SOME PROBABILISTIC PROBLEMS AND METHODS IN SINGULAR PERTURBATIONS

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ABSTRACT. We discuss in detail the asymptotic analysis of deterministic and stochastic problems, within a certain class, from the point of view of order reduction or contraction of description.

1. **Introduction.** Many problems that arise in the asymptotic analysis of stochastic differential equations are singular perturbation problems, sometimes familiar from well studied deterministic problems.

We attempt to give here a unified presentation of such problems emphasizing one aspect they have in common: the order-reduction or contraction in their description that emerges in the limit. One of the simplest examples of this kind in probability is the central limit theorem.

In §2 we review some results on the method of averaging [1, 2]. The general ideas here help motivate later developments. In § 3 we take up the analysis of a class of linear operator-equation problems that arise in random evolutions [3], in the work of Kurtz [4], the work of Ellis and Pinsky [5] and in [6].

The framework of § 3 is not sufficient for the analysis of problems corresponding to the results of [7] as well as those in [8, 9]. In § 4 we introduce a more elaborate framework in order to deal with them. The basic problem under consideration here is the long-time behavior of a dynamical system coupled to a heat bath. The relation to the problem of atoms coupled to radiation in the form treated by Davies [8, 9, 10] and, more generally, to those in [11], is discussed briefly.

The results of § 4 were obtained while the author was visiting the Observatoire de Nice. The hospitality of Uriel and Helene Frisch and their colleagues is gratefully acknowledged. The ideas discussed in [12] and conversations with Uriel Frisch and E. B. Davies provided initial motivation for this work.

2. **Averaging.** The method of averaging for ordinary differential equations has played an important role both as an approximation method and as a theoretical tool [1]. We shall give an account of a

generalization of the averaging method [2] which illustrates the methods and results of later sections.

Many problems in differential equations, after preliminary transformations and scaling, assume the form

(2.1)

$$\frac{dx^\epsilon(t)}{dt} = F(x^\epsilon(t), y^\epsilon(t)), \qquad x^\epsilon(0) = x,$$

$$\frac{dy^\epsilon(t)}{dt} = \frac{1}{\epsilon} G(x^\epsilon(t), y^\epsilon(t)), \qquad y^\epsilon(0) = y.$$

Here $x^\epsilon(t) \in R^n$ and $y^\epsilon(t) \in R^m$ and F and G are vector valued functions of dimension n and m respectively. We refer to $x^\epsilon(t)$ as the slowly varying part of the pair $(x^\epsilon(t), y^\epsilon(t))$ and to $y^\epsilon(t)$ as the rapidly varying part. We assume that the components of F and G and their x and y derivatives are bounded functions of x and y.

The object of interest is the asymptotic behavior of $x^\epsilon(t)$, $0 \leqq t \leqq T(T < \infty)$, as $\epsilon \downarrow 0$. We are not concerned about the behavior of $y^\epsilon(t)$ here. To motivate future developments we study (2.1) by passing to the Liouville equation that corresponds to it. Let $f(x, y)$ be a differentiable function on $R^n \times R^m$ and define

(2.2) $u^\epsilon(t, x, y) = f(x^\epsilon(t, x, y), y^\epsilon(t, x, y)).$

Then we have for $t > 0$

$$\frac{\partial u^\epsilon(t, x, y)}{\partial t} - F(x, y) \frac{\partial u^\epsilon(t, x, y)}{\partial x} - (1/\epsilon)G(x, y) \frac{\partial u^\epsilon(t, x, y)}{\partial y} = 0,$$

(2.3)

$$u^\epsilon(0, x, y) = f(x, y).$$

Here $\partial/\partial x$ stands for the x-gradient and $F\partial/\partial x$ stands for the dot product of F and the gradient; similarly for $G\partial/\partial y$. We shall study the behavior of $u^\epsilon(t, x, y)$ as $\epsilon \downarrow 0$ when $f = f(x)$, i.e., the initial data depend on x only.

Let $Y^x(t)$ be the solution of

(2.4) $$\frac{dY^x(t)}{dt} = G(x, Y^x(t)), Y^x(0) = y,$$

where x is a parameter. For small t, of order ϵ, $y^\epsilon(t)$ is approximated well by $Y^x(t/\epsilon)$. Suppose that

(2.5) $$\bar{F}(x) = \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T F(x, Y^x(s)) \, ds,$$

exists uniformly in x and y and defines a bounded vector function $\overline{F}(x)$ independent of y as shown. We also assume that the derivatives of $\overline{F}(x)$ are bounded functions of x. Let $\overline{x}(t)$ be defined by

$$(2.6) \qquad \frac{d\overline{x}(t)}{dt} = \overline{F}(\overline{x}(t)), \, \overline{x}(0) = x,$$

and let $\overline{u}(t, x)$ be defined by

$$(2.7) \qquad \overline{u}(t, x) = f(\overline{x}(t)).$$

For $f(x)$ differentiable, (2.6) and (2.7) imply that

$$(2.8) \qquad \frac{\partial \overline{u}(t, x)}{\partial t} - \overline{F}(x) \frac{\partial \overline{u}(t, x)}{\partial x} = 0, \, t > 0$$

$$\overline{u}(0, x) = f(x).$$

THEOREM. *Under the above hypotheses and with $f = f(x)$ in (2.3), $u^\epsilon(t, x, y)$ tends to $\overline{u}(t, x)$, $0 \leqq t \leqq T$, as $\epsilon \downarrow 0$ uniformly in x and y on compact sets.*

PROOF. It is enough to have $f(x)$ differentiable with bounded derivatives. With this f fixed we define

$$(2.9) \quad \chi^{(\lambda)}(x, y)$$

$$= \int_0^\infty e^{-\lambda s} \left[F(x, Y^x(s)) \frac{\partial f(x)}{\partial x} - \overline{F}(x) \frac{\partial f(x)}{\partial x} \right] \, ds, \, \lambda > 0.$$

Because of (2.5) it follows by elementary computations that

$$(2.10) \qquad \lim_{\lambda \downarrow 0} \lambda \chi^{(\lambda)}(x, y) = 0$$

uniformly in x and y. This says that the Abel limit exists if the Cesaro limit does, a well known fact. From our differentiability hypotheses it follows that derivatives of $\chi^{(\lambda)}$ exist and are bounded.

From (2.3) and with initial data equal to $f(x) + \epsilon \chi^{(\lambda)}(x, y)$ we obtain the identity

$$f(x^\epsilon(t)) + \epsilon \chi^{(\lambda)}(x^\epsilon(t), y^\epsilon(t)) - f(x) - \epsilon \chi^{(\lambda)}(x, y)$$

$$(2.11) \quad - \int_0^t \left(F(x^\epsilon(s), y^\epsilon(s)) \frac{\partial}{\partial x} + \frac{1}{\epsilon} G(x^\epsilon(s), y^\epsilon(s)) \frac{\partial}{\partial y} \right)$$

$$\cdot \, [f(x^\epsilon(s)) + \epsilon \chi^{(\lambda)}(x^\epsilon(s), y^\epsilon(s))] \, ds = 0.$$

In (2.11) we first differentiate under the integral sign and then evaluate the result as indicated. Now from the definition of $\chi^{(\lambda)}$ it follows that

$$(2.12) \quad G(x, y) \frac{\partial \chi^{(\lambda)}(x, y)}{\partial y} + [F(x, y) - \bar{F}(x)] \frac{\partial f(x)}{\partial x}$$
$$= \lambda \chi^{(\lambda)}(x, y), \; \lambda > 0.$$

Thus from (2.12) and (2.11) we obtain

$$f(x^\epsilon(t)) - f(x) - \int_0^t \bar{F}(E(x^\epsilon(s)) \frac{\partial f(x^\epsilon(s))}{\partial x} \, ds$$

$$(2.13) \quad = -\epsilon \chi^{(\lambda)}(x^\epsilon(t), y^\epsilon(t)) + \epsilon \chi^{(\lambda)}(x, y)$$

$$+ \int_0^t \left[\lambda \chi^{(\lambda)}(x^\epsilon(s), y^\epsilon(s)) \right.$$

$$\left. + \epsilon F(x^\epsilon(s), y^\epsilon(s)) \frac{\partial \chi^{(\lambda)}(x^\epsilon(s), y^\epsilon(s))}{\partial x} \right] ds.$$

By first choosing λ sufficiently small, using (2.10) and then letting ϵ go to zero, we deduce from (2.13) that, for $0 \leqq t \leqq T(T < \infty)$,

$$(2.14) \quad \left| f(x^\epsilon(t)) - f(x) - \int_0^t \bar{F}(x^\epsilon(s)) \frac{\partial f(x^\epsilon(s))}{\partial x} \, ds \right| \to 0$$

as $\epsilon \downarrow 0$, uniformly in x and y (recall x^ϵ depends on y).

We must now deduce from (2.14) that, for $0 \leqq t \leqq T$,

$$(2.15) \quad |f(x^\epsilon(t)) - f(\bar{x}(t))| \to 0,$$

as $\epsilon \downarrow 0$, uniformly in x and y on compact sets. From (2.8) it follows that

$$(2.16) \quad f(\bar{x}(t)) - f(x) - \int_0^t \bar{F}(\bar{x}(s)) \frac{\partial f(\bar{x}(s))}{\partial x} \, ds = 0.$$

From (2.1) it follows that for (x, y) in a compact set, $x^\epsilon(t)$ form a uniformly bounded and equicontinuous family of functions of t, $0 \leqq t \leqq T(T < \infty)$ with, say, $0 \leqq \epsilon \leqq 1$. The results (2.14) and (2.16) show that any limit $\bar{x}(t)$ of the $x^\epsilon(t)$ is a solution of (2.6). But the latter has a unique solution. Hence (2.15) follows.

The usual method of averaging [1] is a special case of the above. Specifically, in (2.1) $y^\epsilon(t)$ is one-dimensional, $G(x, y)$ is identically equal to one and $F(x, y)$ is almost periodic in y.

Recall that $\chi^{(\lambda)}$ was defined by (2.9). If $\chi^{(0)}$ exists and is bounded, i.e.,

$$\int_0^\infty \left[F(x, Y^x(s)) \frac{\partial f(x)}{\partial x} - \bar{F}(x) \frac{\partial f(x)}{\partial x} \right] ds < \infty,$$

then it is easy to see that the right side of (2.14) will be $0(\epsilon)$. In general, we have $o(1)$ as shown.

In the following section we shall be concerned, directly or indirectly, with problems of the form (2.1) where $y^\epsilon(t)$, the rapidly varying part, is a stochastic process. We shall see that one can proceed there in much the same way as above.

3. **A general class of linear problems.** Let us first begin with the following linear sytem of ordinary differential queations

$$\frac{dx^\epsilon(t)}{dt} = A_{11}x^\epsilon(t) + A_{12}y^\epsilon(t), \, x^\epsilon(0) = x,$$

(3.1)

$$\frac{dy^\epsilon(t)}{dt} = \frac{-\Lambda}{\epsilon}y^\epsilon(t) + A_{21}x^\epsilon(t) + A_{22}y^\epsilon(t), \, y^\epsilon(0) = y.$$

Here A_{11}, A_{12}, A_{21} and A_{22} are matrices with appropriate dimensions, $x^\epsilon(t)$ and $y^\epsilon(t)$ being n-vectors and m-vectors respectively. The matrix Λ is assumed to have eigenvalues with positive real parts.

The result of section 2 yields easily that

(3.2) $$x^\epsilon(t) \sim e^{A_{11}t}x, \, 0 \leqq t \leqq T,$$

which is clear since $y^\epsilon(t)$ decays rapidly to zero by our hypotheses.

What can we say about $x^\epsilon(t)$ for $0 \leqq t \leqq T/\epsilon$, with $T < \infty$ but arbitrary? In general, we have that

(3.3) $$x^\epsilon(t) \sim e^{(A_{11}+\epsilon V)t}x, \, 0 \leqq t \leqq T/\epsilon,$$

where

(3.4) $$V = A_{12}\Lambda^{-1}A_{21}.$$

If in addition A_{11} is skew-symmetric then, with

(3.5) $$\bar{V} = \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T e^{-A_{11}s}V e^{A_{11}s} ds,$$

we have

(3.6) $$x^\epsilon(t) \sim e^{A_{11}t}e^{\epsilon\bar{V}t}x, \, 0 \leqq t \leqq T/\epsilon.$$

These approximations hold with error estimates $O(\epsilon)$ uniformly on compact x, y sets. The second one, (3.6) is easier to implement and hence potentially most useful. The passage from (3.3) to (3.6) is simply an application of the averaging method and the arguments of section 2 apply. They also apply in greater generality for linear operator equations but we shall concentrate on the approximation (3.3) here.

We shall formulate below a general result corresponding to (3.3) in an operator framework. Our motivation for this is the work on random evolutions [3], transport theory [13], the linearized hydrodynamical limit of the Boltzmann equation [5, 6, 14], stochastic differential equations with Markovian coefficients and homogenization problems [15], to name a few applications. In the form given here, the results generalize a theorem of Kurtz [4]. A systematic presentation of several applications is given in [16].

Before stating the theorem we shall consider briefly the case of stochastic equations with Markovian coefficients in order to illustrate the framework of the theorem that follows.

Let $y(t)$ be an R^m valued Markov process and let B be its infinitesimal generator defined on smooth functions by

$$Bf(y) = \lim_{h \downarrow 0} \frac{E\{f(y(h)) \mid y(0) = y\} - f(y)}{h}.$$

Let $x^\epsilon(t)$ be the R^n-valued process satisfying

$$\frac{dx^\epsilon(t)}{dt} = \frac{1}{\epsilon} F(x^\epsilon(t), y^\epsilon(t)) + G(x^\epsilon(t), y^\epsilon(t)), \, x^\epsilon(0) = x,$$

$$y^\epsilon(t) \equiv y(t/\epsilon^2).$$

We assume that F and G are smooth vector functions; the discussion of this example will be informal. If $f(x, y)$ is a smooth function and if

$$u^\epsilon(t, x, y) = E\{f(x^\epsilon(t), y^\epsilon(t))\},$$

then it follows that

$$\frac{\partial u^\epsilon}{\partial t} = (1/\epsilon)F(x, y) \, \frac{\partial u^\epsilon}{\partial x} + G(x, y) \, \frac{\partial u^\epsilon}{\partial x} + (1/\epsilon^2)Bu^\epsilon, \, t > 0,$$

(3.7)
$$u^\epsilon(0, x, y) = f(x, y),$$

which is the backward Kolmogorov equation for the Markov process $(x^\epsilon(t), y^\epsilon(t))$, $y^\epsilon(t) \equiv y(t/\epsilon^2)$. The scaling in (3.7) is motivated by the fact that the intermediate term $(1/\epsilon)F \partial/\partial x$ is assumed, sometimes, to average to zero in an appropriate sense.

Our interest is in $x^\epsilon(t)$ so we wish to find the limit as $\epsilon \downarrow 0$ of $E\{f(x^\epsilon(t))\}$ for a sufficiently rich class of functions f on R^n. This is the same as studying the asymptotic limit of the solution of (3.7) with $f = f(x)$ for initial data. This problem is of the form covered by the theorem below (with $B = B$, $A = F\partial/\partial x$, $C = G\partial/\partial x$). A basic hypothesis for the asymptotic analysis that follows is that the process $y(t)$ be ergodic in a sufficiently strong sense. We return to this after the statement of the theorems.

Let L be a Banach space and let B be the infinitesimal generator of an ergodic contraction semigroup e^{Bt} that is,

$$(3.8) \qquad \lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda t} e^{Bt} f \, dt = Pf, \; f \in L.$$

Here Pf is projection into the nullspace of B; (see [17, p. 516] and [4]). Let $L_0 \equiv PL$.

Let A and C be linear operators on L such that for each $\epsilon > 0$

$$(3.9) \qquad H^\epsilon = (1/\epsilon^2)B + (1/\epsilon)A + C$$

has a closure that generates a continuous contraction semigroup on L which we denote by $T^\epsilon(t)$. Let D be a subspace of L_0, dense in L_0, contained in the intersection of the domains of A, B and C and such that the equation

$$(3.10) \qquad B\chi_1 + (A - PA)f = 0 \qquad (\chi_1 = -B^{-1}(A - PA)f)$$

has a solution in L for each $f \in D$ (so $Pf = f$). We assume in addition that χ_1 belongs to the intersection of the domains of A and C. With $f \in D$ and $\lambda > 0$ define $\chi_2^{(\lambda)}$ by

$$(3.11) \qquad \chi_2^{(\lambda)} = \int_0^\infty e^{-\lambda s} e^{Bs} [A\chi_1 + Cf - PA\chi_1 - PCf] \, ds.$$

We assume that $\chi_2^{(\lambda)}$ also belongs to the intersection of the domains of A and C.

Let G^ϵ be defined by the closure of

$$(3.12) \qquad \begin{aligned} G^\epsilon f &= (1/\epsilon)\, PAPf + PA\chi_1 + PCPf \\ &\equiv (1/\epsilon)\, PAPf + Gf, \quad f \in D, \quad \epsilon > 0, \end{aligned}$$

in L_0. Assume that it generates a continuous contraction semigroup in L_0 which we denote by $S^\epsilon(t)$. Assume also that the closure in L_0 of the set

$$(3.13) \qquad \bigcap_{\epsilon > 0} (\lambda - G^\epsilon)D$$

is L_0 for $\lambda > 0$.

THEOREM 1. *Under the above hypotheses, for $f \in L_0$ and $\lambda > 0$,*

$$(3.14) \qquad \lim_{\epsilon \downarrow 0} \| R_\lambda^\epsilon f - Q_\lambda^\epsilon f \| = 0,$$

where

$$R_\lambda^\epsilon = \int_0^\infty e^{-\lambda s} T^\epsilon(t) \, dt,$$

$$Q_\lambda^\epsilon = \int_0^\infty e^{-\lambda s} S^\epsilon(t) \, dt, \quad \lambda > 0.$$

PROOF. Let $f \in D$, let X_1 be the solution of (3.10) and for $\lambda > 0$ let $X_2^{(\lambda)}$ be defined by (3.11). From our hypotheses and (3.8) it follows that

(3.15) $\qquad BX_2^{(\lambda)} + AX_1 + Cf - PAX_1 - PCf = \lambda X_2^{(\lambda)}$

and

(3.16) $\qquad\qquad\qquad \lim_{\lambda \downarrow 0} \lambda X_2^{(\lambda)} = 0.$

Moreover, $X_2^{(\lambda)}$, $\lambda > 0$, belongs to the intersection of the domains of A and C.

Now with $f \in D$, $f + \epsilon X_1 + \epsilon^2 X_2^{(\lambda)}$, $\lambda > 0$, is in the domain of H^ϵ and hence

$$T^\epsilon(t)(f + \epsilon X_1 + \epsilon^2 X_2^{(\lambda)}) - (f + \epsilon X_1 + \epsilon^2 X_2^{(\lambda)}) -$$

(3.17)
$$- \int_0^t T^\epsilon(s)((1/\epsilon^2)B + (1/\epsilon)A + C)(f + \epsilon X_1 + \epsilon^2 X_2^{(\lambda)}) \, ds = 0.$$

Using the definitions of X_1, $X_2^{(\lambda)}$ and G^ϵ we rewrite (3.17) in the following form

$$T^\epsilon(t)f - f - \int_0^t T^\epsilon(s) G^\epsilon f \, ds$$

(3.18)
$$= - T^\epsilon(t)(\epsilon X_1 + \epsilon^2 \, X_2^{(\lambda)}) + \epsilon X_1 + \epsilon^2 X_2^{(\lambda)}$$

$$+ \int_0^t T^\epsilon(s)[\lambda X_2^{(\lambda)} + \epsilon(CX_1 + AX_2^{(\lambda)}) + \epsilon^2 CX_2^{(\lambda)}] \, ds.$$

Thus, since $T^\epsilon(t)$ is a contraction and (3.16) holds we obtain for $f \in D$,

(3.19) $\qquad \lim_{\epsilon \downarrow 0} \left\| T^\epsilon(t)f - f - \int_0^t T^\epsilon(s) G^\epsilon f \, ds \right\| = 0.$

It remains to show that (3.19) implies (3.14).

From (3.19), it follows that for $f \in D$

(3.20) $\qquad\qquad \lim_{\epsilon \downarrow 0} \| f - R_\lambda^\epsilon(\lambda - G^\epsilon)f \| = 0, \quad \lambda > 0.$

Let $g \in \bigcap_{\epsilon > 0} (\lambda - G^\epsilon)D$. Let f^ϵ be defined by

(3.21) $$f^\epsilon = Q_\lambda^\epsilon g.$$

Since f^ϵ is in D for all $\epsilon > 0$ the corresponding χ_1^ϵ and $\chi_2^{(\epsilon,\lambda)}$ have the requisite properties independently of $\epsilon > 0$. Thus the right hand side of (3.16), with χ_1 and $\chi_2^{(\lambda)}$ replaced by χ_1^ϵ and $\chi_2^{(\epsilon,\lambda)}$, goes to zero again and so (3.17) is valid with $f = f^\epsilon$. This leads to (3.20) with $f = f^\epsilon$ and hence, by (3.21),

(3.22) $$\lim_{\epsilon \downarrow 0} \|Q_\lambda^\epsilon g - R_\lambda^\epsilon g\| = 0, \lambda > 0.$$

Since the closure of $\bigcap_{\epsilon > 0} (\lambda - G^\epsilon)D$ is L_0, (3.20) holds for all $g \in L_0$ and hence (3.14).

REMARK 1. In the case $PAP \equiv 0$ the above theorem is essentially Kurtz's result [4]. In this case G^ϵ is independent of ϵ and it is given (dropping the superscript) by

(3.23) $$Gf = PA\chi_1 + PCPf, \quad f \in D.$$

The condition stated immediately above the theorem simplifies to: the closure of $(\lambda - G)D$ is L_0 for each $\lambda > 0$. It is well known, see for example [18], that since both H^ϵ and G of (3.23) are generators of contractions, the result (3.14) implies that

(3.24) $$\lim_{\epsilon \downarrow 0} \sup_{0 \le t \le T} \|T^\epsilon(t)f - S(t)f\| = 0, \, f \in L_0, T < \infty.$$

In general, it is not true that (3.24) holds with $S(t)$ replaced by $S^\epsilon(t)$. This observation is due to E. B. Davies.

We give next a theorem that leads directly to the result (3.24) in the general case. Instead of (3.13) the basic requirement is now smoothness as follows.

For $f \in D \subset L_0$ let

(3.25) $$v^\epsilon(t) = S^\epsilon(t)f$$

so that

(3.26) $$\frac{dv^\epsilon(t)}{dt} = G^\epsilon v^\epsilon(t), \, t > 0, \, v^\epsilon(0) = f.$$

Similarly, let

(3.27) $$u^\epsilon(t) = T^\epsilon(t)f.$$

We shall assume that $D \subset L_0$ is dense in L_0 and that for $f \in D$, $v^\epsilon(t)$ and powers of A, C and G acting on $v^\epsilon(t)$ have finite norm for $0 \le t \le T$ independently of $\epsilon > 0$. This is the smoothness requirement. We also assume that B^{-1} exists and is bounded on all $f \in L$

such that $Pf = 0$ and, moreover, that B^{-1} does not alter the smoothness properties.

THEOREM 2. *Under the above hypotheses and with $f \in D$*

$$(3.27) \qquad \sup_{0 \leq t \leq T} \| T^\epsilon(t)f - S^\epsilon(t)f \| \leq c\epsilon,$$

where $T < \infty$ is arbitrary and c is a constant that depends on T and $f \in D$ but is independent of ϵ.

PROOF. With $f \in D$ fixed let $v^\epsilon(t)$ be given by (3.25) and let $\chi_1^\epsilon(t)$ be defined by

$$(3.28) \qquad \chi_1^\epsilon(t) = - B^{-1}(A - PA)v^\epsilon(t).$$

Since $S^\epsilon(t)$ is a semigroup on L_0, B^{-1} is well defined here because $(PA - PA)v^\epsilon(t) = 0$. Clearly

$$(3.29) \qquad B\chi_1^\epsilon(t) + (A - PA)v^\epsilon(t) = 0.$$

Similarly let $\chi_2^\epsilon(t)$ be given by

$$\chi_2^\epsilon(t) = - B^{-1}[A\chi_1^\epsilon(t) + Cv^\epsilon(t) - Gv^\epsilon(t) + B^{-1}(A - PA)PAPv^\epsilon(t)].$$
$$(3.30)$$

Again, this is well defined because P acting on the expression in the brackets equals zero by definition of G in (3.12) or (3.23) and since B^{-1} commutes with P. Thus

$$B\chi_2^\epsilon(t) + A\chi_1^\epsilon(t) + Cv^\epsilon(t) - Gv^\epsilon(t) + B^{-1}(A - PA)PAPv^\epsilon(t) = 0.$$
$$(3.31)$$

By our smoothness hypotheses $\chi_1^\epsilon(t)$ and $\chi_2^\epsilon(t)$ are bounded on finite t intervals independently of ϵ. We may therefore estimate $u^\epsilon(t) - v^\epsilon(t) - \epsilon\chi_1^\epsilon(t) - \epsilon^2\chi_2^\epsilon(t)$, instead of $u^\epsilon(t) - v^\epsilon(t)$, and show that its norm is $0(\epsilon)$. But, using (3.30), (3.31) and (3.26), we have

$$\left(\frac{d}{dt} - H^\epsilon \right) [u^\epsilon(t) - v^\epsilon(t) - \epsilon\chi_1^\epsilon(t) - \epsilon^2\chi_2^\epsilon(t)]$$

$$= \left(\frac{1}{\epsilon^2} B + \frac{1}{\epsilon} A + C - \frac{d}{dt} \right) [v^\epsilon(t) + \chi_1^\epsilon(t) + \epsilon^2\chi_2^\epsilon(t)]$$

$$= \epsilon [A\chi_2^\epsilon(t) + C\chi_1^\epsilon(t) + B^{-1}(A - PA)Gv^\epsilon(t)$$

$$\qquad + AB^{-1}(A - PA)PAPv^\epsilon(t) - CPAPv^\epsilon(t)$$

$$(3.32) \qquad + GPAPv^\epsilon(t) - B^{-1}(A - PA)PAPAPv^\epsilon(t)]$$

$$\qquad + \epsilon^2 [C\chi_2^\epsilon(t) + B^{-1}AB^{-1}(A - PA)Gv^\epsilon(t) - B^{-1}Cv^\epsilon(t)$$

$$\qquad + B^{-1}G^2v^\epsilon(t) - B^{-2}(A - PA)PAPGv^\epsilon(t)].$$

Our smoothness hypotheses imply now that for $T < \infty$

$$(3.33) \qquad \sup_{0 \le t \le T} \left\| u^\epsilon(t) - v^\epsilon(t) - \epsilon \chi_1^\epsilon(t) - \epsilon^2 \chi_2^\epsilon(t) \right\| \le c\epsilon,$$

where c depends clearly on the terms in the brackets on the right side of (3.32). This completes the proof.

REMARK 2. It was realized in [5] that in the hydrodynamical limit for the linearized Boltzmann equation the condition PAP = 0 is not satisfied. The present theorem covers that case. We should note that both results in [4] and the results in [5], or the present ones, constitute a formalized and streamlined way to do "second order perturbation theory" (see for example [19], Chapter IV). In [5] the smoothness requirements are easily verified because G^ϵ has constant coefficients and so commutes with differentiations.

REMARK 3. The theorem above was formulated in such a way that the simplicity of the proof remains transparent, i.e., the correctors χ_1 and χ_2 are chosen so that the singular in ϵ terms cancel and (3.18) or (3.32) follow. We have not attempted to optimize the various regularity conditions (domains) or the notions of convergence (other than strong) so that some interesting cases may be ruled out. The idea is, however, that in specific problems as for example [13], one should simply adapt the present method of analysis rather than try to fit the situation to the present (or other) theorems verbatim. The method shows also how higher order corrections can be obtained.

4. **Dynamical systems coupled to a heat bath.** It was pointed out by Davies in [20] that results such as the one of § 3 are not sufficient for problems that he considers [8, 9] and that arise in quantum mechanical contexts [11]. They are also not sufficient for the results in [7, 21] and, in addition, it is of interest to have a method of analysis that does not rely on expansions of infinite order [8, 9]. The purpose of this section is to extend the analysis of § 3 appropriately so that such problems are covered. We consider a classical dynamical system coupled to a heat bath and analyze it asymptotically in the weak coupling limit. The problem is equivalent to the asymptotics for stochastic equations with non-Markovian coefficients ([21] and references therein). Our approach is motivated by Davies' ideas in [20], the work of Kurtz in [22] and the work in [21]. A more general approach using Martingale theory is given in [27].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let

$$(4.2) \qquad\qquad \omega \to \omega(t), \quad -\infty < t < \infty$$

be a measure preserving group of transformations on Ω. Let $H = L^2(\Omega, P)$ be the real Hilbert space of square integrable random variables on Ω with inner product

$$(X, Y) = E\{X, Y\} = \int_\Omega X(\omega)Y(\omega)P(d\omega).$$

Let $F(x, \omega)$ be a measurable function on $R^n \times \Omega \to R^n$ such that

(4.3) $E^{1/2}\{(\sup_x |F(x, \cdot)|)^2\} = \| \sup_x |F(x, \cdot)| \|_H < \infty$

and

(4.4) $E^{1/2} \left\{ \left(\sup_x \left| \dfrac{\partial F(x, \cdot)}{\partial x} \right| \right)^2 \right\} < \infty.$

Here $| \ |$ stands for the Euclidean norm of vectors or matrices.

Consider the solution $x^\epsilon(t) = x^\epsilon(t, x, \omega)$ of

(4.5) $\dfrac{dx^\epsilon(t)}{dt} = (1/\epsilon)F(x^\epsilon(t), \omega^\epsilon(t)), \quad x^\epsilon(0) = x.$

Here $\omega^\epsilon(t) \equiv \omega(t/\epsilon^2)$, $\omega(0) = \omega$ and (4.5) has, in view of (4.3) and (4.4), solutions for almost all $\omega \in \Omega$ that exist for all time if $\epsilon > 0$.

We shall assume that

(4.6) $\int_\Omega F(x, \omega)P(d\omega) = E\{F(x, \cdot)\} = 0,$

which corresponds to the case $PAP \equiv 0$ of the previous section. Just as in the theorem of the previous section it can be removed if necessary but the limit problem will then depend on ϵ. See remark 3.

The motion $\omega(t)$ in Ω will be referred to as the bath. P is the probability measure with respect to which the initial state of the bath is distributed. The motion $x^\epsilon(t)$, scaled appropriately already, will be referred to as the system. It depends on the initial state of the system x as well as on the initial state of the bath ω. Thus, the only way randomness enters in the system is via the initial distribution of the bath.

For $f \in H$ define

(4.7) $U(t)f(\omega) = f(\omega(t)) = f(t, \omega), t \in (-\infty, \infty).$

Since $\omega \to \omega(t)$ is measure preserving, $U(t)$ is a unitary group of operators on H. The function $f(t, \omega)$ is a stationary random function generated by $U(t)$. If we define

(4.8) $(U(t)F(x, \cdot))(\omega) = F(x, t, \omega)$

(abbreviated $F(x, t)$ frequently), then (4.5) takes the form

$$(4.9) \qquad \frac{dx^\epsilon(t)}{dt} = (1/\epsilon) F(x^\epsilon(t), t/\epsilon^2), \quad x^\epsilon(0) = x,$$

which is the familiar form for stochastic equations ([21] and references therein).

We now proceed to show the connection of the above problem with the framework of [8, 9, 11]. The actual treatment we shall give later does not use all of the assumptions made here for this comparison.

Assume that the flow $\omega(t)$ is ergodic so that

$$(4.10) \quad \lim_{\lambda \downarrow 0} \lambda \int_0^\infty e^{-\lambda s} U(s) f ds = E\{f\} = \int_\Omega f(\omega) P(d\omega), \, f \in H,$$

the limit being taken in the H norm. Since $U(t)$ is unitary, there exists a possibly unbounded selfadjoint operator B on H such that

$$(4.11) \qquad\qquad U(t) = e^{iBt}.$$

Consider now the space L of functions $f(x, \omega)$ on $R^n \times \Omega \to R$, measurable, bounded continuous in x for almost all ω and in H as functions of ω. Let

$$(4.12) \qquad\qquad \|f\|_L = \| \sup_x |f(x, \cdot)| \|_H.$$

Let L_0 be the Banach space of bounded continuous functions on R^n with $\|f\|_{L_0} = \sup_x |f(x)|$. We define on L the group $T^\epsilon(t)$ by

$$(4.13) \qquad\qquad T^\epsilon(t) f(x, \omega) = f(x^\epsilon(t), \omega^\epsilon(t)),$$

where $x^\epsilon(t)$ is the solution of (4.5). This group is a contraction on L and for appropriately restricted functions in L

$$(4.14) \lim_{h \downarrow 0} \frac{T^\epsilon(h) f - f}{h} \equiv H^\epsilon f = (1/\epsilon) F(x, \omega) \frac{\partial f(x, \omega)}{\partial x} + (i/\epsilon^2) B f(x, \omega).$$

Here B acts on f as a function of ω alone.

The operator H^ϵ of (4.14) has the same form as the operator H^ϵ of (3.8) except that $C \equiv 0$ here. Motivated by this formal similarity we seek to analyze the asymptotic behavior of $T^\epsilon(t)$ for $\epsilon \downarrow 0$ in the manner of Section 3. If we define A formally by

$$(4.15) \qquad\qquad A = F(x, \omega) \frac{\partial}{\partial x},$$

then, because of (4.6), we have $PAP \equiv 0$ with the projection P defined by (4.10): $P = E\{ \cdot \}$. Let $u^\epsilon(t) = T^\epsilon(t) f$ and put

$$v^\epsilon(t) = PT^\epsilon(t)f$$

$$w^\epsilon(t) = (1 - P)T^\epsilon(t)f.$$

Then the formal evolution equation for $T^\epsilon(t)f$ becomes

(4.16)
$$\frac{dv^\epsilon(t)}{dt} = (1/\epsilon)P(Aw^\epsilon),$$

$$\frac{dw^\epsilon(t)}{dt} = (i/\epsilon^2)Bw^\epsilon + (1/\epsilon)Av^\epsilon + (1/\epsilon)(1 - P)Aw^\epsilon.$$

In this form the problem looks similar to the problems in [8, 9, 10, 11]. Note however that A in (4.15) is unbounded, being a differential operator, and hence infinite expansion procedures are not suitable here. Note that the first equation in (4.16) has no $1/\epsilon^2$ term on the right hand side. All problems that at first have such terms there, can be transformed into the form (4.16) (provided that the original problem did admit the kind of asymptotics we are contemplating). See remark 2.

The difficulty with H^ϵ of (4.14), or the system (4.16), is that on attempting to use the procedures of Section 3 the correctors χ_1 and χ_2 do not have the required properties. This difficulty is in the nature of the problem and cannot be avoided. With $f = f(x) \in L_0$ solutions χ_1 of

(4.17) $i B\chi_1 + Af = 0$

exist only for special $F(x, \omega)$ in (4.15) despite the fact that (4.6) holds. If (4.17) has solutions in L then $PA\chi_1 \equiv 0$ and our problem has a trivial limit. On the other hand $PA\chi_1 \not\equiv 0$ implies that (4.17) does not have solutions in L. Said another way, $PA\chi_1 \not\equiv 0$ requires B to have 0 in the continuous spectrum and then (4.17) is solvable for special F for which $PA\chi_1 \equiv 0$. Using (4.8) we have

$$PA\chi_1 = \int_0^\infty E \left\{ F(x, 0) \frac{\partial}{\partial x} \left(F(x, t) \frac{\partial f(x)}{\partial x} \right) \right\} dt$$

and

$$\chi_1 = \int_0^\infty F(x, t) \frac{\partial f(x)}{\partial x} dt,$$

formally, from which the above statements can be verified.

Now we proceed to what seems to be the proper way to treat the asymptotics of $PT^\epsilon(t)f$, with $f \in L_0$. We shall not adhere to the Hilbert space framework above since it is not necessary to do so.

Let $F(x, \omega)$ be a measurable function on $R^n \times \Omega \to R^n$ and let $\mathfrak{F}_t$ $t \geqq 0$, be the σ-algebra generated by $F(x, \omega(s)) = F(x, s, \omega)$, $0 \leqq s \leqq t$. Assume that

$$E\{\sup_x |F(x, \cdot)|\} < \infty, E\left\{\sup_x \left| \frac{\partial F(x, \cdot)}{\partial x} \right| \right\} < \infty,$$

which implies

$$\sup_{t \geqq 0} E\{\sup_x |F(x, t, \cdot)|\} < \infty,$$

$$\sup_{t \geqq 0} E\left\{\sup_x \left| \frac{\partial F(x, t, \cdot)}{\partial x} \right| \right\} < \infty.$$

With these hypotheses, the solution $x^\epsilon(t) = x^\epsilon(t, x, \omega)$ of

$$(4.18) \qquad \frac{dx^\epsilon(t)}{dt} = (1/\epsilon F(x^\epsilon(t), \omega^\epsilon(t)), \quad x^\epsilon(0) = x,$$

with $\omega^\epsilon(t) \equiv \omega(t/\epsilon^2)$, is well defined and it is $\mathfrak{F}_{t/\epsilon^2}$ measurable.

Let $\tilde{L}$ be the collection of measurable functions $f(x, t, \omega)$ on $R^n \times [0, \infty) \times \Omega \to R$ such that $f(x, t, \cdot)$ is $\mathfrak{F}_t$ measurable $t \geqq 0$, $x \in R^n$ and

$$\|f\|_{\tilde{L}} \equiv \sup_{t \geqq 0} E\{\sup_x |f(x, t, \cdot)|\} < \infty.$$

On $\tilde{L}$ we define the semigroup of operators [22]

$$(4.19) \qquad \tilde{U}(s)f(x, t, \omega) = E\{f(x, t + s, \cdot) | \mathfrak{F}_t\},$$

the semigroup property being an immediate consequence of the properties of conditional expectation. We also have that

$$\|\tilde{U}(s)f\|_{\tilde{L}} \leqq \|f\|_{\tilde{L}}$$

so that $\tilde{U}(s)$ is a contraction. Note that $\tilde{U}(s)$ acts on f as a function of t and ω only; x is merely a parameter. The conditional expectations in (4.19) have a version that renders $\tilde{U}(s)f$ an element of $\tilde{L}$ again [22]. On suitably restricted elements of $\tilde{L}$ we define the infinitesimal generator $\tilde{B}$ of $\tilde{U}(s)$ by

$$(4.20) \qquad \tilde{B}f = \lim_{h \downarrow 0} \frac{\tilde{U}(h)f - f}{h}$$

in $\tilde{L}$. Let L_0 be the Banach space of bounded continuous functions on R^n, with $\|f\|_{L_0} = \sup_x |f(x)|$.

For $f \in \bar{L}$ we define $\tilde{T}^\epsilon(s)$ by

$$(4.21) \qquad \tilde{T}^\epsilon(s)f(x, t, \omega) = E\{f(x^\epsilon(t + s), t + s/\epsilon^2, \cdot) \mid \mathcal{F}_t\},$$

where $x^\epsilon(t)$ is the solution of (4.18). For functions in $\tilde{L}$ that are appropriately restricted (at least differentiable in x) we have that

$$(4.22) \qquad \lim_{h\downarrow 0} \frac{\tilde{T}^\epsilon(h)f - f}{h} \equiv \tilde{H}^\epsilon f = (1/\epsilon)F(x, t, \omega)\frac{\partial f(x, t, \omega)}{\partial x}$$
$$+ (1/\epsilon^2)\tilde{B}f(x, t, \omega).$$

From this we see that $\tilde{H}^\epsilon$ is again of the form (3.6) so we shall study the asymptotics of $\tilde{T}^\epsilon(s)$ by the methods of § 3.

Let $f(x)$ be a bounded smooth function on R^n, i.e., with bounded derivatives of all orders and define

$$(4.23) \quad \chi_1(x, t, \omega) = \int_0^\infty E\left\{F(x, t + s, \cdot)\,\frac{\partial f(x)}{\partial x} \mid \mathcal{F}_t\right\} ds$$

$$(4.24) \qquad\qquad\qquad \chi_2(x, t, \omega)$$

$$= \int_0^\infty \left[E\left\{F(x, t + s, \cdot)\frac{\partial \chi_1(x, t + s, \cdot)}{\partial x} \mid \mathcal{F}_t\right\} - \bar{L}f(x)\right]\, ds$$

where

$$(4.25) \quad \bar{L}f(x) = \int_0^\infty E\left\{F(x, 0, \cdot)\frac{\partial}{\partial x}\left(F(x, t, \cdot)\frac{\partial f(x)}{\partial x}\right)\right\} dt.$$

We assume that F is such that χ_1 and χ_2 exist as elements of $\tilde{L}$, i.e.,

$$(4.26)\, \sup_{t\geq 0} E\{\sup_x |\chi_1(x, t, \cdot)|\} < \infty,\, \sup_{t\geq 0} E\left\{\sup_x \left|\frac{\partial \chi_1}{\partial x}(x, t, \cdot)\right|\right\} < \infty,$$

$$(4.27) \qquad\qquad \sup_{t\geq 0} E\{\sup_x |\chi_2(x, t, \cdot)|\} < \infty.$$

Recall that (4.6) holds here so that (4.26) is a form of mixing similar to Rosenblatt's [23, 24]. More appropriately, the existence of χ_1, in the sense of (4.26), corresponds to the existence of the recurrent potential [25, 26]. Definition (4.25) is the same as

$$\int_0^\infty ds\, E\left\{ E\left\{ F(x, t + s, \cdot)\frac{\partial \chi_1(x, t + s, \cdot)}{\partial x} \mid \mathcal{F}_t \right\}\right\} = \bar{L}f(x)$$

so that the integrand in (4.24) has zero expectation. The existence of χ_2 is the same sort of affair as the existence of χ_1 but more involved. In addition to the above hypotheses we shall assume that

$$(4.28) \qquad \sup_{t \geq 0} E \left\{ \sup_x \ \left| \ F(x, t, \cdot) \ \frac{\partial \chi_2(x, t, \cdot)}{\partial x} \ \right| \right\} < \infty,$$

and

$$(4.29) \qquad \lim_{h \downarrow 0} (1/h) \int_0^h E\{\chi_1(x, t + s) \mid \mathfrak{I}_t\} \, ds = \chi_1(x, t),$$

$$(4.30) \qquad \lim_{h \downarrow 0} (1/h) \int_0^h E\{\chi_2(x, t + s) \mid \mathfrak{I}_t\} \, ds = \chi_2(x, t),$$

the limits being in the $\tilde{L}$ norm.

LEMMA. *With the above hypotheses and with $f(x)$ smooth we have*

$$(4.31) \qquad E \left\{ \left| \ E\{f(x^\epsilon(t + s)) \mid \mathfrak{I}_t\} - f(x^\epsilon(t)) \right. \right.$$
$$\left. \left. - \int_0^s E\{\bar{L}f(x^\epsilon(t + \sigma)) \mid \mathfrak{I}_t\} \, d\sigma \ \right| \right\} = 0(\epsilon),$$

for all $t \geq 0$ and $0 \leq s \leq T < \infty$.

PROOF. From the definitions of χ_1 and χ_2, (4.26), (4.27) and (4.29) and (4.30) it follows that χ_1 and χ_2 are in the domain of B and satisfy the potential equations ($f(x)$ smooth is fixed)

$$(4.32) \qquad \tilde{B}\chi_1 + F \frac{\partial f}{\partial x} = 0,$$

$$(4.33) \qquad \tilde{B}\chi_2 + F \frac{\partial \chi_1}{\partial x} - \bar{L}f = 0.$$

Thus, we have the identity

$$(4.34) \qquad \begin{aligned} &\tilde{T}^\epsilon(s)(f + \epsilon\chi_1 + \epsilon^2\chi_2) - (f + \epsilon\chi_1 + \epsilon^2\chi_2) \\ &- \int_0^s \tilde{T}^\epsilon(\sigma) \left(\frac{1}{\epsilon^2} \tilde{B} + \frac{1}{\epsilon} F \right) \frac{\partial}{\partial x} (f + \epsilon\chi_1 + \epsilon^2\chi_2) \, d\sigma = 0, \end{aligned}$$

which, in view of (4.32) and (4.33), becomes

$$\tilde{T}^\epsilon(s)f - f - \int_0^s \tilde{T}^\epsilon(\sigma)\bar{L}f \, d\sigma = -\tilde{T}^\epsilon(s)(\epsilon X_1 + \epsilon^2 X_2)$$

$$+ (\epsilon X_1 + \epsilon^2 X_2)$$

$$+ \int_0^s T^\epsilon(\sigma) \left(\epsilon F \frac{\partial X_2}{\partial x} \right) d\sigma.$$

From the hypotheses above, including (4.28), and the fact that $\tilde{T}^\epsilon(s)$ is a contraction in the $\tilde{L}$ norm the conclusion (4.31) follows.

We wish now to use (4.31) to conclude that if $\bar{u}(t, x)$ is the solution of

$$(4.36) \qquad\qquad \frac{\partial \bar{u}}{\partial t} = \bar{L}\bar{u}(t, x), \, t > 0, \bar{u}(o, x) = f(x),$$

assuming it exists in the classical sense, then $E\{f(x^\epsilon(t + s)) \mid \mathcal{F}_t\}$ is approximated well by $\bar{u}(s, x^\epsilon(t))$, i.e., our system $x^\epsilon(t)$ tends in the limit under consideration to the diffusion generated by $\bar{L}$. The most appropriate way to carry out this step is by Martingale theory as in [27]. Here we proceed in an elementary way using smoothness.

Assume that the coefficients of $\bar{L}$ are smooth, i.e., have bounded derivatives of all orders, but $\bar{L}$ need not be uniformly elliptic. If f has bounded derivatives of all orders then $\bar{u}(t, x)$ in (4.36) has a solution with bounded derivatives of all orders [28] in $0 \leqq t \leqq T < \infty$. The result (4.31) remains valid if $f = f(t, x)$ is a smooth function of $t \geqq 0$ and $x \in R^n$. However, in this case the operator $\bar{L}$ should be replaced by $\partial/\partial\sigma + \bar{L}$ under the integral sign in (4.31). If we now apply this modified (4.31) to $f = u(t + s - \sigma, x)$, with σ the "running" variable, the integral term cancels by (4.36) and the following results obtains.

THEOREM. *Under the hypotheses of the Lemma above and the existence of smooth solutions for* (4.36) *with* $f(x)$ *smooth,*

$$(4.37) \qquad E\{|E\{f(x^\epsilon(t + s) \mid \mathcal{F}_t\} - \bar{u}(s, x^\epsilon(t))|\} = 0(\epsilon)$$

for all $t \geqq 0$ *and* $0 \leqq s \leqq T < \infty$.

We conclude this section with some remarks concerning extensions to related problems.

REMARK 1. Let $H = L^2(R^3)$ and consider on $C \oplus H$ the evolution equation [9 and more generally in 10]

$$\frac{dv^\epsilon}{dt} = \frac{i}{\epsilon}(F, w^\epsilon), \quad v^\epsilon(0) = v_0,$$

(4.38)

$$\frac{dw^\epsilon}{dt} = \frac{i}{\epsilon^2}(\Delta + \omega)w^\epsilon + \frac{i}{\epsilon}Fv^\epsilon, \quad w^\epsilon(0) = 0.$$

Here $v^\epsilon(t) \in C$ and $w^\epsilon(t) \in L^2(R^3)$ for each $t \geq 0$, $F \in H$ is a fixed function, $\omega > 0$ and Δ is the Laplacian on $L^2(R^3)$. We are interested in the behavior of $v^\epsilon(t)$ as $\epsilon \downarrow 0$. If $F \in L^2(R^3) \cap L^1(R^3)$ then, as is shown in [9], $v^\epsilon(t)$ tends to $\bar{v}(t)$ and

(4.39) $$\frac{d\bar{v}(t)}{dt} = -\alpha\bar{v}(t), \quad \bar{v}(0) = v_0,$$

where

$$\alpha = \int_0^\infty (F, e^{i(\Delta + \omega)s}F) \, ds$$

is well defined and $\mathrm{Re}\,\alpha > 0$.

This problem is similar to (4.16) so the results of § 3 do not apply directly. It would be interesting to know if procedures such as the one of this section apply to this and, ultimately, more involved problems of this form [11].

REMARK 2. Frequently, the problem of interest is not in the form (4.18) but in the form

(4.40) $$\frac{d\tilde{x}^\epsilon(t)}{dt} = (1/\epsilon^2)A\tilde{x}^\epsilon(t) + (1/\epsilon)\tilde{F}(\tilde{x}^\epsilon(t), \omega^\epsilon(t)), \quad \tilde{x}^\epsilon(0) = x,$$

where A is an oscillatory matrix. In the terminology of [8, 9, 20], the free system (uncoupled from the bath) is not in a trivial constant state but undergoes oscillatory motions. These rapid motions are removed by passing to the slowly varying quantities

$$x^\epsilon(t) = e^{-At/\epsilon^2}\tilde{x}^\epsilon(t)$$

so that

(4.41) $$\frac{dx^\epsilon(t)}{dt} = (1/\epsilon)e^{-At/\epsilon^2}\tilde{F}(e^{At/\epsilon^2}x^\epsilon(t), \omega^\epsilon(t))$$

$$\equiv (1/\epsilon)F(x^\epsilon(t), t/\epsilon^2, \omega), \quad x^\epsilon(0) = x.$$

Here, F is defined by

(4.42) $$F(x, t, \omega) = e^{-At}\tilde{F}(e^{At}x, \omega(t)),$$

with t dependence entering from both the motion of the bath as well as the oscillations. We assume again that (4.6) holds, i.e.,

(4.43) $\qquad \qquad \qquad \text{'} e \cdot f) x, t, \cdot)\} = 0.$

With minor modifications the above theorem remains valid but without the $0(\epsilon)$ estimate. The differences are as follows. First $\bar{L}$ in (4.25) is now given by

$$\bar{L}f(x) = \lim_{T \uparrow \infty} (1/T)$$

(4.44)

$$\cdot \int_{t_0}^{t_0 + T} \int_0^\infty E \left\{ F(x, t, \cdot) \frac{\partial}{\partial x} \left(F(x, t - s, \cdot) \frac{\partial f(x)}{\partial x} \right) \right\} ds \, dt$$

uniformly in x and $t_0 \geqq 0$. Next, χ_2 is not defined by (4.24) but we work with $\chi_2{}^{(\lambda)}$ defined by

$$\chi_2{}^{(\lambda)}(x, t, \omega) =$$

$$\int_0^\infty e^{-\lambda s} \left[E \left\{ F(x, t + s, \cdot) \frac{\partial \chi_1(x, t + s, \cdot)}{\partial x} \mid \mathcal{F}_t \right\} - \bar{L}f(x) \right] \, a$$

This is similar to what we did in sections 2 and 3. We assume that (4.28) and (4.30) hold here with χ_2 replaced by $\chi_2{}^{(\lambda)}$, $\lambda > 0$ and that

$$\lim_{\lambda \downarrow 0} \sup E\{ \sup_x |\lambda \chi_2{}^{(\lambda)}(x, t, \cdot)| \} = 0,$$

which is compatible with the definition (4.44) of $\bar{L}$.

The importance of being able to deal with systems such as (4.40) was emphasized in [21]. To deal with linear problems our other hypotheses above must be modified in the manner of [21] and Lemma 3 of [21] must be used, with minor modifications.

REMARK 3. Consider (4.18) again but this time assume that

(4.45) $\qquad \qquad \qquad E\{F(x, t, \cdot)\} = \bar{F}(x),$

where $F(x, t, \omega) = F(x, \omega(t))$. This corresponds to the case $PAP \neq 0$ in §3 so the approximation under consideration will be ϵ independent. We define

$$\bar{L}^\epsilon f(x) = (1/\epsilon) \bar{F}(x) \frac{\partial f(x)}{\partial x}$$

(4.46)

$$+ \int_0^\infty E \left\{ (F(x, 0, \cdot) - \bar{F}(x)) \frac{\partial}{\partial x} \left[(F(x, t, \cdot) - \bar{F}(x)) \frac{\partial f(x)}{\partial x} \right] \right\} dt$$

and assume that the solution $\bar{u}^\epsilon(t, x)$ of

(4.47) $$\frac{\partial \bar{u}^\epsilon(t, x)}{\partial t} = \bar{L}^\epsilon \bar{u}^\epsilon(t, x), \quad \bar{u}^\epsilon(0, x) = f(x),$$

has bounded derivatives of all orders (up to order 4 is enough) in $0 \leq t \leq T < \infty$, independently of $\epsilon > 0$, if $f(x)$ is smooth. This assumption corresponds to the smoothness assumption in Theorem 2 of § 3. The result is now:

(4.48) $$\lim_{\epsilon \downarrow 0} E\{|E\{f(x^\epsilon(t + s))| \mathfrak{I}_t\} - \bar{u}^\epsilon(s, x^\epsilon(t)|\} = 0$$

for all $t \geq 0$ and, $0 \leq s \leq T < \infty$.

REMARK 4. The problem of stochastic equations with Markovian coefficients discussed in § 3 is a special case of the above problem: the conditional expection given $\mathfrak{I}_t$ reduces in this case to a point function of the state at time t. The results in our asymptotic limit are identical in both cases.

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