ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 4, Fall 1976

INTRODUCTION TO DIFFERENTIAL EQUATIONS WITH MOVING SINGULARITIES

H. GINGOLD

Consider the differential system

$$(1)\qquad \begin{cases} y' = F(t, \epsilon, y, x) \\ \epsilon^h \phi(t, \epsilon) x' = G(t, \epsilon, y, x) \end{cases}$$

with the following hypothesis that we will call H.

H — (i) y is an m dimensional column vector and x is an n dimensional column vector. $F(t, \epsilon, y, x)$, $G(t, \epsilon, y, x)$ are continuously differentiable vector functions in the domain D, where

$$D = \{0 \leqq t \leqq 1, 0 \leqq \epsilon \leqq \epsilon_0, (\epsilon_0 > 0), (\|y\| + \|x\|) < \infty\},$$

(ii) $h \geqq 0$ (h need not be an integer) and $\phi(t, \epsilon)$ is a continuously differentiable scalar function in D,

(iii) $\phi(t, \epsilon) > 0$ for $0 \leqq t \leqq 1$, $0 < \epsilon < \epsilon_0$, $\phi(t, 0) \cdot \phi(0, \epsilon) \not\equiv 0$. If $h = 0$ we demand that $\phi(0, 0) = 0$ and

$$\lim_{\epsilon \to 0^+} \int_0^t \frac{d\eta}{\phi(\eta, \epsilon)} = +\infty \text{ uniformly}$$

for

$$0 < \delta \leqq t \leqq 1.$$

In the case when $h > 0$ and $\phi(t, \epsilon) \equiv 1$, we recognize the singular perturbation systems, in case $h = 0$ and $\phi(t, \epsilon) = (t + \epsilon)^m$, we have by letting $\epsilon = 0$, the familiar singular systems of mathematical physics. We call (1) a differential equation with *moving singularities* since the location of the zeros of $\phi(t, \epsilon)$ may depend on ϵ. For example,

$$(2)\qquad \begin{aligned} y_1' &= y_2 \\ \epsilon(t^2 + \epsilon) y_2' &= +y_1^3 - y_2, \end{aligned}$$

$$(3)\qquad \begin{aligned} y_1' &= y_2 \\ (t + \epsilon)^2 y_2' &= y_1 + y_2, \end{aligned}$$

$$(4)\qquad \begin{aligned} y_1' &= y_1 + 2y_2 \\ \epsilon y_2' &= 4y_1 + (t^2 + \epsilon) y_2 \end{aligned}$$

are special types of differential equations with moving singularities.

Why consider these equations?

(i) It leads to a unified approach to singular differential equations which apparently seem to have different features.

(ii) By transforming the interval $[0, n]$ via the möbius transformation $x = \delta(t - n)/(t - \delta n)$, $\delta > 1$ onto $[0, 1]$, where $n \to +\infty$, we transform singular perturbation problems on the infinite interval into problems associated with the moving singularities equation on the finite interval $[0, 1]$.

(iii) We claim that singular perturbation methods may be applied successfully to differential equations with moving singularities; thus obtaining a new and better insight into singular differential equations. In particular, indispensable new information concerning singular perturbation problems is obtained.

We mention some proven theorems.

THEOREM 1. *Let the linear system*

$$\left\{\begin{aligned} y' &= A(t,\epsilon)y + B(t,\epsilon)x \\ \epsilon^h\phi(t,\epsilon)x' &= C(t,\epsilon)y + D(t,\epsilon)x \end{aligned}\right. \tag{5}$$

satisfy H where y, x are m, n dimensional vectors, respectively. A, B, C, D are matrices of appropriate orders. Let the eigenvalues of D be $\lambda_1 \cdots \lambda_n$ such that

$$\operatorname{Re}\lambda_i < \alpha < 0, i = 1, \cdots, k, 1 < k < n,$$

$$\operatorname{Re}\lambda_j > \beta > 0, j = k + 1, \cdots, n.$$

The eigenvalues $\lambda_1 \cdots \lambda_n$ may depend on t and ϵ and α, β, are constants. Then there exists an invertible transforming matrix,

$$R(t,\epsilon) = \begin{bmatrix} I_m & -\epsilon^h\phi(t,\epsilon)S \\ -T & I_n + \epsilon^h\phi(t,\epsilon)TS \end{bmatrix} \tag{6}$$

the entries of which belong to $C^1[0,1]$ (for every $\epsilon > 0$) and to $C([0,a] \times [0,\epsilon_1])$, ($\epsilon_1 > 0$), where $a = 1$ in case $h > 0$, and $0 < a \leqq 1$ in case $h = 0$. Herein,

I_m is the m dimensional identity matrix,

S is an $m \times n$ matrix,

T is an $n \times m$ matrix.

The transformation (6) takes the system (5) into,

$$u' = [A - BT]u,$$

$$\epsilon^h\phi(t,\epsilon)w = [D + \epsilon^h\phi(t,\epsilon)TB]\,w,$$

where, $\binom{y}{x} = R(t,\epsilon)\binom{u}{w}$.

A discussion of this result may be found in [2]. A result of a similar character holds in the complex plane when the coefficient matrix is an analytic function of t and ϵ. This result is discussed in [3].

A discussion of the next theorem may be found in [1] and [4].

THEOREM 2. *Assume the following system (which is easily observed to be equivalent to an nth order differential equation).*

$$(7)\quad \begin{cases} \begin{pmatrix} y_1' \\ y_2' \\ \cdot \\ \cdot \\ \cdot \\ y_{m-2}' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ x_1 \end{pmatrix} \\ \epsilon^h\phi(t,\epsilon)x_1' = -a(t,\epsilon)x_1 + f(t,\epsilon,y_1,y_2,\cdots,y_{(m-2)}) \end{cases}$$

to satisfy hypothesis H, and $(a(t,\epsilon) \geqq k > 0$, *where k is a constant and* $f(t,\epsilon,y_1,y_2,\cdots,y_{(m-2)})$ *is a scalar function). Let*

$$(8)\qquad y_1(0,\epsilon), y_2(0,\epsilon), \cdots, y_{(m-2)}(0,\epsilon), x_1(0,\epsilon)$$

be $(m-1)$ *preassigned continuous functions of* ϵ *in* $[0,\epsilon_0]$,

$$x_1(0,\epsilon) \in C(0,\epsilon_0], \quad \textit{and} \quad x_1(0,\epsilon) = O(\epsilon^{-r}), r > 0, \textit{as } \epsilon \to 0^+.$$

Then:

(i) *there exists a solution of* (7), (8) *on* $[0,a)$, $0 < a \leqq 1$ *such that*

$$\lim_{\epsilon\to 0^+} x_1(t,\epsilon)$$

exists uniformly on $0 < \delta \leqq t \leqq a$;

(ii) *if* $x_1(0,\epsilon) = O(1)$ *as* $\epsilon \to 0^+$, *then* "$\lim_{\epsilon\to 0^+} y_\nu(t,\epsilon)$, $\nu = 0, 1, \cdots (m-2)$, *exist uniformly on* $0 \leqq t \leqq a$"; *and,*

(iii) $\lim_{\epsilon\to 0^+} x_1(t,\epsilon)$ *exists uniformly on* $0 \leqq t \leqq 1$ *iff*

$$a(0,0)\, x_1(o,0) = f(0,0,y_1(0,0),y_2(0,0),\cdots,y_{(m-2)}(0,0)).$$

This type of theorem is radically different from the usual theorems in singular perturbations in the following respects.

(i) We neither assume the existence of a solution to the reduced problem nor to the full one.

(ii) The reduced equation is not necessarily algebraic, and its dimensions may not decrease.

(iii) It shows that the expectation for a *boundary layer* for $y_1(t,\epsilon)$, $y_2(t,\epsilon)$, $\cdots$, $y_{(m-2)}(t,\epsilon)$ may be *false*. Moreover we are able to formulate necessary and sufficient conditions on the initial values in order to get uniform convergence on $[0, a]$, even for $\lim_{\epsilon \to 0^+} x_1(t, \epsilon)$.

(iv) We are rewarded by this type of theorem since we obtain new information concerning

existence,

uniqueness, and

asymptotic behaviour,

of solutions of the singular differential equation

$$\phi(t, 0)y^{(n)} + a(t, 0)y^{(n-1)} = f(t, 0, y, y^1, \cdots, y^{(n-2)}),$$

at the singular point $t = 0$. This provides us with a tool to prove that for singularly perturbed equations

$$\begin{aligned} &\epsilon(y^{(m)} + a(t)y^{(m-1)} + \cdots + a_{m-n+1}(t)y^{(m-n+1)}), \\ &\quad + \phi(t, \epsilon)y^{(n)} + a(t, \epsilon)y^{(n-1)} = f(t, \epsilon, y, y', \cdots, y^{(n-2)}) \end{aligned}$$

with turning points at $t = 0$ (recall that $\phi(0, 0) = 0$), the reduced problem *has* a solution. This is a sufficient condition to "linearize" the nonlinear equation.

References

1. H. Gingold, *Ordinary differential Equations with moving singularities,* D.S.C. Dissertation Technion, I.I.T. Haifa, Israel, 1974.

2. ——, *Dichotomies and moving singularities* (*Simplification of Linear systems*), to appear.

3. ——, *Simplification of Linear homogeneous differential equations with moving singularities,* to appear in the Funkcialaj Ekvacioj.

4. H. Gingold and S. Rosenblat, *Differential equations with moving singularities,* to appear in SIAM J. on Math. Anal.

5. W. Wasow, *Asymptotic expansions for ordinary differential equations,* John Wiley and Sons, Inc., 1965.

University of Utah, Salt Lake City, Utah 84112