ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 4, Fall 1976

SINGULAR PERTURBATION OF SOME QUASILINEAR PARABOLIC EQUATIONS IN DIVERGENCE FORM

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Let $\Omega \subset E^n$, $n \geqq 2$, be a smooth domain. Consider for every $\epsilon > 0$ and $t > 0$ the initial value problem

$$\epsilon u_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, t, u, u_x, \epsilon) + a(x, t, u, u_x, \epsilon)u + \epsilon f(x, t, \epsilon) = 0.$$

$$u(x, t, \epsilon) = 0, \quad \text{for } x \in \partial\Omega.$$

$$u(x, 0, \epsilon) = \mathring{u}(x, \epsilon), \text{ for } x \in \Omega.$$

$$(I_\epsilon) \quad \sum_{i=1}^{n} a_i(x, t, u, u_x, \epsilon)u_{x_i} \geqq \nu|\nabla u|^2.$$

$$|a_i(x, t, z, p, \epsilon)| \leqq M(|z| + |p|), \text{ and}$$

$$a(x, t, u, u_x, \epsilon) \geqq 0, \text{ for any } x \in \Omega, t > 0, \text{ and } z, p \in R \times R^n.$$

$$|f(x, t, \epsilon)| < M \text{ for any } x \in \Omega, t > 0, \text{ and } 0 < \epsilon < 1.$$

The purpose of this paper is to study the behavior of the solution of (I_ϵ) as $\epsilon \to 0$. The methods employed here are similar to [5], where the stability of some quasilinear parabolic equations in divergence form is considered.

Singular perturbation of quasilinear parabolic equations has been studied by Hoppensteadt [1]. He was able to obtain uniformly valid asymptotic expansions of the solution in terms of inner and outer expansions. However, many hypotheses were required to obtain his results. In particular, he required certain smallness criteria of the initial conditions. In this paper we are able to eliminate these special hypotheses. However, the equations we consider are of a more special type than was considered in [1] and our results are qualitative.

We will assume a unique classical solution exists for equations (I_ϵ).

Existence theorems for these equations can be found in [3]. [4] obtained maximum type principles for these equations.

Our method of proof is first to obtain an energy type inequality to obtain an L^2 estimate on the behavior of the solution in the spaces variables; and then obtain a Di Giorgi-Nash type inequality to improve the L^2 estimate to an L^∞ estimate.

We state here the results of this paper.

LEMMA 1. *Let $u(x, t, \epsilon)$ be the solution of (I_ϵ). Then the following estimate holds*:

$$\int_\Omega u^2(x, t, \epsilon)\, dx \leqq \beta_1(t, \epsilon) + \beta_2(t, \epsilon), \textit{ where}$$

$$\beta_1(t, \epsilon) = \int_\Omega (\mathring{u}(x, \epsilon))^2\, dx\, e^{-\delta t/\epsilon} \textit{ and}$$

$$\beta_2(t, \epsilon) = (1/\delta)\epsilon \int_0^t \int_\Omega f^2(x, \tau, \epsilon) e^{-\delta(t-\tau)/\epsilon}\, dx\, d\tau.$$

$$\delta = \nu/c(\text{mes }\Omega)^{2/n} \textit{ where } c \textit{ is given by the inequality}$$

$$\int_\Omega u^2\, dx \leqq c(\text{mes }\Omega)^{2/n} \int_\Omega |\nabla u|^2\, dx \textit{ for } u \in \mathring{W}_2{}^1(\Omega).$$

(eq. (1.2) [5])

Moreover, $\beta_1(t, \epsilon)$ is of boundary layer type, and $\beta_2(t, \epsilon)$ is $O(\epsilon^2)$.

THEOREM 1. *Let $u(x, t, \epsilon)$ be a solution of (I_ϵ). Then, for $t > \mu\epsilon$, μ arbitrarily small, the following estimate holds*

$$|u(x, t, \epsilon)| \leqq \tilde{\beta}_1(t, \epsilon) + \tilde{\beta}_2(t, \epsilon), \textit{ where}$$

$$\tilde{\beta}_1(t, \epsilon) = (c\epsilon^{-1/\lambda} \int_{t-(3/4)\mu\epsilon}^t \beta_1(\tau, \epsilon)\, d\tau)^{\lambda/(2+2\lambda)}, \text{ and}$$

$$\tilde{\beta}_2(t, \epsilon) = (c\epsilon^{-1/\lambda} \int_{t-(3/4)\mu\epsilon}^t \beta_2(\tau, \epsilon)\, d\tau)^{\lambda/(2+2\lambda)}.$$

$$\lambda = \frac{2}{n+2}.$$

Moreover, $\tilde{\beta}_1(t, \epsilon)$ is still of boundary layer type, and $\tilde{\beta}_2(t, \epsilon)$ is $O(\epsilon^{(3\lambda-1)/(2+\lambda)})$.

PROOF OF LEMMA 1. Multiplying equation (I_ϵ) by u, integrating over Ω, integrating by parts, noting the hypotheses on the coefficients of (I_ϵ) and dividing by ϵ, we obtain the inequality:

$$\frac{\partial}{\partial t}\int_\Omega u^2\,dx + (2\nu/\epsilon)\int_\Omega |\nabla u|^2\,dx$$

$$\leqq 2\int_\Omega |fu|\,dx \leqq (1/\delta_2)\int_\Omega f^2\,dx + \delta_2\int_\Omega u^2\,dx.$$

Since

$$(-2\nu/\epsilon)\int_\Omega |\nabla u|^2\,dx \leqq \frac{-2\nu}{\epsilon c(\text{mes }\Omega)}(2/n)\int_\Omega u^2\,dx,$$

we choose

$$\delta_2 = \frac{\nu}{\epsilon c(\text{mes }\Omega)(2/n)}$$

We obtain the differential inequality

$$y'(t) + \delta_2 y(t) \leqq (\epsilon/\delta)\int_\Omega f^2\,dx,$$

$$\text{where } y(t) = \int_\Omega u^2\,dx.$$

Using a form of Gronwall's inequality, we obtain the desired result.

PROOF OF THEOREM 1. We will only give a brief outline of the proof of this theorem, since it is somewhat technical. However, many of the details of this proof can be found in [5]. In a subsequent paper, we will include all the details and generalize the results.

We need to define the following terms:

(1) $u^k(x, t) = \max\{u(x, t) - k, 0\}$,

(2) $Q_{\sigma-\sigma_1\rho,\mu\epsilon} = \{|x - x_0| < \rho - \sigma_1\rho\colon t_0 - (1 - \sigma_2\mu\epsilon) < t < t_0\}$,

(3) $A_{k,\rho}(t) = \{x|\ |x - x_0| \leqq \rho;\ u(x, t) > k\}$,

(4) $\mu(k, \rho, \mu\epsilon) = \displaystyle\int_{t_0-\mu\epsilon}^{t_0} \text{mes } A_{k,\rho}(\tau)\,d\tau$, and

(5) $|u|^2_{Q_T} = \displaystyle\operatorname*{ess\,max}_{0\leqq t\leqq T}\int_\Omega u^2\,dx + \int_0^T\int_\Omega |\nabla u|^2\,dx.$

Then multiplying equation (I_ϵ) with $u^k\xi^2(x, t)$, where $\xi(x, t)$ is chosen suitably, we eventually can obtain the Di Giorgi-Nash type inequality

$$|u^k|^2_{Q_{\rho-\sigma_1\rho,\mu\epsilon-\sigma_2\mu\epsilon}}$$

$$\leqq (\gamma/\epsilon)\{((\sigma_1\rho)^{-2} + (\sigma_2\mu\epsilon)^{-1}) \int_Q \int_{\rho,\mu\epsilon} (u-k)^2\,dx\,dt$$

$$+ (k^2+1)\mu(k,\rho,\mu\epsilon)\}.$$

We now define the following terms:

(6) $k_h = k + k(1-(1/2)^h)$,

(7) $\rho_h = (1/2 + (1/2)^{h+2})\rho$,

(8) $\tau_h = (1/2 + (1/2)^{h+2})\,\mu\epsilon$, and

(9) $$y_h = \int_{t_0-\tau_h}^{t_0} \int_{A_{k_h,\rho_h}(t)} (u-k_h)^2\,dx\,dt.$$

Using the above inequality, we can show that the following recursion relationship holds

$$y_{h+1} \leqq (\gamma/\epsilon)2^{4h}(1/k^{2+2\lambda})(1/\rho^2 + 1/\mu\epsilon)y_h^{1+\lambda},$$

$$h = 0, \cdots, \text{where } \lambda = 2/(n+2).$$

From [2, p. 66], $y_h \to 0$ as $h \to \infty$ if

$$y_0 < 2^{-4/\lambda^2}\{(\gamma/\epsilon)(1/k^{2+2\lambda})(1/\rho^2 + \mu\epsilon)\}^{-1/\lambda}.$$

Since

$$y_0 \leqq \int_{t_0-(3/4)\mu\epsilon}^{t_0} \int_{A_{k,(3/4)\rho}} (u-k)^2\,dx\,dt \leqq \int_{t_0-(3/4)\mu\epsilon}^{t_0} \int_\Omega u^2\,dx\,dt$$

$$\leqq \int_{t_0-(3/4)\mu\epsilon}^{t_0} (\beta_1(t,\epsilon) + \beta_2(t,\epsilon))\,dt \overset{\mathrm{DEF}}{=} \tilde{\beta}(t,\epsilon),$$

we choose

$$k = \left\{\frac{\epsilon^{-1/\lambda}\tilde{\beta}(t,\epsilon)}{(\gamma(1/\rho + \mu\epsilon))^{-1/\lambda_2 - 4/\lambda_2}}\right\}^{\frac{\lambda}{\lambda/2+2\lambda}}$$

$$= c(\epsilon^{-1/\lambda}\tilde{\beta}(t,\epsilon))^{\lambda/(2+2\lambda)}.$$

Then

$$\int_{t_0-(1/2)\mu\epsilon}^{t_0} \int_{A_{2k,\rho/2}} (u-2k)^2\,dx\,dt = 0.$$

Therefore, $u(x,t,\epsilon) < 2k$ a.e. Since m spheres of radii ρ can cover Ω and $u(x,t,\epsilon)$ is a classical solution, then $u(x,t,\epsilon) < 2mk$.

In a similar manner, we can show that $u(x, t, \epsilon) > -2mk$. We have now proved the desired result.

BIBLIOGRAPHY

1. F. Hoppensteadt, *On Quasilinear Parabolic Equations with a Small Parameter,* Comm. Pure Applied Math. 24 (1971), 17-38.

2. O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Equations,* Academic Press, New York, N.Y., 1968.

3. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type,* A.M.S., Providence, R.I., 1968.

4. D. Aronson and J. Serrin, *Local Behavior of Solutions of Quasilinear Parabolic Equations,* Arch. Rat. Mech. Anal. 25 (1967), 81-122.

5. M. M. Tang, *Asymptotic Stability of Some Quasilinear Parabolic Equations in Divergence From. L^∞ Results,* J. Math. Anal. Appl. (to appear).

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