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BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS AND APPLICATIONS TO SINGULAR PERTURBATION PROBLEMS ON $[a, b] \subset (-\infty, \infty]$

TAI-CHI LEE

Throughout this paper we will assume that $L = D^2 + p(t)D + q(t)$ is disconjugate on $[a, b]$, where $D = d/dt$. Let (u_1, u_2) be a fundamental principal system of solutions of $Ly = 0$ on $[a, b]$.

Let $u(t) = u_1(t) + u_2(t)$. Define

$$(1) \qquad \bar{\mathfrak{D}}^0 y(t) = \lim_{s \to t} \frac{y(s)}{u(s)}, \quad \bar{\mathfrak{D}}^1 y(t) = \lim_{s \to t} \frac{W(u, y)}{W(u_1, u_2)}(s),$$

$$(2) \qquad \mathfrak{D}^0 y(a) = \lim_{s \to a+} \frac{y(s)}{u_1(s)}, \quad \mathfrak{D}^1 y(a) = \lim_{s \to a+} \frac{W(u_1, y)}{W(u_1, u_2)}(s),$$

$$(3) \qquad \mathfrak{D}^0 y(b) = \lim_{s \to b-} \frac{y(s)}{u_2(s)}, \quad \mathfrak{D}^1 y(b) = \lim_{s \to b} \frac{W(y, u_2)}{W(u_1, u_2)}(s),$$

for each $y \in C^1(a, b)$, and $a \leqq t \leqq b$, where $W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2$.

Assume $\alpha, \beta \in C^2(a, b)$, $\alpha \leqq \beta$, $f(t, y, y') \in C(G)$, where $G = \{(t, y, y') : a < t < b, \alpha(t) \leqq y \leqq \beta(t)$, and $|y'| < \infty\}$, $\mathfrak{D}^0\alpha(a)$, $\mathfrak{D}^0\alpha(b)$, $\mathfrak{D}^0\beta(a)$, $\mathfrak{D}^0\beta(b)$ exist, and $\bar{\mathfrak{D}}^1\alpha(t)$, $\bar{\mathfrak{D}}^1\beta(t)$ are bounded on (a, b). Let

$$(4) \qquad \lambda = \max(\mathfrak{D}^0\beta(b) - \mathfrak{D}^0\alpha(a), \mathfrak{D}^0\beta(a) - \mathfrak{D}^0\alpha(b)),$$

$$(5) \qquad F(t, y, z) = f\left(t, y, \frac{u'(t)y + W(u_1, u_2)(t)z}{u(t)}\right).$$

Definition. $f(t, y, y')$ is said to satisfy a generalized Nagumo condition on (a, b) with respect to the pairs (u_1, u_2) and (α, β) if there exists a positive continuous function $\phi(s)$ on $[\lambda, \infty)$ such that

$$(6) \qquad \frac{u^3(t)}{W^2(u_1, u_2)(t)} |F(t, y, z)| \leqq \phi(|z|), \text{ for } \alpha(t) \leqq y \leqq \beta(t),$$

$$a < t < b, \ |z| < \infty,$$

and

$$(7) \qquad \int_\lambda^\infty \frac{s}{\phi(s)}\, ds > \sup_{a<t<b} \frac{\beta(t)}{u(t)} - \inf_{a<t<b} \frac{\alpha(t)}{u(t)} = K,$$

where λ is as in (4).

Let N be such that

$$N > \lambda \tag{8}$$

$$\int_\lambda^N \frac{s}{\phi(s)}\,ds > K \tag{9}$$

$$|\mathfrak{D}^1\alpha(t)|, |\mathfrak{D}^1\beta(t)| < N \text{ on } [a, b). \tag{10}$$

Assumption A. The function $f(t, y, y') \in C((a, b) \times R^2)$, α and $\beta \in C^2(a, b)$, and, for $a < t < b$, $L\beta \leqq f(t, \beta, \beta')$, $L\alpha \geqq f(t, \alpha, \alpha')$, $\alpha(t) \leqq \beta(t)$. Also $f(t, y, y')$ satisfies a generalized Nagumo condition on (a, b) with respect to α and β. Let N be such that (8)–(10) hold, $k(t)$ be any positive continuous function such that $v_i k \in L^1[a, b]$, $i = 1, 2$, and assume $v_i\Lambda$, $v_i\lambda \in L^1[a, b]$, $i = 1, 2$, where

$$\begin{aligned}
v_1(t) &= u_2(t)/W(u_1, u_2)(t),\\
v_2(t) &= u_1(t)/W(u_1, u_2)(t),\\
\lambda(t) &= \inf\{F(t, y, z) : \alpha(t) \leqq y \leqq \beta(t),\\
&\quad |z| \leqq N, a < t < b\} - (\pi/2)k(t),\\
\Lambda(t) &= \sup\{F(t, y, z) : \alpha(t) \leqq y \leqq \beta(t),\\
&\quad |z| \leqq N, a < t < b\} + (\pi/2)k(t).
\end{aligned}$$

Let

$$M_0 = \mathfrak{D}^0\beta(b) - \int_a^b v_2(s)\lambda(s)\,ds,\ N_0 = \mathfrak{D}^0\alpha(b) - \int_a^b v_2(s)\Lambda(s)\,ds, \tag{11}$$

$$M_1 = \mathfrak{D}^0\beta(a) - \int_a^b v_1(s)\lambda(s)\,ds,\ N_1 = \mathfrak{D}^0\alpha(a) - \int_a^b v_1(s)\Lambda(s)\,ds, \tag{12}$$

$$J_0 = \{(x, y) : \mathfrak{D}^0\alpha(a) \leqq x \leqq \mathfrak{D}^0\beta(a), N_0 \leqq y \leqq M_0\}, \tag{13}$$

$$J_1 = \{(x, y) : \mathfrak{D}^0\alpha(b) \leqq x \leqq \mathfrak{D}^0\beta(b), N_1 \leqq y \leqq M_1\}, \tag{14}$$

$$\begin{aligned}
\Gamma_0 = &\{(\mathfrak{D}^0\alpha(a), y) : N_0 \leqq y < \mathfrak{D}^1\alpha(a)\}\\
&\cup \{(\mathfrak{D}^0\beta(a), y) : \mathfrak{D}^1\beta(a) < y \leqq M_0\},
\end{aligned} \tag{15}$$

$$\begin{aligned}
\Gamma_1 = &\{(\mathfrak{D}^0\alpha(b), y) : N_1 \leqq y < \mathfrak{D}^1\alpha(b)\}\\
&\cup \{(\mathfrak{D}^0\beta(b), y) : \mathfrak{D}^1\beta(b) < y \leqq M_1\}.
\end{aligned} \tag{16}$$

THEOREM 1. *Let Assumption A hold and assume that a is finite. If, for $i = 0, 1$, the continuum Ω_i is contained in J_i and intersects both lines $y = M_i$, $y = N_i$, then there exists a solution $y(t)$* of $Ly = f(t, y, y')$ *such that*

$$(\mathfrak{D}^0 y(a), \mathfrak{D}^1 y(a)) \in \Omega_0 - \Gamma_0,$$

$$(\mathfrak{D}^0 y(b), \mathfrak{D}^1 y(b)) \in \Omega_1 - \Gamma_1,$$

and

$$\alpha(t) \leqq y(t) \leqq \beta(t) \text{ for } t \in (a, b),$$

where b is possibly a finite singular point or ∞.

In the proof of theorem, we first obtain a solution to a slightly modified problem by using the Tonelli procedure for demonstrating the existence of solution to initial value problems and the intermediate value theorem for continuous functions. Finally, by means of differential inequalities, we conclude that the solution to the modifying equation is a solution to the problem considered.

COROLLARY 1. *Let Assumption A hold. Then the boundary value problem*

$$Lz = f(t, z, z'), a < t < b,$$

$$a_1 \mathfrak{D}^0 z(a) + a_2 \mathfrak{D}^1 z(a) = A_0,$$

$$b_1 \mathfrak{D}^0 z(b) + b_2 \mathfrak{D}^1 z(b) = A_1,$$

where $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \neq 0$, $A_0, A_1 \in R^1$, *has a solution*

$$\alpha(t) \leqq z(t) \leqq \beta(t) \quad \textit{for} \quad a < t < b,$$

provided

$$a_2^{-1}[a_1 \mathfrak{D}^0 \alpha(a) + a_2 \mathfrak{D}^1 \alpha(a) - A_0] \geqq 0,$$

$$a_2^{-1}[a_1 \mathfrak{D}^0 \beta(a) + a_2 \mathfrak{D}^1 \beta(a) - A_0] \leqq 0,$$

$$\textit{if } a_2 \neq 0$$

or

$$\mathfrak{D}^0 \alpha(a) \leqq A_0/a_1 \leqq \mathfrak{D}^0 \beta(a) \quad \textit{if } a_2 = 0,$$

and

$$b_2^{-1}[b_1 \mathfrak{D}^0 \alpha(b) + b_2 \mathfrak{D}^1 \alpha(b) - A_1] \geqq 0,$$

$$b_2^{-1}[b_1 \mathfrak{D}^0 \beta(b) + b_2 \mathfrak{D}^1 \beta(b) - A_1] \leqq 0,$$

$$\textit{if } b_2 \neq 0,$$

or

$$\mathfrak{D}^0\alpha(b) \leqq A_1/b_1 \leqq \mathfrak{D}^0\beta(b) \quad \text{if } b_2 = 0.$$

COROLLARY 2. *Let Assumption A hold. Then the boundary value problem*

$$Lz = f(t, z, z'), a < t < b,$$

$$a_1 z(a) + a_2 z'(a) = A_0,$$

$$b_1 \mathfrak{D}^0 z(b) + b_2 \mathfrak{D}^1 z(b) = A_1,$$

where $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \neq 0$, $A_0, A_1 \in R^1$, *has a solution*

$$\alpha(t) \leqq z(t) \leqq \beta(t) \quad \text{for} \quad a < t < b,$$

provided

$$a_2^{-1}[a_1\alpha(a) + a_2\alpha'(a) - A_0] \geqq 0,$$

$$a_2^{-1}[a_1\beta(a) + a_2\beta'(a) - A_0] \leqq 0, \text{ if } a_2 \neq 0$$

or

$$\alpha(a) \leqq A_0/a_1 \leqq \beta(a) \text{ if } a_2 = 0 \text{ and } u_1(a) = 1,$$

and

$$b_2^{-1}[b_1\mathfrak{D}^0\alpha(b) + b_2\mathfrak{D}^1\alpha(b) - A_1] \geqq 0,$$

$$b_2^{-1}[b_1\mathfrak{D}^0\beta(b) + b_2\mathfrak{D}^1\beta(b) - A_1] \leqq 0 \text{ if } b_2 \neq 0,$$

or

$$\mathfrak{D}^0\alpha(b) \leqq A_1/b_1 \leqq \mathfrak{D}^0\beta(b) \text{ if } b_2 = 0.$$

These results generalize most of the existence theorems in [1], [2], [4], [7], [8], [10], [11], for the solutions of two-point boundary value problems on the finite interval and also apply to nonlinear boundary conditions as well. Finally, the results are used to deduce the existence of solutions to singular perturbation problems on $[a, b] \subset (-\infty, \infty]$, which are of the general form

$$L_\epsilon y = \epsilon y'' + Ky' + My = f(t, y, y', \epsilon), -\infty < a \leqq t \leqq b \leqq \infty, \tag{17}$$

$$a_1 y(a) + a_2 y'(a) = A_\epsilon, \mathfrak{D}^0 y(b) = B_\epsilon. \tag{18}$$

The procedure used to treat (17)-(18) starts by assuming the existence of an appropriate approximate solution to (17)-(18). Such a solution is usually obtained by inspection or some approximation method and is another interesting and difficult problem in itself. From this approximate solution we show how to construct upper and lower solutions. Therefore, existence of solution to (17) and (18) follows immediately from Corollary 2. Moreover, the true solution obtained lies between the upper and lower solutions and so a measurement of how much the true solution differs from the approximate solution, as $\epsilon \to 0$, is available.

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Louisiana State University, Baton Rouge, Louisiana 70803