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AXIAL INCOMPRESSIBLE VISCOUS FLOW PAST A SLENDER BODY OF REVOLUTION

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ABSTRACT. As an example illustrating expansion procedures for the Navier-Stokes equations involving two parameters, the axisymmetric incompressible large Reynolds number viscous flow of a uniform stream past a semi-infinite slender body of revolution (or 'needle') is studied for the case where the radius of the body is much less than that of the thin boundary layer that the body produces and supports.

Through the application to the Navier-Stokes equations of limit process (or parameter-type) expansions that are valid for both δ, the body thickness parameter, and λ, the boundary layer thickness parameter, going to zero, with the ratio λ/δ going to infinity (specifically, $\lambda^{1+Q}/\delta^{Q} = 1$, with Q a positive constant of order unity), asymptotic solutions are obtained for the three distinct flow regions that span the domain from the body surface to the freestream.

1. **Introduction.** This paper presents an analysis of the axisymmetric incompressible viscous flow of a uniform stream past a slender body of revolution for the case where the cross-sectional radius of this body is much less than the cross-sectional radius of the thin boundary layer that the body produces and supports. This subject has been studied by Stewartson [8], Glauert and Lighthill [3], and Mark [6]. Stewartson treats the case of flow past a cylinder ($r_b{}^* = b_0{}^*$); Mark treats the case of flow past a paraboloid ($r_b{}^* = b_{1/2}{}^* x^{*1/2}$), the case for which self-similar solutions exist; and Glauert and Lighthill treat the (more general) case of flow past a 'power-law' body ($r_b{}^* = b_n{}^* x^{*n}$, with $0 \leqq n \leqq 1$).

In the present paper, rather than employing the more intuitive approach of the above-mentioned papers, a more formal approach is employed. In this approach, through the application of limit process (or parameter-type) expansions (cf., e.g., Lagerstrom and Cole [4]; Cole [2]), solutions are obtained to the axisymmetric incompressible Navier-Stokes equations of motion for high Reynolds number viscous flow past a slender body of revolution. Such a flow is characterized by two small parameters, λ and δ. The parameter λ is defined by $\lambda = R^{-1/2}$, where $R = \rho^* u_\infty{}^* a^*/\mu^*$ is recognized as the Reynolds number (based upon the characteristic axial body length). Further, λ is identified as the boundary layer thickness parameter. The body thickness parameter δ is defined by $\delta = c^*/a^*$, the ratio of the characteristic normal and

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axial body lengths, such that the equation for the body surface takes the form $(r_b^*/a^*) = \delta Z(x^*/a^*)$, with Z a function of order unity. (For a 'power-law' body, $c^* = b_n^* a^{*n}$, such that $\delta = b_n^*/a^{*1-n}$, and $(r_b^*/a^*) = \delta(x^*/a^*)^n$.) With δ and λ identified as the body and boundary layer thickness parameters, respectively, it follows that, physically, the ratio λ/δ must be greater than or equal to order unity. For the case under consideration, where the boundary layer radius is taken to be significantly greater than the body radius, this ratio is greater than order unity.

In the limit of both λ and δ going to zero, subject to the constraint that λ/δ goes to infinity, it is determined that three distinct flow regimes span the domain from the body surface to the freestream, namely: an exterior inviscid region (or layer); an interior viscous region, which is made up of an outer viscous (or boundary) layer and an inner viscous (or surface) layer; and a transition viscous region (or layer), intermediate to these exterior and interior regions. In the papers cited previously, neither the exterior inviscid region nor the intermediate viscous region is treated; while, the interior viscous region is treated as a one-layer region. It should be noted, however, that the exterior inviscid region and the intermediate viscous region solutions do not play a primary role in the determination of such properties as the viscous stress at the body surface (skin friction). Further, in the previous (boundary layer) treatments of the interior viscous region, the two-layer character of this region is implicit in the methods of solution employed. Indeed, it should be emphasized that the representations introduced in the present analysis are motivated, in part, by the results obtained in these previous analyses. It is the purpose of the more systematic approach of this paper to present the results within a framework that both clarifies the problem at hand and offers guidance for the solution of future problems by perturbation techniques involving two parameters.

In the analysis presented, the asymptotic solutions for the exterior, interior, and intermediate regions are shown to match, subject to certain restrictions on the domain of validity of the analysis. Rather than $\lambda(= R^{-1/2})$ and δ being independent small parameters (as is initially implicitly assumed), it is determined that they are related by $\lambda^{1+Q}/\delta^Q = 1$ (and/or $\delta R^{(1+Q)/2Q} = 1$), where Q is a positive constant of order unity. Further, it is found that the 'power-law' body exponent n must satisfy $0 < Q/2(1+Q) \leqq n \leqq 1$, and, thus, the case where the body is a cylinder (with $n = 0$) is excluded from the present analysis.

2. **The Equations of Motion.** Consider the uniform axisymmetric flow of a fluid of constant density and viscosity (ρ^*, $\mu^* =$ consts.) past

a slender body of revolution. Let $x^* = a^*x$ and $r^* = a^*r$ represent the cylindrical polar coordinates along the axis of symmetry from the vertex of the body and normal to this axis, respectively, with the characteristic axial body length a^* chosen so that x is of order unity in the region where the interaction theory presented is valid. The velocity components in the x^*- and r^*-directions are $u^*(x^*, r^*) = u_\infty^* u(x, r)$ and $v^*(x^*, r^*) = u_\infty^* v(x, r)$, and the pressure is $p^*(x^*, r^*) = p_\infty^* + \rho^* u_\infty^{*2} p(x, r)$, with u_∞^* and p_∞^*, respectively, the (constant) velocity in the x^*-direction and pressure in the undisturbed region far from the body surface.

The Navier-Stokes equations of motion governing such a flow are

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 : u = \frac{1}{r}\frac{\partial\psi}{\partial r}, v = -\frac{1}{r}\frac{\partial\psi}{\partial x}; \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} + \frac{\partial p}{\partial x} = \lambda^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial x^2}\right], \tag{2.2a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} + \frac{\partial p}{\partial r} = \lambda^2\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv)}{\partial r}\right) + \frac{\partial^2 v}{\partial x^2}\right]. \tag{2.2b}$$

Here, the parameter λ is defined by

$$\lambda = (\rho^* u_\infty^* a^*/\mu^*)^{-1/2} = R^{-1/2}. \tag{2.3}$$

In this analysis, it is taken that $R \to \infty$, and, thus, $\lambda \to 0$.

The uniform freestream and nonslip surface boundary conditions, respectively, for this system are

$$u \to 1, v \to 0, \text{ and } p \to 0 \text{ as } r \to \infty \quad (0 < x_0 \leqq x); \tag{2.4a}$$

$$u, v \to 0 \text{ as } r \to \delta Z, \text{ with } Z = fnc(x) \quad (0 < x_0 \leqq x). \tag{2.4b}$$

Here, x_0 is the lower bound on x (to be specified). Further,

$$\delta = c^*/a^*, \tag{2.5}$$

with c^* the characteristic normal body thickness. In this analysis, it is taken that $\delta \to 0$. Although a general body shape function $Z(x)$ of order unity is considered, special attention is directed to the 'power-law' body, for which $Z(x) = x^n$ (and $c^* = b_n^* a^{*n}$).

As has been noted, it is taken that both of the parameters — δ, the body thickness parameter (appearing in the boundary conditions), and λ, the 'effective body' (or boundary layer) thickness parameter (appearing in the equations of motion) — go to zero. In the present paper, solutions are developed for the case where the boundary layer thickness is large compared to the body thickness, i.e., for

$$\tau = \tau(\lambda, \delta) = \lambda/\delta \to \infty. \tag{2.6a}$$

In much of what follows, however, rather than to employ the large parameter τ, it is more convenient to employ the small parameter ϵ, defined by

$$\epsilon = \epsilon(\lambda, \delta) = \log^{-1}\tau(\lambda, \delta) = \log^{-1}(\lambda/\delta) \to 0. \tag{2.6b}$$

In the limit of $\lambda, \delta \to 0$, and $\tau(\lambda, \delta) \to \infty$ (and/or $\epsilon(\lambda, \delta) \to 0$), the analysis of the flowfield divides into one for an exterior inviscid region (or layer); an interior viscous region, which consists of an outer and an inner viscous layer; and an intermediate viscous region (or layer). The formulation for the exterior inviscid region (or layer) is presented in § 3; the formulations for the outer and inner layers of the interior viscous region are presented in § 4; while that for the intermediate viscous region (or layer) is presented in § 5. It is determined that the parameters λ and δ are not independent, but, rather, are related by

$$\lambda^{1+Q}/\delta^{Q} = 1, \text{ with } Q = \text{positive const. of } O(1), \tag{2.7a}$$

and, in turn,

$$\begin{aligned} &\tau = \lambda/\delta = \delta^{-1/(1+Q)} = \lambda^{-1/Q}; \\ &\epsilon = \log^{-1}(\lambda/\delta) = (1 + Q)\log^{-1}(1/\delta) = Q\log^{-1}(1/\lambda). \end{aligned} \tag{2.7b}$$

Further, based on equation (2.7), it is noted, for future reference, that

$$\lambda^2/\epsilon^m = Q^{-m}\lambda^2 \log^m(1/\lambda) = (1/\epsilon)^m \exp\{-2Q(1/\epsilon)\} \to 0; \tag{2.8a}$$

$$\begin{aligned} \delta^2/\epsilon^m &= (1 + Q)^{-m}\delta^2 \log^m(1/\delta) \\ &= (1/\epsilon)^m \exp\{-2(1 + Q)(1/\epsilon)\} \to 0. \end{aligned} \tag{2.8b}$$

3. **The Exterior Inviscid Region.** To initiate the analysis, the exterior inviscid region (or layer), produced by the interaction of the slender 'effective body' (i.e., the combination of the actual slender body and the thin viscous layers adjacent to the body surface) and the uniform external flow, is studied.

For this exterior layer, the independent variables x and r are both of order unity; while, the dependent variables are taken to have the representations

$$u = 1 + \sigma U, v = \sigma V, p = \sigma P, \tag{3.1}$$

where the inviscid perturbation parameter σ is taken to be $\sigma = \lambda^2\epsilon \to 0$. (That $\sigma = \lambda^2\epsilon$ is the appropriate inviscid perturbation scaling is determined only after consideration of the solutions in the interior viscous region.) In terms of the variables of equation (3.1), the equations of motion become

$$\frac{\partial(rU)}{\partial x} + \frac{\partial(rV)}{\partial r} = 0; \tag{3.2}$$

$$\frac{\partial U}{\partial x} + \sigma\left(U\,\frac{\partial U}{\partial x} + V\,\frac{\partial V}{\partial r}\right) + \frac{\partial P}{\partial x} = \lambda^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial x^2}\right], \tag{3.3a}$$

$$\frac{\partial V}{\partial x} + \sigma\left(U\,\frac{\partial V}{\partial x} + V\,\frac{\partial V}{\partial r}\right) + \frac{\partial P}{\partial r} = \lambda^2\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial(rV)}{\partial r}\right) + \frac{\partial^2 V}{\partial x^2}\right]. \tag{3.3b}$$

The freestream boundary conditions are

$$U, V, P \to 0 \text{ as } r \to \infty \quad (x \text{ fixed}). \tag{3.4}$$

The perturbation flow quantities U, V, and P are taken to have asymptotic expansions of the forms

$$U \simeq (U_0 + \epsilon U_1 + \epsilon^2 U_2 + \cdots) + \cdots, \tag{3.5a}$$

$$V \simeq (V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots) + \cdots, \tag{3.5b}$$

$$P \simeq (P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots) + \cdots. \tag{3.5c}$$

Then, the governing equations for U_k, V_k, and $P_k (k = 0, 1, 2, \cdots)$ are

$$\frac{\partial(rU_k)}{\partial x} + \frac{\partial(rV_k)}{\partial r} = 0; \tag{3.6}$$

$$\frac{\partial U_k}{\partial x} + \frac{\partial P_k}{\partial x} = 0, \quad \frac{\partial V_k}{\partial x} + \frac{\partial P_k}{\partial r} = 0. \tag{3.7}$$

The momentum equations, equations (3.7), yield the following:

$$\Pi_k = (P_k + U_k) = \Theta_k, \quad \Omega_k = \left(\frac{\partial V_k}{\partial x} - \frac{\partial U_k}{\partial r}\right) = -\frac{d\Theta_k}{dr}, \tag{3.8}$$

with Θ_k functions of r (to be determined). If this flow is taken to be irrotational, i.e., $\Omega_k = 0$, then, $d\Theta_k/dr = 0$, and/or $\Theta_k =$ const. The freestream boundary conditions yield $\Theta_k = 0$. Thus, the equations of motion reduce to the 'small disturbance theory' equations for inviscid irrotational incompressible flow past a slender body of revolution (cf., e.g., Ashley and Landahl [1]; Cole [2]), namely:

(3.9)
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial^2\Phi_k}{\partial r}\right)+\frac{\partial^2\Phi_k}{\partial x^2}=0,$$
$$\text{with } U_k=-P_k=\frac{\partial\Phi_k}{\partial x},\ V_k=\frac{\partial\Phi_k}{\partial r}.$$

As $r \to 0$ (x fixed), it is determined (cf. Cole [2]) that the solutions for Φ_k are of the forms

(3.10a)
$$\Phi_k \sim -S_k\log(1/r)+T_k$$
$$+\tfrac{1}{2}\left[\frac{d^2S_k}{dx^2}\log(1/r)+\tfrac{1}{4}\frac{d^2(S_k+2T_k)}{dx^2}\right]r^2+\cdots,$$

with S_k and T_k functions of x (to be determined). Thus, in this limit, U_k, V_k, and P_k are given by

(3.10b)
$$U_k=-P_k\sim-\frac{dS_k}{dx}\log(1/r)+\frac{dT_k}{dx}+\cdots,\ V_k\sim S_k(1/r)+\cdots.$$

From equation (3.10), it follows that, as $r \to 0$ (x fixed), the asymptotic behaviors for u, v, and p are given by

(3.11a)
$$u\sim 1+\lambda^2\epsilon\left\{\sum_{k=0}\epsilon^k\left[-\frac{dS_k}{dx}\log(1/r)+\frac{dT_k}{dx}+\cdots\right]+\cdots\right\},$$

(3.11b)
$$v\sim\lambda^2\epsilon\left\{\sum_{k=0}\epsilon^k\left[S_k(1/r)+\cdots\right]+\cdots\right\},$$

(3.11c)
$$p\sim\lambda^2\epsilon\left\{\sum_{k=0}\epsilon^k\left[\frac{dS_k}{dx}\log(1/r)-\frac{dT_k}{dx}+\cdots\right]+\cdots\right\}.$$

The appropriate functions S_k are determined only after the matching of the solutions for the outer and inner layers of the interior viscous region has been performed. When the functions S_k are known, the functions T_k are found from the solutions of equation (3.9) (cf. Cole [2]). Here, it is anticipated that this matching yields $S_0 = 1$, and that the resulting solution for Φ_0 is that for flow past a paraboloid, namely:

(3.12a)
$$\Phi_0=\tfrac{1}{2}\log\{(x^2+r^2)^{1/2}-x\},$$

such that

(3.12b)
$$U_0 = -P_0 = \frac{\partial \Phi_0}{\partial x} = -\frac{1}{2(x^2 + r^2)^{1/2}},$$
$$V_0 = \frac{\partial \Phi_0}{\partial r} = \frac{(x^2 + r^2)^{1/2} + x}{2r(x^2 + r^2)^{1/2}}.$$

For this solution, it is found that $T_0 = (1/2)\{\log(1/x) - \log 2\}$, such that $dT_0/dx = -1/2x$. From the above-mentioned matching, it is determined that, for a 'power-law' body, $S_1 = (1/2)\{(1 - 2n)\log(1/x) + \gamma\}, \cdots$.

With $S_0 = 1$, $dS_0dx = 0$, $T_0 = (1/2\{\log(1/x) - \log 2\}$, $dT_0/dx = -1/2x$, $S_1 = (1/2)\{(1 - 2n)\log(1/2) + \gamma\}$, $dS_1/dx = -(1 - 2n)/2x, \cdots$, then, at the 'inner edge' (where $r \to 0$ (x fixed)) of this exterior layer, this exterior layer,

(3.13a)
$$u \sim 1 - \lambda^2\epsilon \left\{\left[\frac{1}{2x} + \cdots\right] - \epsilon\left[\frac{(1-2n)}{2x}\log(1/r) + \cdots\right] + \cdots\right\},$$

(3.13b)
$$v \sim \lambda^2\epsilon\{[(1/r) + \cdots] + \epsilon[\tfrac{1}{2}\{(1 - 2n)\log(1/x) + \gamma\}(1/r) + \cdots] + \cdots\},$$

(3.13c)
$$p \sim \lambda^2\epsilon \left\{\left[\frac{1}{2x} + \cdots\right] - \epsilon\left[\frac{(1-2n)}{2x}\log(1/r) + \cdots\right] + \cdots\right\}.$$

Written in terms of the outer viscous layer coordinates x, y, with $y = r/\lambda$, equation (3.13) yields the following 'outer edge' ($y \to \infty$ (x fixed)) boundary conditions for the outer viscous layer:

(3.14a)
$$u \sim 1 - \lambda^2\epsilon \left\{\left[\frac{1}{2x} + \cdots\right] - \epsilon\log(1/\lambda)\left[\frac{(1-2n)}{2x}\right] + \epsilon\left[\frac{(1-2n)}{2x}\log y + \cdots\right] + \cdots\right\}$$
$$= 1 - Q\lambda^2\log^{-1}(1/\lambda)\left\{\left[\frac{1 - Q(1-2n)}{2x} + \cdots\right] + Q\log^{-1}(1/\lambda)\left[\frac{(1-2n)}{2x}\log y + \cdots\right] + \cdots\right\},$$

$$\begin{aligned}(3.14\text{b})\quad v \sim{}& \lambda\epsilon\{[(1/y) + \cdots]\\ &+\epsilon[\tfrac{1}{2}\{(1-2n)\log(1/x)+\gamma\}(1/y)+\cdots]+\cdots\}\\ ={}& Q\lambda\log^{-1}(1/\lambda)\{[(1/y)+\cdots]\\ &+Q\log^{-1}(1/\lambda)[\tfrac{1}{2}\{(1-2n)\log(1/x)+\gamma\}(1/y)+\cdots]\\ &+\cdots\},\end{aligned}$$

$$\begin{aligned}(3.14\text{c})\quad p \sim{}& \lambda^2\epsilon\left\{\left[\frac{1}{2x}+\cdots\right]-\epsilon\log(1/\lambda)\left[\frac{(1-2n)}{2x}\right]\right.\\ &\left.+\epsilon\left[\frac{(1-2n)}{2x}\log y+\cdots\right]+\cdots\right\},\\ ={}& Q\lambda^2\log^{-1}(1/\lambda)\left\{\left[\frac{1-Q(1-2n)}{2x}+\cdots\right]\right.\\ &\left.+Q\log^{-1}(1/\lambda)\left[\frac{(1-2n)}{2x}\log y+\cdots\right]+\cdots\right\},\end{aligned}$$

for the 'distinguished limit' of

$$(3.15)\qquad \epsilon\log(1/\lambda)=\frac{\log(1/\lambda)}{\log(\lambda/\delta)}=Q=\text{const. of O(1)}.$$

For this 'distinguished limit' it follows that the parameters λ and δ are not independent; rather, they are related by

$$(3.16)\qquad \lambda^{1+Q}/\delta^{Q}=1.$$

4. The Interior Viscous Region

4.1. **The outer viscous layer.** The flow in an outer viscous (or boundary) layer, which is the principal contributor to the slender 'effective body', is studied now. In this layer, the flow is taken to be both (1) 'Prandt-like', in that the spatial variables x and r, respectively, are of order unity and of order $\lambda = R^{-1/2}$ and (2) 'Oseen-like', in that the flow quantities u, v, and p are viscous perturbations with respect to their freestream values.

To formulate the problem for this outer viscous layer, then, the appropriate independent variables of unit order are

$$(4.1)\qquad x,\ y=r/\lambda;$$

while, the representations for the dependent variables are taken to be

$$(4.2)\qquad \begin{aligned}&u=1+\epsilon\bar{F}+\sigma\hat{F}=1+\epsilon(\bar{F}+\lambda^2\hat{F})=1+\epsilon F,\\ &v=(\sigma/\lambda)G=\lambda\epsilon G,\quad p=\sigma H=\lambda^2\epsilon H.\end{aligned}$$

This representation for u is determined to be one that is consistent with the one introduced by Lagerstrom and Cole [4] for an 'Oseen-like' outer viscous layer and is sufficiently general to take into account matching with the exterior layer representation. The representations for v and p are determined to be ones that complement this representation for u. In terms of these outer layer variables, the equations of motion become

$$\frac{\partial(yF)}{\partial x} + \frac{\partial(yG)}{\partial y} = 0; \tag{4.3}$$

$$\frac{\partial F}{\partial x} + \epsilon\left(F\frac{\partial F}{\partial x} + G\frac{\partial F}{\partial y}\right) + \lambda^2\frac{\partial H}{\partial x} = \left[\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial F}{\partial y}\right) + \lambda^2\frac{\partial^2 F}{\partial x^2}\right], \tag{4.4a}$$

$$\frac{\partial G}{\partial x} + \epsilon\left(F\frac{\partial G}{\partial x} + G\frac{\partial G}{\partial y}\right) + \frac{\partial H}{\partial y} = \left[\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial(yG)}{\partial y}\right) + \lambda^2\frac{\partial^2 G}{\partial x^2}\right]. \tag{4.4b}$$

Consider, now, that the perturbation flow quantities F, G, and H have asymptotic expansions of the forms

$$F \simeq (F_0 + \epsilon F_1 + \epsilon^2 F + \cdots) + \lambda^2(\hat{F}_0 + \epsilon\hat{F}_1 + \cdots) + \cdots, \tag{4.5a}$$

$$G \simeq (G_0 + \epsilon G_1 + \epsilon^2 G_2 + \cdots) + \lambda^2(\hat{G}_0 + \epsilon\hat{G}_1 + \cdots) + \cdots, \tag{4.5b}$$

$$H \simeq (H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots) + \lambda^2(\hat{H}_0 + \epsilon\hat{H}_1 + \cdots) + \cdots. \tag{4.5c}$$

Then, subject to the restriction that $\lambda^2/\epsilon^m \to 0$ (cf. equation (2.8a)), the equations of motion for the zeroth- and first-approximations, essentially the Oseen equations in boundary layer form (i.e., the (so-called) defect boundary layer equations), are

$$\frac{\partial(yF_0)}{\partial x} + \frac{\partial(yG_0)}{\partial y} = 0, \tag{4.6}$$

$$\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial F_0}{\partial y}\right) - \frac{\partial F_0}{\partial x} = 0, \tag{4.7}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial(yG_0)}{\partial y}\right) - \frac{\partial G_0}{\partial x} - \frac{\partial H_0}{\partial y} = 0;$$

$$\frac{\partial(yF_1)}{\partial x} + \frac{\partial(yG_1)}{\partial y} = 0, \tag{4.8}$$

$$\text{(4.9a)} \quad \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial F_1}{\partial y}\right) - \frac{\partial F_1}{\partial x} = \left[F_0\frac{\partial F_0}{\partial x} + G_0\frac{\partial F_0}{\partial y}\right],$$

$$\text{(4.9b)} \quad \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial(yG_1)}{\partial y}\right) - \frac{\partial G_1}{\partial x} - \frac{\partial H_1}{\partial y} = \left[F_0\frac{\partial G_0}{\partial x} + G_0\frac{\partial G_0}{\partial y}\right].$$

(The higher order approximations for u, i.e., $\hat{F}_0, \hat{F}_1, \cdots$, are considered in § 4.3.)

Directly, it is seen, from equations (4.6) and (4.7), that the zeroth-approximation has the first integrals

$$\text{(4.10)} \quad y\left(\frac{\partial F_0}{\partial y} + G_0\right) = \Gamma_0,\ H_0 = \Xi_0 - \frac{d\Gamma_0}{dx}\log y,$$

with Γ_0 and Ξ_0 functions of x (to be determined). It is taken that, as $y \to 0$ (x fixed), $F_0 \sim -\log(1/y) \to -\infty$, $G_0 \to 0$ (algebraically); while, as $y \to \infty$ (x fixed), $F_0 \to 0$ (exponentially), then, $G_0 \sim (1/y) \to 0$ (algebraically). Hence, it is taken that $\Gamma_0 = S_0 = 1$. Clearly, in this approximation, the pressure is constant across the layer, i.e., $H_0 = \Xi_0$, a function of x.

In the limit of $y \to 0$ (x fixed), it is found that the asymptotic behaviors for F_0 and G_0 are

$$\text{(4.11a)} \quad F_0 \sim -\log(1/y) + C_0 + \frac{1}{4}\frac{dC_0}{dx}y^2 + \cdots,$$

$$\text{(4.11b)} \quad G_0 \sim -\frac{1}{2}\frac{dC_0}{dx}y + \cdots,$$

with C_0 a function of x (to be determined). It is noted, for future reference, that, in this limit,

$$\text{(4.12a)} \quad F_0\frac{\partial F_0}{\partial x} + G_0\frac{\partial F_0}{\partial y} \sim -\frac{dC_0}{dx}\log(1/y) + \frac{1}{2}\frac{d(C_0\{C_0-1\})}{dx} + \cdots,$$

$$\text{(4.12b)} \quad F_0\frac{\partial G_0}{\partial x} + G_0\frac{\partial G_0}{\partial y} \sim \frac{1}{2}\frac{d^2C_0}{dx^2}y\log(1/y) - \frac{1}{4}\left\{2C_0\frac{d^2C_0}{dx^2} - \left(\frac{dC_0}{dx}\right)^2\right\}y + \cdots$$

The first-approximation has the first integral

$$(4.13)\quad y\left(\frac{\partial F_1}{\partial x}+G_1\right)=\Gamma_1+F_0G_0-\frac{\partial}{\partial x}\left(\int_y^\infty tF_0^2\,dt\right),$$

with $\Gamma_1 = S_1$ a function of x (to be determined). Based, in part, on this first integral, it is taken that, as $y \to 0$ (x fixed), the asymptotic behaviors for F_1, G_1, and H_1 are given by

$$(4.14a)\quad F_1 \sim -K_1\log(1/y)+C_1 - \frac{1}{4}\left[\left\{\frac{dK_1}{dx}+\frac{dC_0}{dx}\right\}\log(1/y) - \left\{\frac{d(C_1-K_1)}{dx}+\frac{1}{2}\frac{d(C_0\{C_0-3\})}{dx}\right\}\right]y^2+\cdots,$$

$$(4.14b)\quad G_1 \sim \frac{1}{2}\left[\frac{dK_1}{dx}\log(1/y)-\frac{d(C_1-\frac{1}{2}K_1)}{dx}\right]y+\cdots,$$

$$(4.14c)\quad H_1 \sim \frac{dK_1}{dx}\log(1/y)+J_1+\cdots,$$

with $K_1=\Gamma_1-d(\int_0^\infty yF_0^2\,dy)/dx$, C_1, and J_1 functions of x (to be determined).

Collecting terms, the asymptotic behaviors for u, v, and p at the 'inner edge' (i.e., $y \to 0$ (x fixed)) of this outer viscous layer are given by

$$(4.15a)\quad u \sim 1+\epsilon\{[-\log(1/y)+C_0+\cdots] + \epsilon[-K_1\log(1/y)+C_1+\cdots]+\cdots\},$$

$$(4.15b)\quad v \sim \lambda\epsilon\left\{\left[-\frac{1}{2}\frac{dC_0}{dx}y+\cdots\right] + \epsilon\left[\frac{1}{2}\left\{\frac{dK_1}{dx}\log(1/y)-\frac{d(C_1-\frac{1}{2}K_1}{dx}\right\}y+\cdots\right]+\cdots\right\},$$

$$(4.15c)\quad p \sim \lambda^2\epsilon\left\{J_0+\epsilon\left[\frac{dK_1}{dx}\log(1/y)+J_1+\cdots\right]+\cdots\right\},$$

with $J_0 = \Xi_0$.

Written in terms of the inner viscous layer coordinates, x, z, with $z = r/\delta = (\lambda/\delta)y$, equation (4.15) yields the following 'outer edge' ($z \to \infty$ (x fixed)) boundary conditions for the inner viscous layer:

(4.16a) $$u \sim 1 - \epsilon \log(\lambda/\delta) + \epsilon(\log z + C_0) - \epsilon^2 \log(\lambda/\delta) K_1 + \epsilon^2(K_1 \log z + C_1) + \cdots = \epsilon\{[\log z + (C_0 - K_1) + \cdots] + O(\epsilon)\},$$

(4.16b) $$v \sim \delta\epsilon \left\{ -\tfrac{1}{2} \frac{dC_0}{dx} z + \tfrac{1}{2} \epsilon \log(\lambda/\delta) \frac{dK_1}{dx} z - \tfrac{1}{2} \epsilon \left(\frac{dK_1}{dx} \log z + \frac{d(C_1 - \frac{1}{2}K_1)}{dx} \right) z + \cdots \right\} = \delta\epsilon \left\{ \left[-\tfrac{1}{2} \frac{d(C_0 - K_1)}{dx} z + \cdots \right] + O(\epsilon) \right\},$$

(4.16c) $$p \sim \lambda^2 \epsilon \left\{ J_0 + \epsilon \log(\lambda/\delta) \frac{dK_1}{dx} + \epsilon \left(-\frac{dK_1}{dx} \log z + J_1 \right) + \cdots \right\} = \lambda^2\epsilon \left\{ \left[\left(\frac{dK_1}{dx} + J_0 \right) + \cdots \right] + O(\epsilon) \right\},$$

for $\epsilon \log(\lambda/\delta) = 1$, i.e., $\epsilon = \log^{-1}(\lambda/\delta)$, as defined originally.

4.2. **The inner viscous layer.** To complete the analysis of the interior viscous region, the flow in an inner viscous (or surface) layer, which is directly adjacent to the slender body, is studied. In this layer, the flow is taken to be 'Stokes-like', in that (1) the radial variable is scaled to be of the order of the body thickness parameter, and (2) the viscous effects are the dominant ones as the velocity goes to zero to satisfy the nonslip boundary conditions at the body surface.

For this inner viscous layer, the appropriate tangential and normal independent variables of unit order are

(4.17) $$x, z = r/\delta (= (\lambda/\delta) y);$$

while, the representations for the dependent variables are taken to be

(4.18) $$u = \epsilon f, v = \delta\epsilon g, p = \lambda^2 \epsilon h.$$

These forms for the variables, which are consistent with those introduced by Lagerstrom and Cole [4] for a 'Stokes-like' inner viscous layer, can be 'derived' from a consideration of the nonuniformity in the solutions at the 'inner edge' of the outer viscous layer. The form of equation (4.15a) for $y \to 0$ (x fixed) suggests that $\epsilon \log(1/y) = O(1)$

or $y = O((\lambda/\delta)^{-q})$, where q must be found. If it is taken that $u \to 0$ for $y \to 0$ (on physical grounds), then, it follows that $q = 1$. Such a 'derivation' yields equation (4.16) from equation (4.15), and, in turn, yields the proper scalings for the dependent variables in this inner layer. Introduction of equations (4.17) and (4.18) into equations (2.1) and (2.2) produces

$$\frac{\partial(zf)}{\partial x} + \frac{\partial(zg)}{\partial z} = 0; \tag{4.19}$$

$$\theta\left(f\frac{\partial f}{\partial x} + g\frac{\partial f}{\partial z}\right) + \delta^2\frac{\partial h}{\partial x} = \left[\frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial z}\right) + \delta^2\frac{\partial^2 f}{\partial x^2}\right], \tag{4.20a}$$

$$\theta\left(f\frac{\partial g}{\partial x} + g\frac{\partial g}{\partial z}\right) + \frac{\partial h}{\partial z} = \left[\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial(zg)}{\partial z}\right) + \delta^2\frac{\partial^2 g}{\partial x^2}\right]. \tag{4.20b}$$

Here, $\theta = \epsilon\exp(-2/\epsilon) = (1 + Q)\delta^{2/(1+Q)}\log^{-1}(1/\delta)$. The boundary conditions to be satisfied are

$$f, g \to 0 \text{ as } z \to Z \quad (x \text{ fixed}). \tag{4.21}$$

It is taken, as suggested by equation (4.16), that the perturbation flow quantities f, g, and h have asymptotic expansions of the forms

$$f \simeq (f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots) + \cdots, \tag{4.22a}$$

$$g \simeq (g0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots) + \cdots, \tag{4.22b}$$

$$h \simeq (h_0 + \epsilon h_1 + \epsilon^2 h_2 + \cdots) + \cdots. \tag{4.22c}$$

Then, subject to the restrictions that θ/ϵ^m and $\delta^2/\epsilon^m \to 0$ (cf. equation (2.8b)), the equations of motion and the boundary conditions, for $k = 0, 1, 2, \cdots$, reduce to

$$\frac{\partial(zf_k)}{\partial x} + \frac{\partial(zg_k)}{\partial z} = 0, \tag{4.23}$$

$$\frac{\partial}{\partial z}\left(z\frac{\partial f_k}{\partial z}\right) = 0, \quad \frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial(zg_k)}{\partial z}\right) - \frac{\partial h_k}{\partial z} = 0; \tag{4.24}$$

$$f_k, g_k \to 0 \text{ as } z \to Z \quad (x \text{ fixed}). \tag{4.25}$$

The solutions of these equations, the Stokes equations in boundary layer form, are determined to be

(4.26a) $$f_k = A_k(\log z - \log Z),$$

(4.26b) $$g_k = -\tfrac{1}{2}\frac{dA_k}{dx} z\log z + \tfrac{1}{2}\frac{d(A_k\{\log Z + 1/2\}}{dx} z - \tfrac{1}{4}\frac{d(A_k Z^2)}{dx}(1/z),$$

(4.26c) $$h_k = -\frac{dA_k}{dx}\log z + \frac{d(A_k \log Z)}{dx} + B_k.$$

Here, A_k and B_k are functions of x (to be determined).

Collecting terms, the solutions for u, v, and p can be written as

(4.27a) $$u \simeq \epsilon\left\{\left[\left(\sum_{k=0}\epsilon^k A_k\right)(\log z - \log Z)\right] + \cdots\right\} = \epsilon\left\{\sum_{k=0}\epsilon^k[A_k(\log z - \log Z)] + \cdots\right\},$$

(4.27b) $$v \simeq \delta\epsilon\left\{\sum_{k=0}\epsilon^k\left[-\tfrac{1}{2}\frac{dA_k}{dx}z\log z + \tfrac{1}{2}\frac{d(A_k\{\log Z + \frac{1}{2}\})}{dx}z - \tfrac{1}{4}\frac{d(A_k Z^2)}{dx}(1/z)\right] + \cdots\right\},$$

(4.27c) $$p \simeq \lambda^2\epsilon\left\{\sum_{k=0}\epsilon^k\left[-\frac{dA_k}{dx}\log z + \frac{d(A_k \log Z)}{dx} + B_k\right] + \cdots\right\}.$$

Written in terms of the outer viscous layer coordinates x, y, equation (4.27) yields the following 'inner edge" boundary conditions for the outer viscous layer:

(4.28a) $$\begin{aligned} u &\simeq \epsilon\left\{\sum_{k=0}\epsilon^k[A_k(\log(\lambda/\delta) - \log(1/y) - \log Z)] + \cdots\right\} \\ &= A_0 - \epsilon\{[A_0\log(1/y) + (A_0\log Z - A_1) + \cdots] \\ &\quad + \epsilon[A_1\log(1/y) + (A_1\log Z - A_2) + \cdots] + \cdots\} \\ &= 1 + \epsilon\{[-\log(1/y) + (A_1 - \log Z) + \cdots] \\ &\quad + \epsilon[-A_1\log(1/y) + (A_2 - A_1\log Z) + \cdots] + \cdots\},\end{aligned}$$

(4.28b)
$$v \simeq \lambda\epsilon \Big\{ \sum_{k=0} \epsilon^k \Big[-\tfrac{1}{2} \frac{dA_k}{dx} y(\log(\lambda/\delta) - \log(1/y))$$
$$+ \tfrac{1}{2} \frac{d(A_k\{\log Z + \tfrac{1}{2}\})}{dx} y - \tfrac{1}{4} (\delta/\lambda)^2 \frac{d(A_k Z^2)}{dx} (1/y) \Big]$$
$$+ \cdots \Big\}$$
$$= \lambda\epsilon \Big\{ \Big[-\tfrac{1}{2} \frac{d(A_1 - \log Z)}{dx} y + \cdots \Big]$$
$$+ \epsilon \Big[\tfrac{1}{2} \Big\{ \frac{dA_1}{dx} \log(1/y) - \frac{d(A_2 - A_1\{\log Z + \tfrac{1}{2}\})}{dx} \Big\} y + \cdots \Big]$$
$$+ \cdots \Big\},$$

(4.28c)
$$p \simeq \lambda^2\epsilon \Big\{ \sum_{k=0} \epsilon^k \Big[-\frac{dA_k}{dx} (\log(\lambda/\delta) - \log(1/y))$$
$$+ \frac{d(A_k \log Z)}{dx} + B_k \Big] + \cdots \Big\}$$
$$= \lambda^2\epsilon \Big\{ \Big[B_0 - \frac{d(A_1 - \log Z)}{dx} \Big]$$
$$+ \epsilon \Big[\frac{dA_1}{dx} \log(1/y)$$
$$+ \Big\{ B_1 - \frac{d(A_2 - A_1 \log Z)}{dx} \Big\} \cdots \Big] + \cdots \Big\},$$

for $A_0 = 1$(and $\epsilon \log(\lambda/\delta) = 1$).

From a comparison of equation (4.15) and (4.28) it is seen that the outer layer-inner layer matching requires

(4.29a) $$A_1 = C_0 + \log Z, \quad A_2 = C_1 + (C_0 + \log Z) \log Z, \cdots,$$

(4.29b) $$B_0 = J_0 + \frac{dC_0}{dx}, \quad B_1 = J_1 + \frac{dC_1}{dx}, \cdots;$$

(4.30) $$K_1 = A_1 = C_0 + \log Z, \cdots.$$

Thus, to complete the velocity problem (to the order of approximation considered), the functions C_0 and C_1 and/or A_1 and A_2 must be determined.

4.3. **The velocity boundary value problem.** From the results of § 4.1 and § 4.2, it follows that the zeroth- and first-approximation outer viscous layer velocity boundary value problems take the forms

(4.31a) $$\frac{1}{y}\frac{\partial}{\partial y}\left(y\ \frac{\partial F_0}{\partial x}\right) - \frac{\partial F_0}{\partial x} = 0:$$

(4.31b) $$F_0 \to 0 \text{ (exponentially) as } y \to \infty \ (x \text{ fixed}),$$

(4.31c) $$F_0 \sim -\log(1/y) + C_0 + \cdots \text{ as } y \to 0 \ (x \text{ fixed});$$

(4.32a) $$\frac{1}{y}\frac{\partial}{\partial y}\left(y\ \frac{\partial F_1}{\partial y}\right) - \frac{\partial F_1}{\partial x} = \left[F_0\ \frac{\partial F_0}{\partial x} + \left(\frac{1}{y} - \frac{\partial F_0}{\partial y}\right)\frac{\partial F_0}{\partial y}\right]:$$

(4.32b) $$F_1 \to 0 \text{ (exponentially) as } y \to \infty \ (x \text{ fixed}),$$

(4.32c) $$F_1 \sim -(C_0 + \log Z)\log(1/y) + C_1 + \cdots \text{ as } y \to 0 \ (x \text{ fixed}).$$

Solutions of these boundary value problems are presented now for the case of a 'power-law' body, i.e., for $Z = x^n$.

Consider the transformation of independent variables from (x, y) to (ξ, ζ), where

(4.33) $$\xi = x, \quad \zeta = y^2/4x.$$

Under this transformation, the differential equations become

(4.34) $$\frac{\partial}{\partial \zeta}\left(\zeta\ \frac{\partial F_0}{\partial \zeta}\right) + \zeta\ \frac{\partial F_0}{\partial \zeta} - \xi\ \frac{\partial F_0}{\partial \xi} = 0;$$

(4.35) $$\frac{\partial}{\partial \zeta}\left(\zeta\ \frac{\partial F_1}{\partial \zeta}\right) + \zeta\frac{\partial F_1}{\partial \zeta} - \xi\ \frac{\partial F_1}{\partial \xi} = \left[\frac{\partial}{\partial \zeta}\left(F_0\left\{\zeta\ \frac{\partial F_0}{\partial \zeta} + \frac{1}{2}\right\}\right) - 2\zeta\left(\frac{\partial F_0}{\partial \zeta}\right)^2\right].$$

The solutions of equations (4.34) and (4.35) are determined (cf. Glauert and Lighthill [3]; Lagerstrom and Cole [4]) to be

(4.36) $$F_0 = -\tfrac{1}{2}E_0(\zeta);$$

(4.37) $$F_1 = \tfrac{1}{4}(1 - 2n)[\log \xi E_0(\zeta) + \{\log(1/\zeta) - \gamma\}E_0(\zeta) - 2E_1(\zeta)]$$

$$+\tfrac{1}{4}[2E_0(2\zeta) - E_1(\zeta)$$
$$- E_0(\zeta)\{\tfrac{1}{2}E_0(\zeta) - \log(1/\zeta)$$
$$+ \exp(-\zeta) + (1 + 2\gamma - \log 4)\}].$$

The functions $E_0(\zeta)$ and $E_1(\zeta)$ introduced in these equations are defined by

$$E_0(\zeta) = -Ei(-\zeta) = \int_\zeta^\infty (1/\nu)\exp(-\nu)\,d\nu; \tag{4.38}$$

$$E_1(\zeta) = \int_\zeta^\infty (1/\nu)E_0(\nu)\,d\nu. \tag{4.39}$$

From these solutions, it is found that

$$C_0 = -\tfrac{1}{2}\{\log\xi + (\log 4 - \gamma)\}, \tag{4.40a}$$

$$A_1 = -\tfrac{1}{2}\{(1 - 2n)\log\xi + (\log 4 - \gamma)\}; \tag{4.40b}$$

$$C_1 = \tfrac{1}{4}\{\log\xi + (\log 4 - \gamma)\}\{(1 - 2n)\log\xi + (\log 4 - \gamma)\} - \tfrac{1}{4}\left\{(3 - 4n)\frac{\pi^2}{12} + \log 4\right\}, \tag{4.41a}$$

$$A_2 = \tfrac{1}{4}\{(1 - 2n)\log\xi + (\log 4 - \gamma)\}^2 - \tfrac{1}{4}\left\{(3 - 4n)\frac{\pi^2}{12} + \log 4\right\}. \tag{4.41b}$$

(Here, γ = Euler's const. = 0.577 · · ·.) (The presence of the term containing $\log\xi$ in equation (4.37) introduces nonuniformities for $\xi \to 0$ and $\xi \to \infty$.)

To complement the solutions of equations (4.36) and (4.37), it is determined that the normal velocity functions are

$$(4\xi\zeta)^{1/2}G_0 = 1 - \exp(-\zeta); \tag{4.42}$$

$$(4\xi\zeta)^{1/2}G_1 = -\tfrac{1}{2}(1 - 2n)[\log\xi\{1 - \exp(-\zeta)\} + E_0(\zeta) - \{\log(1/\zeta) - \gamma\}\exp(-\zeta)]$$
$$+\tfrac{1}{2}[2E_0(2\zeta) + \{\log(1/\zeta) + (\log 4 - \gamma)\}\exp(-\zeta) - E_0(\zeta)\{1 + (2 + \zeta)\exp(-\zeta)\}]$$
$$+\tfrac{1}{2}[\{\gamma - \exp(-\zeta)\}\{1 - \exp(-\zeta)\}]. \tag{4.43}$$

In turn, the pressure functions are found to be

$$H_0 = J_0 = \Xi_0; \tag{4.44}$$

$$H_1 = J_1 - \frac{(1-2n)}{4\xi}\,[\log(1/\zeta) - \log\xi - \log 4] + \frac{1}{4\xi}\,[\{E_0(2\zeta) + \log 2 - E_0(\zeta)\} - \tfrac{1}{2}(1/\zeta)\{1 - \exp(-\zeta)\}^2]. \tag{4.45}$$

Here, J_0 and J_1 are functions of ξ, namely:

$$\begin{aligned} J_0 &= B_0 + \frac{1}{2\xi}; \\ J_1 &= B_1 - \frac{1}{2\xi}\{(1-2n)\log\xi + (1-n)(\log 4 - \gamma)\}. \end{aligned} \tag{4.46}$$

Based upon the solutions obtained here for F_0, G_0, H_0 and F_1, G_1, H_1, at the 'outer edge' of the outer viscous layer, it is found that

$$\begin{aligned} u &\sim 1 - \tfrac{1}{2}\,\epsilon\{(1/\zeta)\exp(-\zeta)(1 + \cdots)\} \times \\ &\qquad \{1 + \epsilon[(1-n)\log\zeta + \Delta_1{}' + \cdots] + \cdots\} \\ &= 1 - \tfrac{1}{2}\,\epsilon\{(y^2/4x)^{-1}\exp(-y^2/4x)(1 + \cdots)\} \times \\ &\qquad \{1 + \epsilon[2(1-n)\log y + \Delta_1 + \cdots] + \cdots\}. \end{aligned} \tag{4.47a}$$

$$\begin{aligned} v &\sim \tfrac{1}{2}\,\lambda\epsilon\,(1/\xi\zeta)^{1/2}\{[1 + \cdots] + \epsilon[\Gamma_1{}' + \cdots] + \cdots\} \\ &= \lambda\epsilon(1/y)\{[1 + \cdots] + \epsilon[\Gamma_1 + \cdots] + \cdots\}, \end{aligned} \tag{4.47b}$$

$$\begin{aligned} p &\sim \lambda^2\epsilon\left\{\Xi_0{}' + \epsilon\left[\frac{(1-2n)}{4\xi}\log\zeta + \Xi_1{}' + \cdots\right] + \cdots\right\} \\ p &= \lambda^2\epsilon\left\{\Xi_0 + \epsilon\left[\frac{(1-2n)}{2x}\log y + \Xi_1 + \cdots\right] + \cdots\right\}, \end{aligned} \tag{4.47c}$$

where

$$\begin{aligned} \Delta_1{}' &= \tfrac{1}{2}\,\{(1-2n)\log(1/\xi) + 1 - (\log 4 - (3-2n)\gamma))\}, \\ \Delta_1 &= \tfrac{1}{2}\,\{(3-4n)\log(1/x) + 1 - (3-2n)(\log 4 - \gamma)\}, \end{aligned} \tag{4.48a}$$

$$\begin{aligned} \Gamma_1{}' &= \tfrac{1}{2}\,\{(1-2n)\log(1/\xi) + \gamma\}, \\ \Gamma_1 &= \tfrac{1}{2}\,\{(1-2n)\log(1/x) + \gamma\}, \end{aligned} \tag{4.48b}$$

$$(4.48c)\qquad \Xi_0{}' = B_0 + \frac{1}{2\xi},$$

$$\Xi_1{}' = B_1 + \frac{1}{4\xi}\{(1-2n)\log(1/\xi) - (\log 2 - 2(1-n)\gamma)\},$$

$$\Xi_0 = B_0 + \frac{1}{2x},$$

$$\Xi_1 = B_1 + \frac{1}{2x}\{(1-2n)\log(1/x) - (1-n)(\log 2 - \gamma)\}.$$

From inspection of (3.13b) and/or (3.14b) and (4.48b), it follows directly that the exterior region-interior region matching for v is accomplished if $\Gamma_0 = S_0 = 1$, $\Gamma_1 = S_1 = \frac{1}{2}\{(1-2n)\log(1/x) + \gamma\}, \cdots$. Similarly, inspection of (3.13c) and/or (3.14c) and (4.48c) indicates that such matching for p is accomplished if $\Xi_0 = \{1 + 2Q(n - \frac{1}{2})\}/2x$ and/or $B_0 = Q(n - \frac{1}{2})/x, \cdots$. It is noted that, based on the foregoing, the pressure at the 'power-law' body surface is

$$(4.49)\qquad p_b \simeq Q(1+Q)\lambda^2 \log^{-1}(1/\lambda)\left\{\left[\frac{(n-n_0)}{x}\right] + \cdots\right\},$$

with $n_0 = Q/2(1+Q)$.

This result indicates that, in order for p_b to be non-negative to leading order of approximation, the analysis is applicable only to a 'power-law' body for which the exponent n satisfies $0 < n_0 = Q/2(1+Q) \leqq n \leqq 1$.

From inspection of (3.13a) and/or (3.14a) and (4.48a), it is seen that the exterior region-interior region matching for u does not follow directly. It is suggested that, to accomplish this matching, it is necessary to consider the contributions of the terms of order λ^2 and higher in the outer viscous layer asymptotic representations (equation (4.5)) for the flow quantities. From such a consideration, it is determined that the governing equations for $\hat{F}_0$, $\hat{G}_0$ and $\hat{F}_1$, $\hat{G}_1$ are

$$(4.50a)\qquad \frac{\partial(y\hat{F}_0)}{\partial x} + \frac{\partial(y\hat{G}_0)}{\partial y} = 0,$$

$$(4.50b)\qquad \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial \hat{F}_0}{\partial y}\right) - \frac{\partial \hat{F}_0}{\partial x} = \frac{\partial H_0}{\partial x} - \frac{\partial^2 F_0}{\partial x^2};$$

$$(4.51a)\qquad \frac{\partial(y\hat{F}_1)}{\partial x} + \frac{\partial(y\hat{G}_1)}{\partial y} = 0,$$

$$(4.51b)\qquad \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial \hat{F}_1}{\partial y}\right) - \frac{\partial \hat{F}_1}{\partial x}$$

$$= \frac{\partial H_1}{\partial x} - \frac{\partial^2 F_1}{\partial x^2} + \left(F_0\frac{\partial \hat{F}_0}{\partial x} + G\frac{\partial \hat{F}_0}{\partial x} + \frac{\partial F_0}{\partial x}\quad \hat{F}_0\quad \frac{\partial F_0}{\partial y} + \hat{G}_0\right).$$

In principle, equations (4.50) and (4.51) can be solved to determine the behaviors of $\hat{F}_0$, $\hat{G}_0$ and $\hat{F}_1$, $\hat{G}_1$ throughout the outer viscous layer. However, here, it is sufficient to determine the behaviors of $\hat{F}_0$ and $\hat{F}_1$ at the 'outer edge' of this layer. From inspection of the above equations, it is seen that these asymptotic behaviors are those that result from a velocity-pressure balance, namely:

$$\begin{aligned}\hat{F}_0 &\sim -[\Xi_0 + \cdots];\\ \hat{F}_1 &\sim -\left[\frac{(1-2n)}{2x}\log y + \Xi_1 + \cdots\right].\end{aligned} \tag{4.52}$$

With the introduction of these higher order contributions, the expression for u at the 'outer edge' of the outer viscous layer becomes

$$\begin{aligned}u \sim\ & 1 - \tfrac{1}{2}\epsilon\{(1/\zeta)\exp(-\zeta)(1+\cdots)\}\times\\ &\{1 + \epsilon[(1-n)\log\zeta + \Delta_1{}' + \cdots] + \cdots\}\\ &- \lambda^2\epsilon\left\{[\Xi_0{}' + \cdots] + \epsilon\left[\frac{(1-2n)}{4\xi}\log\zeta + \Xi_1{}'\right.\right.\\ &\left.\left.+ \cdots\right] + \cdots\right\} + \cdots\\ =\ & 1 - \tfrac{1}{2}\epsilon\{(y^2/4x)^{-1}\exp(-y^2/4x)(1+\cdots)\}\times\\ &\{1 + \epsilon[2(1-n)\log y + \Delta_1 + \cdots] + \cdots\}\\ &-\lambda^2\epsilon\left\{[\Xi_0 + \cdots] + \epsilon\left[\frac{(1-2n)}{2x}\log y + \Xi_1\right.\right.\\ &\left.\left.+ \cdots\right] + \cdots\right\} + \cdots.\end{aligned} \tag{4.53}$$

Now, an examination of equations (3.13a) and/or (3.14a) and (4.53) reveals that these higher order boundary layer contributions behave consistently with the leading order inviscid layer contributions. However, a further examination of equation (4.53) reveals that this solution for u has a nonuniformity when $\lambda^2\zeta\exp(\zeta) = O(1)$. This nonuniformity in the solution for u at the 'outer edge' of the interior region prevents direct matching of this solution with that for u at the 'inner edge' of the exterior region. How this matching is accomplished is discussed in the following section on the intermediate viscous region (or layer), which is introduced in order to treat the above-mentioned nonuniformity in the interior viscous region solution.

5. **The Intermediate Viscous Region.** The motivation (cf. Lee and Cheng [5]) for the consideration of an intermediate viscous region

arises from the results of equation (4.53), which indicate that, while the leading order correction terms for the interior region velocity u decay exponentially near the 'outer edge', the higher order correction terms remain fixed near this 'edge'. Thus, near this 'edge', although the combined leading order-higher order boundary layer solution (of equation 4.53)) has the proper behavior to match that of the inviscid layer, the leading order boundary layer solution (of equation (4.47a)) fails to be valid as a 'first approximation'. Indeed, it is seen from equation (4.53) that the given solution ceases to be the uniformly valid asymptotic solution for the interior region in a region where $\lambda^2\zeta\exp(\zeta) = O(1)$. To treat this region, a new set of independent variables of unit order is defined as follows:

$$s = \xi = x,$$
$$t = \lambda^2\zeta\exp(\zeta) = \lambda^2(y^2/4x)\exp(y^2/4x) \tag{5.1}$$
$$= (r^2/4x)\exp(r^2/4\lambda^2 x).$$

For the interior region, with x, y fixed, $t = O(\lambda^2) \to 0$; while, for the exterior region, with x, r fixed, $t = O(\exp(1/\lambda^2)) \to \infty$. As $t \to 1$, $\zeta \to \zeta_*$, where

$$\zeta_*\exp(\zeta_*) = (1/\lambda^2):$$

$$\zeta_* \simeq \log(1/\lambda^2)\left[1 - \frac{\log\log(1/\lambda^2)}{\log(1/\lambda^2)}\left\{1 - \frac{1}{\log(1/\lambda^2)} + \cdots\right\}\right] \to \infty. \tag{5.2a}$$

Equation (5.2a), thus, relates the small parameter λ to a new large parameter ζ_*. For future reference, it is appropriate to note that

$$\lambda = \zeta_*^{-1/2}\exp(-\tfrac{1}{2}\zeta_*);$$
$$\epsilon = Q\log^{-1}(1/\lambda) = 2Q\zeta_*^{-1}(1 + \zeta_*^{-1}\log\zeta_*)^{-1}. \tag{5.2b}$$

In turn, it is determined that

$$\zeta \simeq \zeta_* + \log t(1 - \zeta_*^{-1} + \cdots) \to \infty, \tag{5.3}$$

such that

$$y = 2(\xi\zeta)^{1/2} \simeq 2\zeta_*^{1/2}s^{1/2}(1 + \tfrac{1}{2}\zeta_*^{-1}\log t + \cdots), \tag{5.4a}$$
$$\log y \simeq \tfrac{1}{2}\{\log\zeta_* - (\log(1/s) - \log 4) + \zeta_*^{-1}\log t + \cdots\};$$
$$(1/r) = \tfrac{1}{2}(1/\lambda)(\xi\zeta)^{-1/2} \tag{5.4b}$$
$$\simeq \tfrac{1}{2}\exp(\tfrac{1}{2}\zeta_*)(1/s)^{1/2}(1 - \tfrac{1}{2}\zeta_*^{-1}\log t + \cdots),$$

$$\log(1/r) \simeq \tfrac{1}{2}\{\zeta_* + (\log(1/s) - \log 4) - \zeta_*^{-1}\log t + \cdots\}.$$

Introduction of equation (5.4a) into equations (4.53) and (4.47b,c) yields

$$(5.5a)\qquad u \sim 1 - Q\zeta_*^{-2}\exp(-\zeta_*)\{[(1/t) + 2\,\Xi_0] + O(\zeta_*^{-1}(\log\zeta_* + 1))\},$$

$$(5.5b)\qquad v \sim Q\zeta_*^{-2}\exp(-\tfrac{1}{2}\zeta_*)\{[(1/s)^{1/2}] + O(\zeta_*^{-1}(\log\zeta_* + 1))\},$$

$$(5.5c)\qquad p \sim Q\zeta_*^{-2}\exp(-\zeta_*)\{[2\,\Xi_0] + O(\zeta_*^{-1}(\log\zeta_* + 1))\},$$

the boundary (or matching) conditions for the flow quantities at the 'inner edge' (defined by $t \to 0$, s fixed) of this intermediate layer. In a similar manner, introduction of equation (5.4b) into equation (3.13) yields the 'outer edge' (defined by $t \to \infty$, s fixed) boundary (or matching) conditions

$$(5.6a)\qquad u \sim 1 - Q\zeta_*^{-2}\exp(-\zeta_*)\{[2\,\Xi_0] + O(\zeta_*^{-1}(\log\zeta_* + 1))\},$$

$$(5.6b)\qquad v \sim Q\zeta_*^{-2}\exp(-\tfrac{1}{2}\zeta_*)\{[(1/s)^{1/2}] + O(\zeta_*^{-1}(\log\zeta_* + 1))\},$$

$$(5.6c)\qquad p \sim Q\zeta_*^{-2}\exp(-\zeta_*)\{[2\,\Xi_0] + O(\zeta_*^{-1}(\log\zeta_* + 1))\}.$$

From the forms that the boundary conditions of equations (5.5) and (5.6) take, the following asymptotic representations for the flow quantities in this intermediate layer are suggested:

$$(5.7a)\qquad u \simeq 1 - Q\zeta_*^{-2}\exp(-\zeta_*)(U_0^+ + \zeta_*^{-1}\log\zeta_* U_{11}^+ + \zeta_*^{-1}U_1^+ + \cdots),$$

$$(5.7b)\qquad v \simeq Q\zeta_*^{-2}\exp(-\tfrac{1}{2}\zeta_*)(V_0^+ + \zeta_*^{-1}\log\zeta_* V_{11}^+ + \zeta_*^{-1}V_1^+ + \cdots),$$

$$(5.7c)\qquad p \simeq Q\zeta_*^{-2}\exp(-\zeta_*)(P_0^+ + \zeta_*^{-1}\log\zeta_* P_{11}^+ + \zeta_*^{-1}P_1^+ + \cdots).$$

Substitution of equations (5.1) and (5.7) into equations (2.1) and (2.2) yields

$$(5.8)\qquad \frac{\partial P_0^+}{\partial t} = 0,\quad \frac{\partial V_0^+}{\partial t} = 0,\quad \frac{\partial^2 U_0^+}{\partial t^2} + \frac{2}{t}\frac{\partial U_0^+}{\partial t} = 0;\ \cdots.$$

The solutions for the zeroth-order quantities U_0^+, V_0^+, and P_0^+ satisfying the boundary conditions are determined to be

$$U_0{}^+ = 2\,\Xi_0 + (1/t),\ V_0{}^+ = (1/s)^{1/2},\ P_0{}^+ = 2\Xi_0. \tag{5.9}$$

The details of the work needed to obtain higher order solutions for the flow quantities in such an intermediate layer are not presented here. It is important, however, to note how the technique outlined here is able to treat matching problems involving decaying exponentials.

Appendix. The Local Skin Friction Coefficient. The local skin friction coefficient C_f, is defined by

$$C_f = \frac{2\mu^*}{\rho^* u_\infty{}^{*2}} \left(\frac{\partial u^*}{\partial r^*}\right)_{r^* = r_b{}^*}, \tag{A.1a}$$

or, in terms of the previously-introduced nondimensional quantities, by

$$\begin{aligned} C_f &= 2\lambda^2 \left(\frac{\partial u}{\partial r}\right)_{r=\delta Z}; \\ C_f R^{1/2} &= \frac{2\tau}{\log \tau}\left(\frac{\partial f}{\partial z}\right)_{z=Z} = 2\epsilon \exp(1/\epsilon)\left(\frac{\partial f}{\partial z}\right)_{z=Z}. \end{aligned} \tag{A.1b}$$

It follows from the present analysis that, for $Z = x^n$, with $0 < Q/2(1 + Q) \leqq n \leqq 1$,

$$C_f R^{1/2} \simeq 2\epsilon \exp(1/\epsilon)\left[\frac{1}{x^n}(1 + \epsilon A_1 + \epsilon^2 A_2 + \cdots) + \cdots\right], \tag{A.2a}$$

with

$$\begin{aligned} A_0 = 1,\ A_1 &= -\tfrac{1}{2}\{(1 - 2n)\log x + (\log 4 - \gamma)\}, \\ A_2 &= \tfrac{1}{4}\ \{(1 - 2n)\log x + (\log 4 - \gamma).\}^2 \\ &\quad - \tfrac{1}{4}\left\{(3 - 4n)\ \frac{\pi^2}{12} + \log 4\right\}, \cdots. \end{aligned} \tag{A.2b}$$

It is noted that this expression may be rewritten as

$$\begin{aligned} C_f R_x{}^{1/2} &\simeq 2\exp(\tfrac{1}{2}\gamma)(1/\chi)\exp(\tfrac{1}{2}\chi) \times \\ &\left[1 - \left\{(3 - 4n)\ \frac{\pi^2}{12} + \log 4\right\}(1/\chi)^2 + O((1/\chi)^3)\right], \end{aligned} \tag{A.3a}$$

where $R_x = Rx(= \rho^* u_\infty{}^* x^*/\mu^*)$, and

$$\begin{aligned} \chi &= \log\{4\exp(-\gamma)\tau^2 x^{1-2n}\} \\ &= 2(1/\epsilon) + \{(1 - 2n)\log x + (\log 4 - \gamma)\} \\ &= 2\{(1/\epsilon) - A_1\}. \end{aligned} \tag{A.3b}$$

Essentially this expression is presented by Glauert and Lighthill [3], Wu and Wu [9], and Mark [7].

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