*

Proof. A compact element c, with $c \leqq x$, is an unavoidable lower bound for x. A similar observation for co-compact elements and Corollary 2.10 complete the proof. //

We shall now proceed to attempt a description of those subsets of a convergence lattice that are themselves convergence lattices (in the induced order). A sublattice A of a convergence lattice X is called a *sub-convergence lattice* of X if A is a complete lattice in the induced order and if the induced order convergence (i.e., o_A-convergence) agrees with Θ-convergence in X.

For example, if $X = [-1, 1]$ and $B = [-1, 0) \cup \{1\}$, then B is a complete lattice (indeed, a convergence lattice) in the induced order, but o_B-convergence does not agree with Θ-convergence (viz., the sequence x_n, where $x_n = -1/n$, is such that x_n o_B-converges to 1, but x_n Θ-converges to 0 in X. Therefore, B is not a sub-convergence lattice of X. However, if we let $A = B \cup \{0\}$, then A is a sub-convergence lattice of X.

THEOREM 2.13. *Let X be a convergence lattice and A a sublattice of X. Then the following statements are equivalent:*

(1) *A is Θ-closed.*

(2) *A is a sub-complete lattice of X, i.e., if $S \subset A$, then* sup S *and* inf S *are in A.*

(3) *A is a sub-convergence lattice of X.*

PROOF. (1) implies (2): Suppose A is Θ-closed. Let $S \subset A$ and set $s = \sup S$. If S is finite, $s \in A$ since A is a sublattice. If S is infinite, let $D = \{J \subset S \mid J \text{ is finite}\}$ and direct D by setting $K \leqq J$ if $K \subset J$. Define a net Φ on D in A by $\Phi(J) = \sup J$. Then Φ is isotone and if we set $s = \sup S$, then $\sup \Phi(J) = s$. For if $t = \sup \Phi(J)$, then $t \leqq s$. If $t < s$, then for any $x \in S$, $\{x\} \in D$ and $\Phi(\{x\}) = x \leqq t$. Thus $s = \sup A \leqq t$. Therefore, by antisymmetry, $t = s = \sup S$ and Φ o-converges to s. Hence, Φ Θ-converges to s. Since A is Θ-closed, $s \in S$. Similar considerations show that $\inf S \in A$. Therefore, A is a sub-complete lattice of X.

(2) implies (3). Suppose A is a sub-complete lattice of X. Then by definition A is a complete lattice in the induced order. Let $\Phi(\alpha)$ be a net in A that Θ-converges to x. Thus, $\text{lim-inf}\,\Phi(\alpha) = x = \text{lim-sup}\,\Phi(\alpha)$. As all infima and suprema taken in A agree with those taken in X, we have that $\Phi(\alpha)$ o_A-converges to x. Conversely, if $\Phi(\alpha)$ o_A-converges to x, then $\Phi(\alpha)$ o-converges to x, hence Θ-converges to x. Thus both types of convergence agree and A is a sub-convergence lattice of X.

(3) implies (1). Let A be a sub-convergence lattice of X. Let Φ be a net in A Θ-converging to x. Since Θ-convergence agrees with o_A-convergence, Φ o_A-converges to x. Since A is complete in the induced order, $x \in A$. Thus, A is Θ-closed. //

COROLLARY 2.14. *Let X be a convergence lattice and A a sub-convergence lattice of X. Then Θ_A-convergence (where Θ_A is the intrinsic order topology on A) and $\Theta|_A$-convergence (where $\alpha|_A$ is the relative topology induced on A by Θ) coincide.*

PROOF. By Theorem 2.13, A is Θ-closed. It is well known (and easy to show) that for a net Φ in A, Φ Θ-converges to x if and only if Φ $\Theta|_A$-converges to x, that is, for a closed subspace of a topological space, convergence in the subspace coincides with convergence in the whole space. Thus, $\Theta|_A$-convergence agrees with o_A-convergence. Therefore, o-convergence is derived from a topology on A and Θ_A-convergence agrees with o_A-convergence as asserted. //

The foregoing corollary says that the relative topology on a sub-convergence lattice agrees with its induced order topology. Thus, each sub-convergence lattice is a convergence lattice (in the induced order). Examples of sub-convergence lattices are singletons, intervals, and maximal chains.

THEOREM 2.15. *Any complete chain is a convergence lattice.*

PROOF. It is known (e.g., [2″, pg. 241]) that in a (complete) chain, the order topology coincides with the intrinsic neighborhood topology obtained by taking the open intervals $(a, b) = [a, b] \nabla \{a, b\}$ as a base for the open sets. Now suppose $\Phi(\alpha)$ is a net Θ-converging to x. This means that for any interval (a,b) containing x, there exists β such that $\alpha \geqq \beta$ implies that $\Phi(\alpha) \in (a, b)$. Thus, $\bigwedge_{\alpha>\beta}\Phi(\alpha) \geqq a$, which in turn implies that lim-inf$\,\Phi(\alpha) \geqq a$. Since this is true for any a with $a < x$, we have that lim-inf$\,\Phi(\alpha) \geqq x$ by the completeness of X. Dual observations show that lim-sup$\,\Phi(\alpha) \leqq x$, so that we have $x \leqq$ lim-inf$\,\Phi(\alpha) \leqq$ lim-sup$\,\Phi(\alpha) \leqq x$. Therefore, Φ o-converges to x. //

THEOREM 2.16. *Let X_λ be a complete lattice for each $\lambda \in \Lambda$. Let $X = \prod \{X_\lambda \mid \lambda \in \Lambda\}$ and order X with the product (i.e., coordinatewise) ordering. Then X is a convergence lattice if and only if each X is a convergence lattice.*

PROOF. Let $|\cdot|_\lambda$ denote the (order-preserving) projection of X into X_λ. A direct verification shows that X is a complete lattice. Furthermore, if A is a subset of X, then $\sup|A|_\lambda = |\sup A|_\lambda$ and $\inf|A|_\lambda = |\inf A|_\lambda$. To show the first equality, for instance, let $x \in A$. Then $|x|_\lambda \leqq |\sup A|_\lambda$ whence $\sup|A|_\lambda \leqq |\sup A|_\lambda$. If $c \in X$ is such that $|c|_\lambda \geqq |x|_\lambda$ for each $x \in A$ and each $\lambda \in \Lambda$, then $c \geqq x$, wherefore, $c \geqq \sup A$. Thus, $|c|_\lambda \geqq |\sup A|_\lambda$. Therefore, $\sup|A|_\lambda = |\sup A|_\lambda$.

Now assume that X is a convergence lattice. For each β in Λ, set $B_\beta = \prod \{A_\lambda \mid \lambda \in \Lambda\}$ where $A_\beta = X_\beta$ and $A_\lambda = \{O_\lambda\}$ for $\lambda \neq \beta$ (O_λ is the zero in X_λ). Then B_β is a sub-complete lattice of X, which, by Corollary 2.14, has its induced order topology coinciding with the relative topology from X. As X_β and B_β are order homomorphic and homemorphic, we have that each X_β is a convergence lattice.

To show the converse, we assume that each X_λ is a convergence lattice. We now proceed to show that for a net $\Phi(\alpha)$ in X, Φ o-converges to x in X if and only if $|\Phi(\alpha)|_\lambda$ o-converges to $|x]_\lambda$ in X_λ for each λ in Λ.

First suppose that $\Phi(\alpha)$ o-converges to x in X. Then there exist subsets A and B of X such that A is up-directed, B is down-directed, $\sup A = x = \inf B$, and for any $a \in A$ and $b \in B$, there exists β such that $\alpha \geqq \beta$ implies that $a \leqq \Phi(\alpha) \leqq b$. Then for each λ, $|A|_\lambda$ is up-directed, $|B|_\lambda$ is down-directed, and, by the first paragraph of this proof, $\sup|A|_\lambda = |x|_\lambda = \inf|B|_\lambda$. Moreover, if $s \in |A|_\lambda$ and $t \in |B|_\lambda$, then there exists $a \in A$ and $b \in B$ such that $|a|_\lambda = s$ and $|b|_\lambda = t$. Hence, there exists β such that $\alpha \geqq \beta$ implies that $a \leqq \Phi(\alpha) \leqq b$. Consequently, $s = |a|_\lambda \leqq |\Phi(\alpha)|_\lambda \leqq |b|_\lambda = t$. Therefore, $|\Phi(\alpha)|_\lambda$ o-converges to $|x|_\lambda$ for each λ in Λ.

On the other hand, assume that $\Phi(\alpha)$ is a net in X such that for each λ, $|\Phi(\alpha)|_\lambda$ o-converges to x_λ in X_λ. We define x in X such that $|x|_\lambda = x_\lambda$ for each $\lambda \in \Lambda$ and intend to show that $\Phi(\alpha)$ o-converges to x. For each λ, there exist subsets A_λ and B_λ of X_λ such that A_λ is up-directed, B_λ is down-directed, and $\sup A_\lambda = |x|_\lambda = \inf B_\lambda$. Define $A = \{a \in X \mid |a|_\lambda \in A_\lambda$ for finitely many λ and $|a|_\lambda = O_\lambda$ for all other $\lambda\}$ and $B = \{b \in X \mid |b|_\lambda \in B_\lambda$ for finitely many λ and $|b|_\lambda = I_\lambda$ for all other $\lambda\}$.

Claim: A is up-directed and B is down-directed and $\sup A = x = \inf B$. For let s and t be in A. Then there exist finite subsets S and T of Λ for which $|s|_\lambda \in A_\lambda$ if $\lambda \in S$ and $|s|_\lambda = O_\lambda$ otherwise, and $|t|_\lambda \in A_\lambda$ if $\lambda \in T$ and $|s|_\lambda = O_\lambda$ otherwise. Setting $R = S \cup T$, R is a finite subset of Λ. Define a in X such that $|a|_\lambda = O_\lambda$ for $\lambda \notin R$ and $|a|_\lambda \geqq |s|_\lambda$ and $|a|_\lambda \geqq |t|_\lambda$ for $\lambda \in R$ (this can be done since R is finite and each A_λ is up-directed). Then $a \in A$, $a \geqq s$, and $a \geqq t$. In a similar fashion, B is shown to be down-directed. Since $|\sup A|_\lambda = \sup|A|_\lambda = |x|_\lambda$, $\sup A = x$. Similarly, $\inf B = x$.

Now let $a \in A$ and $b \in B$ and let C and D be finite subsets of Λ such that $|a|_\lambda \in A_\lambda$ for $\lambda \in C$, $|b|_\lambda \in B_\lambda$, $|a|_\lambda = O_\lambda$ for $\lambda \notin C$, and $|b|_\lambda = I_\lambda$ for $\lambda \notin D$. Set $E = C \cup D$. Then for any $\lambda \in E$, there exists β_λ such that for $\alpha \geqq \beta_\lambda$, $|a|_\lambda \leqq |\Phi(\alpha)|_\lambda \leqq |b|_\lambda$. Let $\beta \geqq \beta_\lambda$ for $\lambda \in E$ (note that E is finite). Then $|a|_\lambda \leqq |\Phi(\alpha)|_\lambda \leqq |b|_\lambda$ for $\alpha \geqq \beta$ and $\lambda \in E$. But for $\lambda \notin E$, $|a|_\lambda = O_\lambda \leqq |\Phi(\alpha)|_\lambda \leqq I_\lambda = |b|_\lambda$ for $\alpha \geqq \beta$. Thus, $a \leqq \Phi(\alpha) \leqq b$ for $\alpha \geqq \beta$. Therefore, Φ o-converges to x in X.

In summary, $\Phi(\alpha)$ o-converges to x in X if and only if $|\Phi(\alpha)|_\lambda$ o-converges to $|x|_\lambda$ in X_λ for each λ in Λ. But $|\Phi(\alpha)|_\lambda$ o-converges to $|x|_\lambda$ in X_λ if and only if $|\Phi(\alpha)|_\lambda$ Θ-converges to $|x|_\lambda$ since each X_λ is a convergence lattice. As a net in the product topology converges if and only if each projected net converges, we have that $|\Phi(\alpha)|_\lambda$ Θ-converges to $|x|_\lambda$ if and only if $|\Phi(\alpha)|_\lambda$ T-converges to $|x|_\lambda$ where T is the product topology on X generated by the order topology on each X_λ. Frink [7] has shown that the order topology of a lattice product is the product topology generated by the order topology on each coordinate lattice, i.e., $T = \Theta$ on X. Thus, Φ o-converges to x in X if and only if Φ Θ-converges to x in X. //

As an application of Theorems 2.13 and 2.16, we present the following example of a generalized Helly space (cf. Kelly [10, Problem 5.M, pg. 164]).

COROLLARY 2.17. *Let X and Y be convergence lattices and define H as the set of all isotone functions on X into Y. Then H is a convergence lattice.*

PROOF. Let Z denote $\prod\{Y_x \mid Y_x = Y \text{ and } x \in X\}$. By Theorem 2.16, Z is a convergence lattice. Claim: H is a sublattice of Z. For if f and g belong to H, then $f \wedge g$ may be defined by $[f \wedge g](x) = f(x) \wedge g(x)$ for $x \in X$. Then $f \wedge g$ is in H for if $x \leqq y$, then $f(x) \wedge g(x) \leqq f(y) \wedge g(x) \leqq f(y) \wedge g(y)$. It is clear that $f \wedge g \leqq f$ and $f \wedge g \leqq g$. If $h \leqq f$ and $h \leqq g$, then $h(x) \leqq f(x) \wedge g(x) = [f \wedge g](x)$. Thus, $f \wedge g = \inf\{f, g\}$. Furthermore, H is Θ-closed in Z. For if $f \in Z \nabla H$, then f is not isotone, i.e., there exist x and y in X such that $x < y$, but $f(x) \not\leqq f(y)$. Since Y is Θ-Hausdorff, there exist open sets U and V such that $f(x) \in U, f(y) \in V$, $U \cap V = \emptyset$, and if $z \in U$ and $w \in V$, then $z \not\leqq w$. The last statement follows from a result of Nachbin [13, § 1, Prop. 1] that states that in a convergence lattice, if $a \not\leqq b$, then there exist $U \in \eta(a)$ and $V \in \eta(b)$ such that U is increasing, V is decreasing, and $U \cap V = \emptyset$. Letting $\bar{\pi}_z$ denote the inverse set mapping of the z-th projection from Z into X, we see that $N = \bar{\pi}_x(U) \cap \bar{\pi}_y(V)$ is a neighborhood of f in Z. If $g \in N$, then $g(x) \not\leqq g(y)$ since $g(x) \in U$ and $g(y) \in V$. Hence $N \cap H = \emptyset$. Therefore, H is closed in Z. By Theorem 2.13, H is a convergence lattice. //

Given a collection of complete lattices, there are several standard ways of constructing new partially ordered sets from the old. Theorem 2.16 states that the cardinal product [2″, pg. 55] of an arbitrary collection of convergence lattices is also a convergence lattice. However, since the cardinal sum of two lattices [2″, pg. 55] is not even a lattice, the cardinal sum of two convergence lattices is not a convergence lattice. The cardinal power [2″, pg. 55] Y^X with base Y and exponent X has been shown to be a convergence lattice whenever X and Y are convergence lattices (Corollary 2.17).

The ordinal product [2″, pg. 199] and the ordinal sum [2″, pg. 198] of two complete lattices are easily shown to be complete lattices [2″, pg. 201, exercise 10]. The next two propositions show that the ordinal product and sum of two convergence lattices are also convergence lattices.

PROPOSITION 2.18. *Let X and Y be two convergence lattices. Then the ordinal product $Z = X \circ Y$ of X and Y is also a convergence lattice.*

PROOF. Let (x, y) be an element of Z and C any up-directed subset of Z such that $\sup C = (x, y)$. If $D = \{d \in X \mid (d, e) \in C$ for some e in $Y\}$, then D is an up-directed subset of X and $\sup D = x$. We intend to use Corollary 2.10 to effect a proof, but actually show only that $\sup \lambda(x, y) = (x, y)$.

We first assume that $x \in \lambda(x)$. Claim: if $b \in \lambda(y)$, then $(x, b) \in \lambda(x, y)$. Notice that since $x \in \lambda(x)$, $x \in D$. Now let $E = \{e \in Y \mid (x, e) \in C\}$. Then E is an up-directed subset of Y and $\sup E = y$. Therefore, there exists some e in E such that $b \leqq e$. Thus, $(x, b) \leqq (x, e)$ and $(x, b) \in \lambda(x, y)$. If we let $B = \{(x, b) \mid b \in \lambda(y)\}$, then $\sup B = (x, \sup \lambda(y)) = (x, y) \geqq \sup \lambda(x, y) \geqq \sup B$. Therefore, $\sup \lambda(x, y) = (x, y)$.

We now assume that $x \notin \lambda(x)$. Claim: if $a \in \lambda(x)$, then $(a, y) \in \lambda(x, y)$. Since $\sup D = x$ and $a \in \lambda(x)$, there exists some d in D with $a \leqq d$. Letting $F = \{f \in D \mid f \geqq d\}$, then $\sup F = x$ and $f \geqq a$ for each $f \in F$. Clearly, there exists some particular $g \in F$ such that $g > a$ since $a < x$. For this g, there exists some $e \in Y$ such that $(g, e) \in C$. Thus, $(a, y) \leqq (g, e)$. Therefore, $(a, y) \in \lambda(x, y)$. If we let $A = \{(a, y) \mid a \in \lambda(x)\}$, then $\sup A = (\sup \lambda(x), y) = (x, y) \geqq \sup \lambda(x, y) \geqq \sup A$. Therefore, $\sup \lambda(x, y) = (x, y)$. //

PROPOSITION 2.19. *For two convergence lattices X and Y, the ordinal sum $Z = X \oplus Y$ is also a convergence lattice.*

PROOF. Observe that X and Y are disjoint sub-convergence lattices of Z whose union is Z. Now suppose that $\Phi(\alpha)$ is a net in Z Θ-converging to z. Assume first that $z \leqq I_X$. Since X is open in Z, $\Phi(\alpha)$ is eventually in X, that is, there exists δ in the domain of Φ such that $\alpha \geqq \delta$ implies $\Phi(\alpha)$ is in X. Then the net $(\Phi(\alpha); \alpha \geqq \delta)$ in X Θ-converges to z. Thus, by Theorem 2.13, $\bigvee_{\gamma \geqq \delta} \bigwedge_{\alpha \geqq \gamma} \Phi(\alpha) = z = \bigwedge_{\gamma \geqq \delta} \bigvee_{\alpha \geqq \gamma} \Phi(\alpha)$. This gives $z = \bigvee_{\gamma \geqq \delta} \bigwedge_{\alpha \geqq \gamma} \Phi(\alpha) \leqq \bigvee_{\gamma} \bigwedge_{\alpha \geqq \gamma} \Phi(\alpha) \leqq \bigwedge_{\gamma} \bigvee_{\alpha \geqq \gamma} \Phi(\alpha) \leqq \bigwedge_{\gamma \geqq \delta} \bigvee_{\alpha \geqq \gamma} \Phi(\alpha) = z$. Therefore, Φ o-converges to z in Z. A similar argument works if $z \geqq O_Y$. //

We now give another example of a family of complete lattices each member of which is a convergence lattice. (See Examples 3.1 and 3.3.)

THEOREM 2.20. *A lattice in which each chain is finite is a convergence lattice.*

PROOF. When all chains in a lattice are finite, both the ascending and descending chain conditions are satisfied. In this case, each element is both compact and co-compact. Applying Corollary 2.10 completes the proof. //

COROLLARY 2.21. *Each lattice of finite length is a convergence lattice.*

§ 3. **Examples.** This section contains four examples of convergence lattices (Examples 3.1 to 3.4) and four examples of complete lattices that fail to be convergence lattices (Examples 3.5 to 3.8). Some of these examples are discussed in previous sections of this paper.

EXAMPLE 3.1. Every finite lattice is a discrete compact convergence lattice. These lattices include examples which display the presence or absence of various lattice properties (e.g., modularity, distributivity, and complementedness) in reasonable combination.

EXAMPLE 3.2. Complete chains and products of complete chains are all convergence lattices. Thus, we have the following examples of convergence lattices (in their natural order):

(1) The unit interval $J = [0, 1]$.

(2) The real subset $\{1/n \mid n \text{ is a natural number}\} \cup \{0\}$.

(3) The Cantor ternary set.

(4) The closed ordinal space $[0, \Omega]$ where Ω is the first uncountable ordinal.

(5) The Euclidean n-cell J^n.

(6) The Hilbert Cube J^ω where ω is the first infinite ordinal.

(7) The Tychonoff plank $[0, \Omega] \times [0, \omega]$.

EXAMPLE 3.3. Let W be an infinite set. Let $X = W \cup \{O, I\}$ and order X as follows: any two elements in W are incomparable and $O \leqq w \leqq I$ for every $w \in W$. Then X is a non-compact discrete convergence lattice by Theorem 2.19. X is modular and complemented, but not distributive. Also, X is a complete lattice in which every element is both compact and co-compact and X itself is bi-compactly generated.

EXAMPLE 3.4. Let $X = \{(a, b) \mid a = 0, 1, \text{ or } b;\ b \in [0, 1]\}$. As a subset of the plane with the usual ordering, X is a distributive compact connected convergence lattice that is not a topological lattice. Hence, X is not infinitely-distributive. This example contradicts a claim made by Frink [7, Theorem 2] stating that any distributive lattice is a topological lattice with respect to the order topology.

EXAMPLE 3.5. Let I denote the interior of the unit square in the Euclidean plane. Let $Y = I \cup \{(0,0), (1,1)\}$. Then Y is not a convergence lattice since $(0,0)$ and $(1,1)$ cannot be topologically separated (i.e., Θ is not Hausdorff).

Although Y is not a Boolean algebra, it does represent a conceptually simpler example of a non-Θ-Hausdorff lattice than the classic one due to Floyd [**6**, Theorem 1] that deals with the complete Boolean algebra B of all regular open subsets of the unit interval. Incidentally, Floyd's example answers (negatively) a question recently posed by Strauss [**16**, pg. 230, question (iii)]: "Is every [topological] lattice Hausdorff in its order topology?"

EXAMPLE 3.6. Let $A = \{(x, 0) \mid x \in [0, 1)\}$, $B = \{(0, y) \mid y \in [0, 1)\}$, and $Y = A \cup B \cup \{(1, 1)\}$. Under the coordinate-wise order, Y is a complete lattice. Θ is Hausdorff and compact and $I = \Theta$, yet Y is not a convergence lattice. Notice that Y is homeomorphic, but not homomorphic, to the unit circle of the plane.

We use this example to show that, in general, o-convergence does not form a convergence class (see § 1 for definitions). We may show that C4 is not satisfied in Y by first letting D denote the positive integers, P the non-negative integers, and $E_m = D$ for each m in D. Define the sets E and B and the net ψ as in the statement of condition C4. Define the net Φ by $\Phi(m, n) = (1/n, 0)$ if m is even and $(0, 1/n)$ otherwise. Then $o\text{-}\lim_m o\text{-}\lim_n \Phi(m, n) = O$. However, $\Phi \circ \psi$ fails to o-converge since $\text{lim-inf}\, \Phi \circ \psi = O$ and $\text{lim-sup}\, \Phi \circ \psi = I$. It is also easy to show that o-convergence in Y fails to satisfy C3.

A curious thing about this example is that o-closure is indeed a closure operator. To demonstrate this we first define the sets $V = \{(a, b) \in Y \mid a = 0\} \cup \{I\}$ and $H = \{(a, b) \in Y \mid b = 0\} \cup \{I\}$. Then both V and H are order homomorphic and homeomorphic to the unit interval $J = [0, 1]$ with the usual (metric) topology μ on J. It is well known that o-convergence in V and H agrees with convergence with respect to μ in J. Therefore, o-closure of a subset B of V (or H) is the same as its μ-closure in J. Hence $o\text{-cl}(o\text{-cl}(B)) = o\text{-cl}(\mu\text{-cl}(B)) = \mu\text{-cl}(\mu\text{-cl}(B)) = \mu\text{-cl}(B) = o\text{-cl}(B)$. Now let A be any subset of Y. Since $V \cup H = Y$, we have $A = A \cap Y = A \cap (V \cup H) = (A \cap V) \cup (A \cap H)$. Thus, $o\text{-cl}(A) = o\text{-cl}(A \cap V) \cup o\text{-cl}(A \cap H)$ and $o\text{-cl}(o\text{-cl}(A)) = o\text{-cl}(o\text{-cl}(A \cap V) \cup o\text{-cl}(o\text{-cl}(A \cap H)) = o\text{-cl}(A \cap V) \cup o\text{-cl}(A \cap H) = o\text{-cl}(A)$. Therefore, o-closure is idempotent and satisfies all of the axioms for a closure operator.

EXAMPLE 3.7. Let Y denote the collection of all non-negative real sequences $\{x_n\}$ (partially ordered component-wise) satisfying the condition $\sup x_n < 1$ together with the element I that has 1 for each component. Then Y is a complete lattice. Since o-closure is a (topo--logical) closure operator in a convergence lattice, we show that Y is not a convergence lattice by showing that o-closure in Y is not idempotent. Let A denote the collection of elements $y(m, n)$ in Y for m, $n = 1, 2, 3, \cdots$, where the k-th component of $y(m, n)$ is defined as defined as follows:

$$\begin{array}{ll} 1/n & \text{if } k = 1, \\ 0 & \text{if } 2 \leqq k \leqq m, \\ 1 - 1/n & \text{if } k \geqq m + 1. \end{array}$$

Then for a fixed n, $y(m, n)$ o-converges to $x_n = (1/n, 0, 0, \cdots)$. Thus, x_n belongs to $o\text{-cl}(A)$ for each n. Also, x_n o-converges to $O = (0, 0, 0, \cdots)$. Hence O belongs to $o\text{-cl}(o\text{-cl}(A))$. But $O \notin o\text{-cl}(A)$ as no infinite sequence of distinct elements in A is o-convergent to O. Therefore, $o\text{-cl}(o\text{-cl}(A)) \neq o\text{-cl}(A)$.

Example 3.8. The following sets are constructed in the Euclidean plane: $H = \{(x, y) \mid 0 \leqq x \leqq 5,\ 0 \leqq y \leqq 2\}$, $K = \{x, y) \mid 0 \leqq y \leqq 5,\ 0 \leqq x \leqq 2\}$, and $L = \{(4, 4), (5, 5)\}$. Let $Y = H \cup K \cup L$. Then Y is a complete lattice that is not a convergence lattice since it contains a homeomorphic copy of the lattice of Example 3.6. This example is drawn from a paper by Kent [11, example 2] and corrects an error made there. (It is claimed in [11] that Y is a convergence lattice.)

§ 4. Unsolved Problems.

1. Show that conditions C3 and C4 of Section 1 are independent in a complete lattice.
2. A complete lattice is Θ-Hausdorff if and only if each Θ-convergent net contains an o-convergent subnet.
3. Every (distributive?) convergence lattice is a normal topological space.
4. A complemented convergence lattice is not connected.

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IBM Boulder Laboratory, Boulder, Colorado 80302