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A CLASS OF BOUNDARY VALUE PROBLEMS WHOSE SOLUTIONS POSSESS ANGULAR LIMITING BEHAVIOR

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ABSTRACT. For small positive values of the parameter ϵ, solutions $y = y(t, \epsilon)$ of the problem $\epsilon^2 y'' = h(t, y)$, $-1 < t < 1$, $y(\pm 1, \epsilon)$ prescribed, are shown to exist and to satisfy (as $\epsilon \to 0^+$) $y(t, \epsilon) \to u_1(t)$ on $(-1, 0]$ and $y(t, \epsilon) \to u_2(t)$ on $[0, 1)$. The functions u_1, u_2 are solutions of the associated reduced equation $0 = h(t, u)$ on the intervals $[-1, 0]$, $[0, 1]$, respectively, which satisfy $u_1(0) = u_2(0)$ and $u_1'(0) \neq u_2'(0)$. The results are established by using an extension of the classical Nagumo theorem for second-order boundary value problems.

1. **Introduction.** We consider in this note the singularly perturbed boundary value problem

(1.1) $$\epsilon^2 y'' = h(t, y), \ -1 < t < 1,$$

(1.2) $$y(-1, \epsilon) = A, \ y(1, \epsilon) = B,$$

for small positive values of the parameter ϵ. The principal assumption is that the corresponding reduced or degenerate equation

(1.3) $$0 = h(t, u)$$

has a pair of solutions $u_1 = u_1(t)$ and $u_2 = u_2(t)$ defined and suitably smooth on $[-1, 0]$ and $[0, 1]$, respectively, with $u_1(0) = u_2(0)$ and $u_1'(0) \neq u_2'(0)$. Under additional assumptions which guarantee that u_1 and u_2 are stable roots of equation (1.3), it will be proved that for each sufficiently small value of $\epsilon > 0$, the problem (1.1), (1.2) has a solution $y = y(t, \epsilon)$ which satisfies $y(t, \epsilon) \to u_1(t)$, $t \in (-1, 0]$, and $y(t, \epsilon) \to u_2(t)$, $t \in [0, 1)$, as $\epsilon \to 0^+$. The nonuniform convergence which occurs near the boundaries $t = \pm 1$ (i.e., boundary layer behavior) is, of course, characteristic of solutions of a wide class of singular perturbation problems.

Similar observations have been made by Haber and Levinson [7] for the case of the more general boundary value problem

$$\epsilon y'' = f(t, y, y', \epsilon), \ -1 < t < 1,$$
$$y(-1, \epsilon), y(1, \epsilon) \text{ prescribed.}$$

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However, in their discussion, a crucial assumption is that the function $\partial f/\partial y'$ is never zero along appropriate solutions of the reduced equation $0 = f(t, u, u', 0)$. In addition, their assumptions preclude the occurrence of boundary layer behavior near $t = \pm 1$.

We remark that various aspects of the problem (1.1), (1.2) have been studied by several authors, including Briš [2], Vasileva [13, Chap. 3], Carrier and Pearson [3, Chap. 18], Boglaev [1], Fife [4], [5], [6], and most recently, Habets [8], and O'Malley [12]. The theorems proved below are most closely related to the work in [1] and [2].

The principal tool of our investigation is an extension of the classical Nagumo theory of two-point boundary value problems due to Habets and Laloy [9]. For convenience of the reader, we state this result in the next section before commencing our study of (1.1), (1.2).

2. **Nagumo-type Inequalities.** Our study of the problem (1.1), (1.2) is made possible by the following generalization of a theorem of Nagumo [**11**], [**10**, Thrm. 7.3] due to Habets and Laloy [**9**].

Consider the general boundary value problem

$$x'' = F(t, x, x'), a < t < b, \tag{2.1}$$

$$x(a), x(b) \text{ prescribed,} \tag{2.2}$$

where F is continuous on $[a, b] \times R^2$. Suppose first that there exist two continuous functions $\alpha = \alpha(t)$, $\beta = \beta(t)$ on $[a, b]$, $\alpha \leqq \beta$, which are piecewise $-C^{(2)}$ on $[a, b]$, i.e., there is a partition $\{t_i\}$ of $[a, b]$, $1 \leqq i \leqq n + 1$, with $a = t_1 < t_2 < \cdots < t_{n+1} = b$, such that on each subinterval $[t_i, t_{i+1}]$, α and β are twice continuously differentiable. At the partition points t_i, t_{i+1}, the derivatives are the righthand, respectively, lefthand derivatives. Suppose next that on each subinterval $[t_i, t_{i+1}]$,

$$\alpha''(t) \geqq F(t, \alpha(t), \alpha'(t))$$

and

$$\beta''(t) \leqq F(t, \beta(t), \beta'(t)),$$

and that for each $t \in [a, b]$, $D_\ell\alpha(t) \leqq D_r\alpha(t)$ and $D_\ell\beta(t) \geqq D_r\beta(t)$, where D_ℓ, D_r denote lefthand, respectively, righthand, differentiation. Finally suppose that $\alpha(a) \leqq x(a) \leqq \beta(a)$ and $\alpha(b) \leqq x(b) \leqq \beta(b)$. Then if the function F satisfies a Nagumo condition with respect to α, β, i.e., if there exists a positive continuous function $\phi = \phi(s)$ such that $\int_0^\infty s\,ds/\phi(s) = \infty$ and $|F(t, x, x')| \leqq \phi(|x'|)$, for $t \in [a, b]$, $\alpha(t) \leqq x(t) \leqq \beta(t)$ and $|x'| < \infty$, the problem (2.1), (2.2) has a solution $x = x(t)$ satisfying $\alpha(t) \leqq x(t) \leqq \beta(t)$, for $t \in [a, b]$.

We remark that since the function $h = h(t, y)$ in (1.1) is independent of y', it trivially satisfies a Nagumo condition. Consequently, the problem of studying the existence and asymptotic behavior of solutions of (1.1), (1.2) for small values of ϵ reduces to the construction of suitable bounding functions α and β satisfying the above inequalities.

Finally we state that in the following sections all derivatives of the functions u_1 and u_2 at the point $t = 0$ are to be understood as the appropriate one-sided derivatives. Also partial derivatives will be indicated as follows: $\partial_y{}^k h(t, y) \equiv \partial^k h(t, y)/\partial y^k$.

3. **The Case** $u_1{}'(0) < u_2{}'(0)$. Consider now the boundary value problem

$$\epsilon^2 y'' = h(t, y), \ -1 < t < 1, \tag{3.1}$$

$$y(-1, \epsilon) = A, \ y(1, \epsilon) = B, \tag{3.2}$$

and the corresponding reduced equation

$$0 = h(t, u). \tag{3.3}$$

We assume in this section that (3.3) has solutions $u = u_1(t)$, $u = u_2(t)$ on $[-1, 0]$, $[0, 1]$, respectively, satisfying $u_1(0) = u_2(0)$ and $u_1{}'(0) < u_2{}'(0)$.

THEOREM 3.1. *Assume*

(1) *there exist functions* $u_1 = u_1(t)$ *and* $u_2 = u_2(t)$ *defined on* $I_1 = [-1, 0]$ *and* $I_2 = [0, 1]$, *respectively, satisfying for* $i = 1, 2 : h(t, u_i(t)) = 0$, *with* $|u_i''(t)| \leqq M$, $t \in I_i$; *moreover*, $u_1(0) = u_2(0)$ *and* $u_1{}'(0) < u_2{}'(0)$;

(2) *for a nonnegative integer* q, *the function* h *is continuous in* (t, y) *and of class* $C^{(2q+1)}$ *with respect to* y *in* $D : -1 \leqq t \leqq 1, |y - u_i(t)| \leqq d$, $i = 1, 2$, *with* $d \geqq \max\{|A - u_1(-1)|, |B - u_2(1)|\}$;

(3) *for* $t \in I_i$, $i = 1, 2$, $\partial_y{}^k h(t, u_i(t)) = 0$, $k = 1, \cdots, 2q$; $\partial_y{}^{2q+1} h \geqq m^2 > 0$ *in* D, *for a positive constant* m.

Then there exists an $\epsilon_0 > 0$ *such that for each* ϵ, $0 < \epsilon \leqq \epsilon_0$, *there exists a solution* $y = y(t, \epsilon)$ *of* (3.1), (3.2). *In addition, for*

$$u(t) = \begin{cases} u_1(t), & -1 \leqq t \leqq 0, \\ u_2(t), & 0 \leqq t \leqq 1, \end{cases} \quad \textit{and } t \in [-1, 1],$$

$$|y(t, \epsilon) - u(t)| \leqq |A - u_1(-1)| \exp[-m\epsilon^{-1}(1 + t)] + |B - u_2(1)| \exp[-m\epsilon^{-1}(1 - t)] + c_0\epsilon,$$

if $q = 0$;

$$\begin{aligned}|y(t,\epsilon) - u(t)| \leqq\ & |A - u_1(-1)|(1 + \sigma_1(q)\epsilon^{-1}(1 + t))^{-q^{-1}} \\ & + |B - u_2(1)|(1 + \sigma_2(q)\epsilon^{-1}(1 - t))^{-q^{-1}} \\ & + c_q\epsilon^{(q+1)^{-1}},\end{aligned}$$

if $q \geqq 1$. *Here*

$$\sigma_1(q) = mq((q + 1)(2q + 1)!)^{-1/2}|A - u_1(-1)|^q,$$
$$\sigma_2(q) = mq((q + 1)(2q + 1)!)^{-1/2}|B - u_2(1)|^q,$$

and c_0, c_q *are positive, computable constants independent of* ϵ.

This theorem and the two that follow will be proved at the end of this section. We note that in Theorem 3.1 there is no restriction placed on the relative position of $u_1(-1)$ and A, and $u_2(1)$ and B. In the next two theorems, however, the weaker assumptions will require that appropriate restrictions be placed on these quantities.

THEOREM 3.2. *Assume* (1) *and* (2) *as in Theorem* 3.1 *with the exception that in* (1) $u_i'' \geqq 0$, $t \in I_i$, $i = 1, 2$, *and* $u_1(-1) \leqq A$, $u_2(1) \leqq B$, *and in* (2) *h is of class* $C^{(n)}$ *with respect to y, for* $n \geqq 2$. *Assume also* (3)$'$ *for* $t \in I_i$, $i = 1, 2$, $\partial_y{}^k h(t, u_i(t)) \geqq 0$, $k = 1, \cdots, n-1$; $\partial_y{}^n h \geqq m^2 > 0$ *in D, for a positive constant m.*

Then there exists an $\epsilon_0 > 0$ *such that for each* ϵ, $0 < \epsilon \leqq \epsilon_0$, *there exists a solution* $y = y(t,\epsilon)$ *of* (3.1), (3.2). *In addition, for* $t \in [-1, 1]$,

$$\begin{aligned}0 &\leqq y(t,\epsilon) - u(t) \\ &\leqq (A - u_1(-1))(1 + \sigma_1(n)\epsilon^{-1}(1 + t))^{-2(n-1)^{-1}} \\ &\quad + (B - u_2(1))(1 + \sigma_2(n)\epsilon^{-1}(1 - t))^{-2(n-1)^{-1}} \\ &\quad + c_n\epsilon^{2(n+1)^{-1}},\end{aligned}$$

where

$$\sigma_1(n) = m(n - 1)(2(n + 1)n!)^{-1/2}(A - u_1(-1))^{(n-1)/2},$$
$$\sigma_2(n) = m(n - 1)(2(n + 1)n!)^{-1/2}(B - u_2(1))^{(n-1)/2},$$

and the c_n *are positive, computable constants independent of* ϵ.

THEOREM 3.3. *Make the same assumptions as in Theorem* 3.2 *with the exception that in* (3)$'$, *for* $t \in I_i$, $i = 1, 2$, $\partial_y{}^k h(t, u_i(t)) \geqq 0$, $k = 0, 1, \cdots, \ell - 1$; $\partial_y{}^\ell h(t, u_i(t)) \geqq m^2 > 0$, $1 \leqq \ell < n$, *for some positive constant m; and finally,* $\partial_y{}^{\ell+1} h \geqq 0$ in D.

Then there exists an $\epsilon_0 > 0$ *such that for each* ϵ, $0 < \epsilon \leqq \epsilon_0$, *there exists a solution* $y = y(t,\epsilon)$ *of* (3.1), (3.2). *In addition, for* $t \in [-1, 1]$,

$$\begin{aligned} 0 &\leqq y(t,\epsilon) - u(t) \\ &\leqq (A - u_1(-1)) \exp[-m\epsilon^{-1}(1+t)] \\ &\quad + (B - u_2(1)) \exp[-m\epsilon^{-1}(1-t)] + c_0\epsilon, \end{aligned}$$

if $\ell = 1$;

$$\begin{aligned} 0 &\leqq y(t,\epsilon) - u(t) \\ &\leqq (A - u_1(-1))(1 + \sigma_1(\ell)\epsilon^{-1}(1+t))^{-2(\ell-1)^{-1}} \\ &\quad + (B - u_2(1))(1 + \sigma_2(\ell)\epsilon^{-1}(1-t))^{-2(\ell-1)^{-1}} \\ &\quad + c_\ell \epsilon^{2(\ell+1)^{-1}}, \end{aligned}$$

if $\ell \geqq 2$. *Here* $\sigma_1(\ell)$, $\sigma_2(\ell)$ *are the constants appearing in the statement of Theorem* 3.2 *with* n *replaced by* ℓ.

Before proceeding to an outline of the proofs of Theorems 3.1, 3.2 and 3.3, it is perhaps worthwhile to examine briefly the differences between these results. In Theorem 3.1, the functions u_1, u_2 are assumed to be roots of the reduced equation (3.3) of multiplicity $2q + 1$. The partial derivative, $\partial_y^{2q+1}h$, is required to be strictly positive in the entire region D; consequently, as noted above, the validity of the result is independent of the relative positions of $u_1(-1)$, A and $u_2(1)$, B. On the other hand, in Theorem 3.2 it is only required that $\partial_y^k h(t, u_i(t))$ be nonnegative for t in I_i and $1 \leqq k \leqq n - 1$. Moreover, the first partial derivative, $\partial_y^n h$, which is strictly positive in all of D, may be of even or odd order, whereas in Theorem 3.1, an odd order derivative was required to be strictly positive. These weaker assumptions necessitate the restrictions that $u_1(-1) \leqq A$, $u_2(1) \leqq B$ and $u_i'' \geqq 0$, $i = 1, 2$. Finally, Theorem 3.3 is a variant of Theorem 3.2 in the sense that the first strictly positive partial derivative, $\partial_y^\ell h$, is only required to possess this property along the reduced path formed by u_1 and u_2, provided the next higher partial derivative, $\partial_y^{\ell+1} h$, is nonnegative in all of D. These various nuances are illustrated by several representative examples in § 6.

The theorems are proved by defining the following functions α, β and verifying that the inequalities of the Habets-Laloy theorem are satisfied. In the case of Theorem 3.1, define for $t \in [-1, 1]$ and $\epsilon > 0$,

$$\alpha_1(t,\epsilon) = \begin{cases} u_1(t) - |A - u_1(-1)|E(t,\epsilon) \\ \qquad - |B - u_2(1)|F(0,\epsilon) - \Gamma(\epsilon), \ -1 \leqq t \leqq 0, \\ u_2(t) - |B - u_2(1)|F(t,\epsilon) \\ \qquad - |A - u_1(-1)|E(0,\epsilon) - \Gamma(\epsilon), 0 \leqq t \leqq 1; \end{cases}$$

$$\beta_1(t,\epsilon) = \begin{cases} u_1(t) + |A - u_1(-1)|E(t,\epsilon) + H(t,\epsilon) + \Gamma(\epsilon) \\ \qquad + (1-t)|A - u_1(-1)|E'(0,\epsilon) \\ \qquad + |B - u_2(1)|(F(0,\epsilon) \\ \qquad + F'(0,\epsilon)), \ -1 \leqq t \leqq 0, \\ u_2(t) + |B - u_2(1)|F(t,\epsilon) + H(0,\epsilon) + \Gamma(\epsilon) \\ \qquad + (1-t)|B - u_2(1)|F'(0,\epsilon) \\ \qquad + |A - u_1(-1)|(E(0,\epsilon) \\ \qquad + E'(0,\epsilon)), 0 \leqq t \leqq 1. \end{cases}$$

Here

$$\begin{aligned} E(t,\epsilon) &= \exp[-m\epsilon^{-1}(1+t)], && \text{if } q = 0, \\ E(t,\epsilon) &= (1+\sigma_1(q)\epsilon^{-1}(1+t))^{-q^{-1}}, && \text{if } q \geqq 1; \\ F(t,\epsilon) &= \exp[-m\epsilon^{-1}(1-t)], && \text{if } q = 0, \\ F(t,\epsilon) &= (1+\sigma_2(q)\epsilon^{-1}(1-t))^{-q^{-1}}, && \text{if } q \geqq 1; \\ \Gamma(\epsilon) &= (\epsilon^2\gamma m^{-2})^{(2q+1)^{-1}}, && \text{if } q \geqq 0, \end{aligned}$$

for γ a positive constant to be determined below; and finally,

$$\begin{aligned} H(t,\epsilon) &= \epsilon m^{-1}(u_2'(0) - u_1'(0))\exp[m\epsilon^{-1}t], \quad \text{if } q = 0, \\ H(t,\epsilon) &= qk^{-1}\epsilon^{(q+1)^{-1}}(u_2'(0) - u_1'(0)) \cdot \\ &\quad \cdot (1 - k\epsilon^{-(q+1)^{-1}}t)^{-q^{-1}}, \qquad \text{if } q \geqq 1, \end{aligned}$$

with

$$k = |mq^{(q+1)}((q+1)(2q+1)!)^{-1/2}(u_2'(0) - u_1'(0))^q|^{(q+1)^{-1}}.$$

In the case of Theorem 3.2, define

$$\alpha_2(t,\epsilon) = \begin{cases} u_1(t), & -1 \leqq t \leqq 0, \\ u_2(t), & 0 \leqq t \leqq 1; \end{cases}$$

$$\beta_2(t,\epsilon) = \begin{cases} u_1(t) + (A - u_1(-1))E_{(n)}(t,\epsilon) + H_{(n)}(t,\epsilon) + \Gamma(\epsilon) \\ \quad + (1-t)(A - u_1(-1))E'_{(n)}(0,\epsilon) \\ \quad + (B - u_2(1))(F_{(n)}(0,\epsilon) \\ \quad + F'_{(n)}(0,\epsilon)), \ -1 \leqq t \leqq 0, \\ u_2(t) + (B - u_2(1))F_{(n)}(t,\epsilon) + H_{(n)}(0,\epsilon) + \Gamma(\epsilon) \\ \quad + (1-t)(B - u_2(1))F'_{(n)}(0,\epsilon) \\ \quad + (A - u_1(-1))(E_{(n)}(0,\epsilon) \\ \quad + E'_{(n)}(0,\epsilon)), \ 0 \leqq t \leqq 1. \end{cases}$$

Here

$$E_{(n)}(t,\epsilon) = (1 + \sigma_1(n)\epsilon^{-1}(1+t))^{-2(n-1)^{-1}},$$
$$F_{(n)}(t,\epsilon) = (1 + \sigma_2(n)\epsilon^{-1}(1-1))^{-2(n-1)^{-1}},$$
$$H_{(n)}(t,\epsilon) = (n-1)/2k\,\epsilon^{2(n+1)^{-1}}(u_1'(0) - u_1'(0)) \cdot$$
$$\cdot (1 - k\epsilon^{-2(n+1)^{-1}}t)^{-2(n-1)^{-1}},$$

with

$$k = \left| \frac{m^2}{n!}(n-1)^{n+1}(n+1)^{-1}2^{-n}(u_2'(0) - u_1'(0))^{n-1} \right|^{(n+1)^{-1}}$$

and

$$\Gamma(\epsilon) = (\epsilon^2\gamma m^{-2})^{n-1}.$$

Finally, in the case of Theorem 3.3, define

$$\alpha_3(t,\epsilon) = \alpha_2(t,\epsilon) \text{ and } \beta_3(t,\epsilon) = \beta_2(t,\epsilon)$$

with n replaced by ℓ, if $\ell \geqq 2$. If $\ell = 1$, define $E_{(1)}(t,\epsilon) = \exp[-m\epsilon^{-1}(1+t)]$, $F_{(1)}(t,\epsilon) = \exp[-m\epsilon^{-1}(1-t)]$; $H_{(1)}(t,\epsilon) = \epsilon m^{-1}(u_2'(0) - u_1'(0)] \exp[m\epsilon^{-1}t]$, and $\Gamma(\epsilon) = \epsilon^2\gamma m^{-2}$, then $\beta_3(t,\epsilon) = \beta_2(t,\epsilon)$ with $E_{(n)}$ replaced by $E_{(1)}$, etc.

We observe that for each pair α_j, β_j, $j = 1, 2, 3$, $\alpha_j \leqq \beta_j$, $\alpha_j(-1,\epsilon) \leqq A \leqq \beta_j(-1,\epsilon)$, $\alpha_j(1,\epsilon) \leqq B \leqq \beta_j(1,\epsilon)$, and that $D_r\alpha_j(0) \geqq D_\ell\alpha_j(0)$ and $D_\nu\beta_j(0) = D_\ell\beta_j(0)$, for sufficiently small values of ϵ. It only remains to verify that the differential inequalities

$$\epsilon^2\alpha_j''(t) \geqq h(t,\alpha_j(t)) \text{ and } \epsilon^2\beta_j''(t) \leqq h(t,\beta_j(t))$$

are satisfied on $[-1, 0]$ and $[0, 1]$. Since the arguments are virtually the same, we verify in detail only that $\epsilon^2\beta_2''(t) \leqq h(t, \beta_2(t))$ and

$\epsilon^2\alpha_2''(t) \geqq h(t, \alpha_2(t))$. Recall that

$$\beta_2(t,\epsilon) = \begin{cases} u_1(t) + (A - u_1(-1))E_{(n)}(t,\epsilon) & \\ \qquad + H_{(n)}(t,\epsilon) + \Gamma_1(\epsilon), & -1 \leqq t \leqq 0, \\ u_2(t) + (B - u_2(1))F_{(n)}(t,\epsilon) & \\ \qquad + \Gamma_2(\epsilon), & 0 \leqq t \leqq 1, \end{cases}$$

where $\Gamma_1(\epsilon) = (1-t)(A - u_1(-1))E'_{(n)}(0,\epsilon) + (B - u_2(1))(F_{(n)}(0, \epsilon) + F'_{(n)}(0,\epsilon)) + \Gamma(\epsilon)$ and $\Gamma_2(\epsilon) = H_{(n)}(0,\epsilon) + (1-t)(B - u_2(1))F'_{(n)}(0, \epsilon) + (A - u_1(-1))(E_{(n)}(0, \epsilon) + E'_{(n)}(0, \epsilon)) + \Gamma(\epsilon)$. We note that if $A > u_1(-1)$, the terms $\Gamma_1(\epsilon)$ and $\Gamma_2(\epsilon)$ are nonnegative for ϵ sufficiently small even though they contain the negative terms $(1-t)(A - u_1(-1))E'_{(n)}(0,\epsilon)$ and $(A - u_1(-1))E'_{(n)}(0,\epsilon)$, respectively. On $[-1, 0]$, differentiating β_2, substituting into (3.1) and expanding by Taylor's Theorem we have

$$\begin{aligned} &h(t,\beta_2(t)) - \epsilon^2\beta_2''(t) \\ &= \sum_{k=1}^{n-1} \Big\{\frac{1}{k!}\,\partial_y{}^k h(t,u_1(t))\big[(A - u_1(-1))E_{(n)}(t,\epsilon) \\ &\quad + H_{(n)}(t,\epsilon) + \Gamma_1(\epsilon)\big]^k\Big\} \\ &\quad + \frac{1}{n!}\partial_y{}^n h\{\,\cdot\,\}\big[(A - u_1(-1))E_{(n)}(t,\epsilon) \\ &\quad + H_{(n)}(t,\epsilon) + \Gamma_1(\epsilon)\big]^n \\ &\quad - \epsilon^2 u_1''(t) \\ &\quad - \epsilon^2(A - u_1(-1))E''_{(n)}(t,\epsilon) - \epsilon^2 H''_{(n)}(t,\epsilon), \end{aligned}$$

where $\{\,\cdot\,\} \in D$ is the appropriate intermediate point. We now use assumptions (1) and (3)$'$ and the fact that $\Gamma_1(\epsilon) \geqq 0$ to continue with the inequality:

$$\begin{aligned} &h(t,\beta_2(t)) - \epsilon^2\beta_2''(t) \\ &\geqq \frac{m^2}{n!}\big[(A - u_1(-1))^n E^n_{(n)}(t,\epsilon) + H^n_{(n)}(t,\epsilon) + \Gamma_1{}^n(\epsilon)\big] \\ &\quad - \epsilon^2 M - \epsilon^2(A - u_1(-1))E''_{(n)}(t,\epsilon) - \epsilon^2 H''_{(n)}(t,\epsilon). \end{aligned}$$

We note next that the functions $E_{(n)}$ and $H_{(n)}$ satisfy by construction the differential equations:

$$\epsilon^2(A - u_1(-1))E''_{(n)}(t,\epsilon) = \frac{m^2}{n!}(A - u_1(-1))^n E^n_{(n)}(t,\epsilon)$$

and

$$\epsilon^2 H''_{(n)}(t,\epsilon) = \frac{m^2}{n!} H^n_{(n)}(t,\epsilon).$$

Also, a short computation reveals that for $\gamma > 1$ and ϵ sufficiently small, $\Gamma_1(\epsilon) \geqq (1/2)(\epsilon^2\gamma m^{-2})^{n-1}$. Consequently,

$$\begin{aligned} h(t,\beta_2(t)) - \epsilon^2\beta_2''(t) &\geqq \frac{m^2}{n!}\Gamma_1{}^n(\epsilon) - \epsilon^2 M, \\ &\geqq \frac{m^2}{n!}2^{-n}\epsilon^2\gamma m^{-2} - \epsilon^2 M \geqq 0, \end{aligned}$$

provided the constant $\gamma > 1$ is additionally chosen so that $\gamma \geqq 2^n n! M$. Similarly on $[0, 1]$,

$$\begin{aligned} &h(t,\beta_2(t)) - \epsilon^2\beta_2''(t) \\ &\geqq \frac{m^2}{n!}[(B - u_2(1))^n F^n_{(n)}(t,\epsilon) + \Gamma_2{}^n(\epsilon)] \\ &\quad - \epsilon^2 M - \epsilon^2(B - u_2(1))F''_{(n)}(t,\epsilon) \\ &\geqq \frac{m^2}{n!}\Gamma_2{}^n(\epsilon) - \epsilon^2 M \geqq 0, \end{aligned}$$

since $\gamma \geqq \max\{1, 2^n n! M\}$, and since $\epsilon^2(B - u_2(1))F''_{(n)}(t, \epsilon) = (m^2/n!)(B - u_2(1))^n F^n_{(n)}(t, \epsilon)$. The verification that α_2 satisfies the required inequalities is trivial, since $h(t, \alpha_2(t)) = 0$, $t \in [-1, 1]$ and $\alpha_2'' \geqq 0$, i.e., $\epsilon^2\alpha_2''(t) - h(t, \alpha_2(t)) \geqq 0$. Thus by the theorem of Habets and Laloy, quoted in § 2, under the assumptions of Theorem 3.2, the problem (3.1), (3.2) has a solution $y = y(t,\epsilon)$ satisfying the stated estimates. The proofs of the other two theorems follow analogously.

Remark 1. We note that the above theorems are valid under the assumption that $u_1'(0) = u_2'(0)$. Theorem 3.1 is then a slightly stronger statement than a theorem of Boglaev [1, Thrm. 1]. Similarly, Theorem 3.2 is a generalization of Theorem 2 in [1].

Remark 2. In Theorem 3.3, if $\ell = 1$, then it is not necessary to assume that $u_i'' \geqq 0$, $t \in I_i$, $i = 1, 2$. Simply define

$$\alpha_3(t,\epsilon) = \begin{cases} u_1(t) - \epsilon^2 m^{-2} M, & -1 \leqq t \leqq 0, \\ u_2(t) - \epsilon^2 m^{-2} M, & 0 \leqq t \leqq 1, \end{cases}$$

where $|u_i''| \leqq M$. It follows easily that $\epsilon^2\alpha_3'' \geqq h(t, \alpha_3)$.

REMARK 3. Although we will not discuss it here, it is clear that the derivatives $y'(t,\epsilon)$ behave non-uniformly in a neighborhood of $t = 0$, since $u_1'(0) < u_2'(0)$. In addition, $y'(t,\epsilon)$ possesses boundary layers at $t = \pm 1$.

REMARK 4. Finally we note that we have made no statement concerning the uniqueness of solutions. Under certain circumstances, it may be possible to proceed as in [**1**] to show uniqueness. However, many *nonlinear* problems of interest (see, e.g., [3, Chap. 18] and [**12**]) have nonunique solutions. The theorems in this section and the next only assert that under the stated assumptions there is at least one solution with the specified asymptotic behavior.

4. **The Case** $u_1'(0) > u_2'(0)$. If the derivatives of the functions u_1 and u_2 satisfy the opposite inequality $u_1'(0) > u_2'(0)$, then it is possible to obtain results analogous to those of the previous section. To be precise, we simply make the change of dependent variable $y \to -y$ and apply the theorems of § 3 to the transformed problem. For the sake of clarity we isolate these results as theorems.

THEOREM 4.1. *Make the same assumptions as in Theorem* 3.1 *with the exception that in assumption* (1), $u_1'(0) > u_2'(0)$. *Then the conclusion of Theorem* 3.1 *holds.*

THEOREM 4.2. *Assume* (1) *and* (2) *as in Theorem* 3.1 *with the exception that in assumption* (1), $u_i''(t) \leqq 0$, $t \in I_i$, $i = 1, 2$, $u_1'(0) > u_2'(0)$ *and* $u_1(-1) \geqq A$, $u_2(1) \geqq B$; *and in assumption* (2), h *is of class* $C^{(2q)}$ *with respect to* y, *for* $q \geqq 1$. *Assume also* (3) *for* $t \in I_i$, $i = 1, 2$, $\partial_y^{2k+1}h(t, u_i(t)) \geqq 0$, $\partial_y^{2k}h(t, u_i(t)) \leqq 0$, $k = 0, 1, \cdots, q-1$; *and* $\partial_y^{2q}h \leqq -m^2 < 0$ *in* D, *for* m *a positive constant.*

Then there exists an $\epsilon_0 > 0$ *such that for each* ϵ, $0 < \epsilon \leqq \epsilon_0$, *there exists a solution* $y = y(t,\epsilon)$ *of* (3.1), (3.2). *In addition, if*

$$u(t) = \begin{cases} u_1(t), & -1 \leqq t \leqq 0, \\ u_2(t), & 0 \leqq t \leqq 1, \end{cases}$$

and $t \in [-1, 1]$,

$$\begin{aligned} &(A - u_1(-1))(1 + \sigma_1(q)\epsilon^{-1}(1+t))^{-2(2q-1)^{-1}} \\ &\quad + (B - u_2(1))(1 + \sigma_2(q)\epsilon^{-1}(1-t))^{-2(2q-1)^{-1}} \\ &\quad - c_q\epsilon^{2(2q+1)^{-1}} \leqq y(t,\epsilon) - u(t) \leqq 0. \end{aligned}$$

Here $\sigma_1(q)$, $\sigma_2(q)$ *are the constants* $\sigma_1(n)$, $\sigma_2(n)$, *respectively, of Theorem* 3.2 *with* n *replaced by* $2q$.

THEOREM 4.2′. *Make the same assumptions as in Theorem* 4.2 with *the exception that h is of class $C^{(2q+1)}$ with respect to y, $q \geqq 1$, and in* (3),

$$\partial_y^{\,2k+1}h(t, u_i(t)) \geqq 0, \;\; \partial_y^{\,2k+2}h(t, u_i(t)) \leqq 0,$$
$$k = 0, 1, 2, \cdots, q-1;$$

and $\partial_y^{\,2q+1}h \geqq m^2 > 0$ in D, for m a positive constant.

Then the conclusion of Theorem 4.2 *holds with the estimate, for $t \in [-1, 1]$,*

$$\begin{aligned}&(A - u_1(-1))(1 + \sigma_1(q)\epsilon^{-1}(1+t))^{-q^{-1}}\\ &\quad + (B - u_2(1))(1 + \sigma_2(q)\epsilon^{-1}(1-t))^{-q^{-1}}\\ &\quad - c_q\epsilon^{(q+1)^{-1}} \leqq y(t,\epsilon) - u(t) \leqq 0.\end{aligned}$$

The constants $\sigma_1(q)$, $\sigma_2(q)$ are those appearing in Theorem 3.1.

THEOREM 4.3′. *Make the same assumptions as in Theorem* 4.3 *tion that $u_i'' \leqq 0$, $u_1'(0) > u_2'(0)$, and $u_1(-1) \geqq A$, $u_2(1) \geqq B$. Assume also* (3)′ *for $t \in I_i$, $i = 1, 2$, $\partial_y^{\,2k+1}h(t, u_i(t)) \geqq 0$, $\partial_y^{\,2k}h(t, u_i(t)) \leqq 0$, $k = 0, 1, \cdots, \ell - 1$, $\ell \geqq 1$, $2\ell + 1 \leqq n$; $\partial_y^{\,2\ell}h(t, u_i(t)) \leqq -m^2 < 0$, for m a positive constant; finally, $\partial_y^{\,2\ell+1}h \geqq 0$ in D.*

Then there exists an $\epsilon_0 > 0$ such that for each ϵ, $0 < \epsilon \leqq \epsilon_0$, there exists a solution $y = y(t, \epsilon)$ of (3.1), (3.2). *In addition, for $t \in [-1, 1]$,*

$$\begin{aligned}&(A - u_1(-1))(1 + \sigma_1(\ell)\epsilon^{-1}(1+t))^{-2(2\ell-1)^{-1}}\\ &\quad + (B - u_2(1))(1 + \sigma_2(\ell)\epsilon^{-1}(1-t))^{-2(2\ell-1)^{-1}}\\ &\quad - c_\ell\epsilon^{2(2\ell+1)^{-1}} \leqq y(t,\epsilon) - u(t) \leqq 0.\end{aligned}$$

THEOREM 4.3′. *Make the same assumptions as in Theorem* 4.3 *the exception that* (3′) *should read: For $t \in I_i$, $i = 1, 2$, $\partial_y^{\,2k+1}h(t, u_i(t)) \geqq 0$, $\partial_y^{\,2k+2}h(t, u_i(t)) \leqq 0$, $k = 0, 1, \cdots, \ell - 1$, $\ell \geqq 0$, $2\ell + 2 \leqq n$; $\partial_y^{\,2\ell+1}h(t, u_i(t)) \geqq m^2 > 0$, for m a positive constant; finally, $\partial_y^{\,2\ell+2}h \leqq 0$ in D.*

Then the conclusion of Theorem 4.3 *holds with the estimate, for $t \in [-1, 1]$,*

$$\begin{aligned}&(A - u_1(-1))\exp[-m\epsilon^{-1}(1+t)]\\ &\quad + (B - u_2(1))\exp[-m\epsilon^{-1}(1-t)] - c_0\epsilon\\ &\quad \leqq y(t,\epsilon) - u(t) \leqq 0,\end{aligned}$$

if $\ell = 0$;

$$(A - u_1(-1))(1 + \sigma_1(\ell)\epsilon^{-1}(1 + t))^{-\ell^{-1}} + (B - u_2(1))(1 + \sigma_2(\ell)\epsilon^{-1}(1 - t))^{-\ell^{-1}} - c_\ell \epsilon^{(\ell+1)^{-1}} \leqq y(t,\epsilon) - u(t) \leqq 0,$$

if $\ell \geqq 1$.

Remark. The results stated in this section are also valid if $u_1'(0) = u_2'(0)$. Theorem 4.2 is then a slight generalization of another result of Boglaev [**1**, Thrm. 3].

5. **Discussion.** In this section we want to motivate, at least in part, some of our assumptions in the two previous sections, and also to discuss briefly the relationship of our results with those of Boglaev [**1**], Fife [**4**], and O'Malley [**12**].

In § 3 we assumed that the partial derivatives $\partial_y{}^k h$ were all nonnegative either along the reduced solution paths $u_i(t)$ or in the larger domain D. It is instructive to note that such nonnegativity restrictions imply the following: in each of the theorems of § 3, if $u_1(-1) < A$, then $h(-1, A) > 0$, and similarly, if $u_2(1) < B$, then $h(1, B) > 0$. This is most easily seen by applying Taylor's Theorem:

$$\begin{aligned} h(-1, A) &= h(-1, u_1(-1)) + (A - u_1(-1)) \\ &= \sum_{k=1}^{n-1} \frac{1}{k!} \partial_y{}^k h(-1, u_1(-1))(A - u_1(-1))^k \\ &\quad + \frac{1}{n!} \partial_y{}^n h\{\cdot\}(A - u_1(-1))^n, \end{aligned}$$

for $\{\cdot\} = (-1, \; u_1(-1) + \theta(A - u_1(-1)) \in D, \; 0 < \theta < 1,$ i.e., $h(-1, A) \geqq (1/n!) \; \partial_y{}^n h\{\cdot\}(A - u_1(-1))^n > 0$. Similarly, $h(1, B) \geqq (1/n!) \; \partial_y{}^n h\{\cdot\cdot\}(B - u_2(1))^n > 0$. The positivity of $h(-1, A)$ and $h(1, B)$ then implies that any solution $y = y(t, \epsilon)$ of the problem (3.1), (3.2) is convex near $t = -1$ and $t = 1$, since $\epsilon^2 y''(-1) = h(-1, A) > 0$ and $\epsilon^2 y''(1) = h(1, B) > 0$. Analogously, the inequality restrictions in the theorems of § 4 imply the following: if $u_1(-1) > A$, then $h(-1, A) < 0$, and if $u_2(1) > B$, then $h(1, B) < 0$. That is to say, in the boundary layers at $t = \pm 1$, any solution y is concave, since $\epsilon^2 y''(\pm 1)$ is negative. Again these results follow from Taylor expansions of $h(-1, A)$ and $h(1, B)$.

This discussion leads naturally to a consideration of the following question: if we only assume in the case $u_1(-1) > A$ that $h(-1, A) < 0$, can we conclude that any solution of the problem $\epsilon^2 y'' = h(t, y)$, $-1 < t < 1$, $y(-1, \epsilon) = A$, $y(1, \epsilon) = B$, possesses a "concave" boundary layer at $t = -1$? Analogously, if $u_1(-1) < A$, is the assumption that $h(-1, A) > 0$ strong enough to guarantee the existence of "convex" boundary layers at $t = -1$? It does not seem possible, under these weaker assumptions, to prove any kind of general statement akin to those of the previous sections *using differential inequalities*. There are simply not enough hypotheses to which one can apply the machinery of the Nagumo or Habets-Laloy theorems. However, Boglaev [1], Fife [4] and O'Malley [12] have obtained results under such weaker assumptions by using conditions which involve *integrals* of the righthand side $h(t, y)$ in place of conditions involving partial derivatives. In doing so, they forsake obtaining explicit boundary layer information; however, their results apply to problems which ours do not.

Nevertheless we note that in particular instances, it may be possible to apply differential inequalities to problems of the form (3.1), (3.2) which satisfy *some* but not all of the assumptions in §§ 3 and 4. An example of a problem of this type is given in the next section.

6. **Some Examples.** We present now some applications of the theorems proved above and of the discussion in § 5.

EXAMPLE 1.

$$\epsilon^2 y'' = (y + f(t))^{2q+1}, \; -1 < t < 1,$$
$$y(-1, \epsilon) = A, y(1, \epsilon) = B,$$

where q is a nonnegative integer and

$$f(t) = \begin{cases} t, & -1 \leqq t \leqq 0, \\ -t, & 0 \leqq t \leqq 1. \end{cases}$$

Solving the reduced equation $(u + f(t))^{2q+1} = 0$, we clearly have $u_1(t) = -t$, $u_2(t) = t$. Thus applying Theorem 3.1, we conclude that for each sufficiently small $\epsilon > 0$, there exists a solution $y = y(t, \epsilon)$ satisfying: for $q = 0$,

$$|y(t, \epsilon) + t| \leqq |A - 1| \exp[-\epsilon^{-1}(1 + t)] + c_0\epsilon,$$
$$-1 \leqq t \leqq 0,$$
$$|y(t, \epsilon) - t| \leqq |B - 1| \exp[-\epsilon^{-1}(1 - t)] + c_0\epsilon,$$
$$0 \leqq t \leqq 1;$$

for $q \geqq 1$,

$$|y(t, \epsilon) + t| \leqq |A - 1|(1 + \sigma_1(q)\epsilon^{-1}(1 + t))^{-q^{-1}} + c_q\epsilon^{(q+1)^{-1}},$$
$$-1 \leqq t \leqq 0,$$

$$|y(t,\epsilon) - t| \leqq |B-1|(1+\sigma_2(q)\epsilon^{-1}(1-t))^{-q^{-1}} + c_q\epsilon^{(q+1)^{-1}},$$
$$0 \leqq t \leqq 1.$$

EXAMPLE 2.

$$\epsilon^2 y'' = y^2 + (1+t)y + g(t), \quad -1 < t < 1,$$
$$y(-1,\epsilon) = A,\ y(1,\epsilon) = B,\ A, B \geqq 1,$$

where

$$g(t) = \begin{cases} t, & -1 \leqq t \leqq 0, \\ -(t+2t^2), & 0 \leqq t \leqq 1. \end{cases}$$

The reduced equation $u^2 + (1+t)u + g(t) = 0$ has the solutions $u = -t$, $u \equiv -1$ on $[-1, 0]$ and $u = t$, $u = -(1+2t)$ on $[0, 1]$. Consider now $h_y(t, u(t))$ for $h(t, y) = y^2 + (1+t)y + g(t)$. Clearly, $h_y(t, y) = 2y + 1 + t$ and

$$h_y(t, -t) = 1 - t \geqq 1 \text{ on } [-1, 0],$$
$$h_y(t, -1) = t - 1 \leqq 0 \text{ on } [-1, 0];$$
$$h_y(t, t) = 1 + 3t \geqq 1 \text{ on } [0, 1],$$
$$h_y(t, -(1+2t)) = -(1+3t) \leqq -1 \text{ on } [0, 1].$$

Thus, $u_1 = -t$, $u_2 = t$ form a stable pair with $h_y(t, u_i(t)) \geqq 1$. Since $u_1(-1) = 1 \leqq A$ and $u_2(1) = 1 \leqq B$, we may apply Theorem 3.3 (with $\ell = 1$) to conclude that for each sufficiently small $\epsilon > 0$, there exists a solution $y = y(t,\epsilon)$ satisfying

$$0 \leqq y(t,\epsilon) + t \leqq (A-1)\exp[-\epsilon^{-1}(1+t)] + c_0\epsilon,$$
$$-1 \leqq t \leqq 0,$$
$$0 \leqq y(t,\epsilon) - t \leqq (B-1)\exp[-\epsilon^{-1}(1-t)] + c_0\epsilon,$$
$$0 \leqq t \leqq 1.$$

EXAMPLE 3.

$$\epsilon^2 y'' = y^2 - t^2, \quad -1 < t < 1,$$
$$y(-1,\epsilon) = A, \quad y(1,\epsilon) = B.$$

Clearly the solutions of the reduced equation are $u = \pm t$; and since $h_y(t, y) = 2y$, $h_y(t, -t) = -2t \geqq 0$, on $[-1, 0]$ while $h_y(t, t) = 2t \geqq 0$ on $[0, 1]$. Consequently, $u_1 = -t$, $u_2 = t$ form a stable pair and provided $A \geqq 1$, $B \geqq 1$, we can conclude via Theorem 3.2 (with $n = 2$) that for each $\epsilon > 0$, ϵ sufficiently small, there exists a solution $y = y(t,\epsilon)$ with

$$0 \leqq y(t,\epsilon) + t \leqq (A-1)(1+\sigma_1(2)\epsilon^{-1}(1+t))^{-2} + c_2\epsilon^{1/3},$$
$$-1 \leqq t \leqq 0,$$
$$0 \leqq y(t,\epsilon) - t \leqq (B-1)(1+\sigma_2(2)\epsilon^{-1}(1-t))^{-2} + c_2\epsilon^{1/3},$$
$$0 \leqq t \leqq 1.$$

If on the other hand, we have $-1 < A$, $B < 1$, then $u_1(-1) > A$ and $u_2(1) > B$, and we cannot apply any of the results in § 4, since $h_{yy} \equiv 2$ and we require $h_{yy} < 0$. We note however that with this choice of A and B, $h(-1, A) = A^2 - 1 < 0$ and $h(1, B) = B^2 - 1 < 0$, so there is a possibility that solutions exist which possess "concave" boundary layers at $t = \pm 1$. Indeed, it is possible to describe the asymptotic behavior of a solution of this example for $-1 < A$, $B < 1$ by defining the functions

$$\alpha(t,\epsilon) = \begin{cases} -t + (A-1)\exp[-\sigma_1\epsilon^{-1}(1+t)] - \epsilon\gamma(1-t^2), \\ \qquad -1 \leqq t \leqq 0, \\ t + (B-1)\exp[-\sigma_2\epsilon^{-1}(1-t)] - \epsilon\gamma(1-t^2), \\ \qquad 0 \leqq t \leqq 1; \end{cases}$$

$$\beta(t,\epsilon) = \begin{cases} -t + k^{-1}\epsilon^{2/3}(1 - kt\epsilon^{-2/3})^{-2}, & -1 \leqq t \leqq 0; \\ t + k^{-1}\epsilon^{2/3}, & 0 \leqq t \leqq 1, \end{cases}$$

where $0 < \sigma_1 < A + 1$, $0 < \sigma_2 < B + 1$, $0 < k < 6^{-1/3}$, and $\gamma > 0$ is suitably chosen, and verifying that the required inequalities are satisfied for small positive values of ϵ. It is now easy to treat this problem with boundary data of the form $-1 < A \leqq 1$, $B \geqq 1$ or $A \geqq 1$, $-1 < B \leqq 1$, by using suitable combinations of the bounding functions α, β. Finally, we remark that if either $A < -1$ or $B < -1$, then there is no solution of the problem for small $\epsilon > 0$ of bounded t-variation, since $h(-1, A) = A^2 - 1 > 0$ and $h(1, B) = B^2 - 1 > 0$. Put geometrically, if there were a solution of bounded t-variation, it would have the wrong convexity properties in the boundary layers near $t = -1$ or $t = 1$.

EXAMPLE 4. As our final example we study a problem which is not amenable to treatment by the integral methods of [1]. Consider

$$\epsilon^2 y'' = (y + f(t))^2, \quad -1 < t < 1,$$
$$y(-1,\epsilon) = A \geqq 1, \quad y(1,\epsilon) = B \geqq 1,$$

where

$$f(t) = \begin{cases} t, & -1 \leqq t \leqq 0, \\ -t, & 0 \leqq t \leqq 1. \end{cases}$$

Clearly the solutions $u_1 = -t$, $u_2 = t$ form a stable pair with $u_1(-1) = 1 \leqq A$ and $u_2(1) \leqq B$. Thus by Theorem 3.2 (with $n = 2$), this problem has a solution $y = y(t, \epsilon)$ for each sufficiently small $\epsilon > 0$ satisfying

$$0 \leqq y(t,\epsilon) + t \leqq (A - 1)(1 + \sigma_1(2)\epsilon^{-1}(1 + t))^{-2} + c_2\epsilon^{1/3}, \qquad -1 \leqq t \leqq 0,$$

$$0 \leqq y(t,\epsilon) - t \leqq (B - 1)(1 + \sigma_2(2)\epsilon^{-1}(1 - t))^{-2} + c_2\epsilon^{1/3}, \qquad 0 \leqq t \leqq 1.$$

However, the potential energy function

$$U(y, t) = - \int_A^y (s + f(t))^2 \, ds$$

does not possess a maximum along the pair $u_1 = -t$, $u_2 = t$ at the point $t = 0$, and so Theorem 6 of [1] is not applicable to this problem.

7. **An Extension.** We close with the observation that the above results are valid in the case of the more general problem

$$\epsilon^2 y'' = h(t, y)p(t, y, y'), \; -1 < t < 1, \tag{7.1}$$

$$y(\pm 1, \epsilon) \text{ prescribed}, \tag{7.2}$$

where h is as before and where the continuous function p satisfies the two assumptions: $p = p(t, y, y') \geqq \mu^2 > 0$, for $(t, y) \in D$ and $|y'| < \infty$; and $p = O(|y'|^2)$, as $|y'| \to \infty$, for $(t, y) \in D$. This follows again from the Habets-Laloy theorem (using the same α, β as above with m replaced by μm) since the righthand side of (7.1) satisfies a Nagumo growth condition. Thus although the righthand side may depend on y', the problem (7.1), (7.2) still possesses solutions which exhibit boundary layer behavior at *both* endpoints.

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