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A(X) AND GB-NOETHERIAN RINGS

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ABSTRACT. Three theorems concerning a Noetherian ring A are proved: (1) $A(X)$ is a GB-ring (that is, adjacent prime ideals in integral extension rings of A contract in A to adjacent prime ideals) if and only if $A[X]$ is a GB-ring; (2) if A is local and altitude $A = n + 2$ $(n \geqq 0)$, then $A(X_1, \cdots, X_n)$ is a GB-ring if and only if A satisfies the second chain condition for prime ideals (s.c.c.); and, (3) each GB-local domain A is such that $A(X)$ is a GB-ring if and only if each GB-local domain satisfies the s.c.c.

1. **Introduction.** All rings in this article are assumed to be commutative with identity, and the undefined terminology is the same as that in [**6**].

GB-rings were considered in their own right for the first time in [**11**], and therein a number of properties of such rings were proved. The reason such rings are of interest is that they are closely related to the (catenary) chain conjectures. Specifically, these conjectures are concerned with whether or not certain rings satisfy the chain condition for prime ideals (c.c.) (see (3.6.4) for the definition), and it is known [**11**, (3.8)] that a ring A satisfies the c.c. if and only if A is catenary and a GB-ring. Now, catenary rings have been deeply investigated in a number of papers, but, except for [**11**], GB-rings seem not to have been considered as an object of study in their own right. Even so, the literature does contain scattered information on GB-rings. For instance, it is easily seen that M. Nagata's example [**6**, Example 2, pp. 203–205] is not a GB-ring. And, probably the most important fact obtained on such rings is I. Kaplansky's 1972 paper [**3**] in which he gave a negative answer to the following question asked by W. Krull in 1937 in [**4**, p. 755]: is every integrally closed integral domain a GB-ring? (However, it is still an open problem if the integral closure of a Noetherian domain is necessarily a

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GB-ring, and this problem is closely related to the chain conjectures.)

Thus, GB-rings are of importance for the chain conjectures (and even in their own right). In this paper, we concentrate on the relationships between a GB-Noetherian ring A and the rings $A(X_1, \cdots, X_n)$. A brief summary of the contents of this paper will now be given.

The main result in Section 2, (2.7), is (1) above. This result greatly sharpens [**11**, (5.3)], and it has the following result as a corollary: for a local domain (R, M), $R(X)$ is a GB-ring if and only if $R[X]$ is a GB-ring if and only if $R[X]_N$ is a GB-ring, for some maximal ideal N in $R[X]$ such that $N \cap R = M$ (2.8). This is a quite unexpected result, since for almost all of the other chain conditions on a local domain, R has the conditions if and only if $R(X)$ has it, and $R[X]$ and $R[X]_{(M,X)}$ do not inherit the condition from R. (Concerning this, see the end of Section 4.)

Section 3 contains (2) above, (3.7). This result is a corollary to a more general theorem from which it follows that if a Noetherian ring A is such that $A(X_1, \cdots, X_n)$ is a GB-ring, for some $n \geqq$ altitude $A - 2$, then every locality over A is also a GB-ring (3.11).

In Section 4, the main result, (4.2), is (3) above. This result shows that two of the most important open problems concerning GB-rings are, in fact, equivalent problems.

Section 5 contains two new characterizations of GB-rings ((5.2) and (5.3)), and then a characterization is given for when a GB-local domain R is such that $R(X)$ is a GB-ring.

Four results on a GB-ring A of altitude three are proved in Section 6. Among these is: $A(X)$ is a GB-ring if and only if A satisfies the chain condition for prime ideals (6.2).

Finally, Section 7 contains three questions on GB-rings which the author has been unable to answer.

2. **$A[X]$ is a GB-ring if and only if $A(X)$ is.** The main result in this section, (2.6), is a considerable improvement of [**11**, (5.3)], and yields the title of this section (for Noetherian rings) as a corollary (2.7). In order to prove (2.6), a few preliminaries are needed. We begin by fixing two notational conventions, in order to avoid continual repetitions throughout this paper.

NOTATION (2.1). A' denotes the integral closure of a ring A in its total quotient ring, and L^* denotes the J-adic completion of a semi-local ring L, where J is the Jacobson radical of L.

The next result, which is an easy consequence of the Going Down Theorem [**6**, (10.13)], is of some interest in itself, and will be used in the proof of (2.6).

PROPOSITION (2.2). *Let A be an integral domain, X an indeterminate, and C an integral extension domain of $A[X]$. Let P' be a prime ideal in C, $P = P' \cap A[X]$, and $p = P \cap A$. If $pA[X] \subset P$, then there exists a prime ideal P_1 in C such that $P_1 \subset P'$ and $P_1 \cap A[X] = pA[X]$.*

PROOF. Let $D = C[A']$ (2.1), so D is integral over $A'[X]$. Let Q' be a prime ideal in D such that $Q' \cap C = P'$, let $Q = Q' \cap A'[X]$, and let $q = Q \cap A'$, so $q \cap A = p$. Then $qA'[X] \subset Q$ (since $pA[X] \subset P$ and $qA'[X] \cap A[X] = pA[X]$). Therefore, by the Going Down Theorem, there exists a prime ideal Q'' in D such that $Q'' \subset Q'$ and $Q'' \cap A'[X] = qA'[X]$, so $P_1 = Q'' \cap C$ is such that $P_1 \subset P'$ and $P_1 \cap A[X] = pA[X]$.

We now begin to consider GB-rings specifically.

DEFINITION (2.3). A ring A is a *GB-ring* (going between ring) in case, for all integral extension rings B of A and for all prime ideals $P \subset Q$ in B, if there exists a prime ideal p in A such that $P \cap A \subset p \subset Q \cap A$, then there exists a prime ideal P' in B such that $P \subset P' \subset Q$. (That is, adjacent prime ideals in B lie over adjacent prime ideals in A—see (2.5.3).)

The next two remarks summarize the known facts about GB-rings which are needed in what follows.

REMARK (2.4). Let A be a GB-ring. Then the following statements hold:

(2.4.1) [11, (3.11)]. If B is an integral extension ring of A, then B is a GB-ring.

(2.4.2) [11, (3.9.1)]. If A is a domain and altitude $A \leqq 2$, then A satisfies the c.c. (see (3.6.4)).

(2.4.3) [11, (2.2.3)]. If A is a domain and P is a height one prime ideal in an integral extension domain of A, then height $P \cap A = 1$.

REMARK (2.5). The following statements hold for a ring A:

(2.5.1) [11, (3.3)]. The following statements are equivalent: A is a GB-ring; A/I is a GB-ring, for all ideals I in A; and, A/z is a GB-ring, for all minimal prime ideals z in A.

(2.5.2) [11, (3.5)]. The following statements are equivalent: A is a GB-ring; A_S is a GB-ring, for all m.c. sets S in A; and, A_M is a GB-ring, for all maximal ideals M in A.

(2.5.3) [11, (2.2.1)]. A is a GB-ring if and only if, for all integral extension rings B of A and for all prime ideals $P \subset Q$ in B, if height $Q/P = 1$, then height $(Q \cap A)/(P \cap A) = 1$.

(2.5.4) [11, (4.2)]. If A is a Noetherian domain, then A is a GB-ring if and only if the condition in (2.5.3) holds for all principal integral extension domains $B = A[c]$ of A.

(2.5.5) [11, (5.1)]. If A is Noetherian and $A(X)$ is a GB-ring, then A is a GB-ring.

(2.5.6) [11, (3.10)]. If altitude $A < \infty$, then A satisfies the s.c.c if and only if A satisfies the f.c.c. and is a GB-ring. (See (3.6) for the definitions.)

We can now prove the main result in this section. (2.6) is a considerable improvement of [11, (5.3)], which showed that if A is a Noetherian ring such that $A(X)$ is a GB-ring, then, for all integral extension rings B of A, adjacent prime ideals in $B[X]$ lie over adjacent prime ideals in $A[X]$. (Recall that $A(X) = A[X]_S$, where S is the set of polynomials whose coefficients generate the unit ideal in A. For some details on this, see [6, pp. 17–18].)

THEOREM (2.6). *Let $A \subseteq B$ be rings such that A is Noetherian and B is integral over A, and let X be an indeterminate. If $A(X)$ is a GB-ring, then $B[X]$ is a GB-ring.*

PROOF. By (2.4.1), it suffices to prove that $A[X]$ is a GB-ring. Also, by (2.5.1), it suffices to prove that $(A/z)[X]$ is a GB-ring, for all minimal prime ideals z in A, hence it may be assumed that $B = A$ is a Noetherian domain. Further, by (2.5.2), $A[X]$ is a GB-ring if and only if $A[X]_N$ is a GB-ring, for all maximal ideals N in $A[X]$. Therefore, since $A_{N \cap A}(X)$ is a GB-ring (by (2.5.2)) and $A[X]_N$ is a quotient ring of $A_{N \cap A}[X]$, it suffices, by (2.5.2), to prove that $R[X]$ is a GB-ring whenever R is a local domain such that $R(X)$ is a GB-ring. The proof that this holds will be by induction on $a =$ altitude R, and it may clearly be assumed that $a > 0$.

Let C be a principal integral extension domain of $R[X]$, and let $P' \subset Q'$ be prime ideals in C such that height $Q'/P' = 1$. Let $P = P' \cap R[X]$, $Q = Q' \cap R[X]$, $p = P \cap R$, and $q = Q \cap R$. Then it suffices to prove that height $Q/P = 1$ (2.5.4).

For this, if $q \neq M$ (the maximal ideal in R), then let $S = R - q$, so C_S is an integral extension domain of $R_S[X] = R[X]_S$ and $Q'C_S$ is proper. Therefore, by induction on a (since $R_S(X) \cong R(X)_{qR(X)}$ is a GB-ring (2.5.2)), $R_S[X]$ is a GB-ring, so height $Q/P = 1$ (2.5.3), as desired. Thus it may be assumed that $q = M$. Also, if $Q = MR[X]$, then $R[X]_Q = R(X)$ is a GB-ring, by hypothesis, so height $Q/P = 1$ (2.5.3). Thus it may be assumed that Q is a maximal ideal in $R[X]$ and $q = M$, and then Q' is also maximal.

Assume that $p \neq (0)$. If $P = pR[X]$, then C/P' is integral over $(R/p)[X] \cong R[X]/P$, and Q'/P' is a height one prime ideal in C/P'. Therefore, since $(R/p)(X) \cong R(X)/pR(X)$ is a GB-ring (2.5.1), height $Q/P = 1$, by induction on a. Thus, assuming that $p \neq (0)$, it may be assumed that $pR[X] \subset P$. Therefore, by (2.2), there exists a prime ideal P_1 in C such that $P_1 \subset P'$ and $P_1 \cap R[X] = pR[X]$. Hence, by induction on a, it is readily seen (on considering $(R/p)[X] \subseteq C/P_1$) that height $Q/P = 1$.

Thus it may be assumed that $p = (0)$, so height $P \leqq 1$, and so height $P' \leqq 1$. Therefore $(0) \subseteq P' \subset Q'$ is a *maximal* chain of prime ideals of length $k \leqq 2$ in C, and so there exists a maximal chain of prime ideals of length k in an integral extension domain of $R[X]_Q$. Hence, by [**12**, (2.14.1) $\Longleftrightarrow$ (2.14.5)], there exists a maximal chain of prime ideals of length $k - 1$ in some integral extension domain of R. Therefore, since $a > 0$, it follows that $k = 2$ and there exists a height one maximal ideal in some integral extension domain of R. Thus, since R is a GB-local domain (2.5.5), altitude $R = 1$ (2.4.3), so altitude $R[X]_Q = 2$, hence height $Q/P = 1$.

COROLLARY (2.7). *If A is a Noetherian ring, then $A[X]$ is a GB-ring if and only if $A(X)$ is a GB-ring.*

PROOF. This follows immediately from (2.6) and (2.5.2).

COROLLARY (2.8). *The following statements are equivalent for a local ring (R, M):*

(2.8.1) *$R(X)$ is a GB-ring.*

(2.8.2) *$R[X]$ is a GB-ring.*

(2.8.3) *$R[X]_{(M,X)}$ is a GB-ring.*

(2.8.4) *$R[X]_N$ is a GB-ring for some maximal ideal N in $R[X]$ such that $N \cap R = M$.*

PROOF. (2.8.1) $\Longleftrightarrow$ (2.8.2), by (2.7); (2.8.2) implies (2.8.3) and (2.8.4), by (2.5.2); and, (2.8.3) and (2.8.4) each implies (2.8.1), by (2.5.2), since $R(X)$ is a localization of $R[X]_N$, for each maximal ideal N in $R[X]$ such that $N \cap R = M$.

(I am indebted to the referee for suggesting this proof of (2.8). It is much easier than my original proof.)

(2.8) is a very surprising result to the author. The reason for this will be explained following (4.3) below.

3. $\mathbf{R(X_1, \cdots, X_{a-2})}$ is a GB-ring if and only if R satisfies the s.c.c.

For a local domain R and $a =$ altitude R, the title of this section is a corollary (3.7) to the main theorem in this section (3.5). The proof of

(3.5) seems to be quite deep, so a number of preliminary results will first be given. We begin with the following remark which gives two facts that will quite often be used in what follows.

REMARK (3.1). The following statements hold for a semi-local domain R:

(3.1.1) (cf. [7, Proposition 3.5].) There exists a depth one minimal prime ideal in R^* (2.1) if and only if there exists a height one maximal ideal in R'. (This was proved only for a local domain in [7], but the semi-local domain case follows immediately from the local domain case.)

(3.1.2) [**6**, (33.11)]. If there exists a height one maximal ideal in R', then, for all nonzero $x \in J$, the Jacobson radical of R, xR has some maximal ideal as a prime divisor.

The condition in (3.1.2) that some maximal ideal in R is a prime divisor of xR will often, in the future, be written $xR : J \neq xR$.

The following result, which is of some interest in itself, will help to simplify the proof of the main theorem in this section.

PROPOSITION (3.2). *Let R be a semi-local domain which has infinitely many (height one) prime ideals p such that p has a principal primary ideal and $p \subset J$, the Jacobson radical of R. Assume that each maximal ideal in R has height $\geqq 3$ and that R is a GB-ring. Then, for all minimal prime ideals z in R^*, depth $z \geqq 3$.*

PROOF. Fix a minimal prime ideal z in R^*. Then clearly depth $z > 0$ and, by (3.1.2), and the existence of a principal primary ideal contained in J, depth $z > 1$. Now, if πR is a principal primary ideal contained in J and p is the prime divisor of πR, then $L = R/p$ is a semi-local domain such that each maximal ideal has height $\geqq 2$ (by [**6**, (9.7)] and since maximal ideals in R have height $\geqq 3$) and, by (2.5.1) and hypothesis, L is a GB-ring. Therefore there does not exist a height one maximal ideal in L' (2.4.3), so there are no depth one minimal prime ideals in L^* (3.1.1).

Now suppose that depth $z = 2$ and let p^* be a minimal prime divisor of $(z, \pi)R^*$. ($(z, \pi)R^* \neq R^*$, since $\pi \in J$.) Then, by the Principal Ideal Theorem, height $p^*/z = 1$, hence depth $p^* =$ depth $p^*/z = 1$. Now there are infinitely many choices for π with a different prime divisor, so there are infinitely many such p^*, and at most finitely many of the p^* are such that height $p^* >$ height $z + 1 = 1$ [**5**, Theorem 1]. Therefore there exists $\pi \in J$ such that height $p^* = 1$, where p^* is a minimal prime divisor of $(z, \pi)R^*$. Thus p^* is a depth one minimal prime divisor of πR^*, so there exists a depth one minimal prime ideal in $L^* = (R/p)^*$, where πR is p-primary. But this contradicts the observation in the first paragraph of this proof. Therefore depth $z > 2$.

The following corollary of (3.2) will be strengthened in (3.5) (the case $k = 3$ and $g = 1$), by showing that condition (a) is not actually needed. However, this version of the result is of interest, since it will be used to help shorten the proof of (3.5).

COROLLARY (3.3). *Let R be a semi-local domain such that height $M \geqq 3$, for all maximal ideals M in R. Assume that*: (a) *there exists a nonzero $x \in J$, the Jacobson radical of R, such that $xR : J = xR$; and,* (b) *$R(X)$ is a GB-ring. Then, for all minimal prime ideals z in R^*, depth $z \geqq 3$.*

PROOF. By (a), there exist infinitely many R-sequences a_i, b_i contained in J, and then $p_i = (a_i X - b_i)R(X)$ is a principal prime ideal [**2**, Ex. 3, p. 102] which is contained in the Jacobson radical of $R(X)$. Therefore, since (b) holds, (3.2) applied to $R(X)$ implies that every minimal prime ideal w in $R(X)^*$ is such that depth $w \geqq 3$. Now $R^*(X)$ is a dense subspace of $R(X)^*$ [**8**, Lemma 3.2], hence every minimal prime ideal in $R^*(X)$ has depth $\geqq 3$, and so every minimal prime ideal in R^* has depth $\geqq 3$.

Another corollary of (3.2) will be given in (6.1).

The following definition will, to some extent, simplify the notation in the next few results.

DEFINITION (3.4). For a ring A, let $A_{[n]} = A[X_1, \cdots, X_n]$ and $A_{(n)} = A(X_1, \cdots, X_n)$. Also, let $A_{[0]} = A = A_{(0)}$.

THEOREM (3.5). *Let $(R; M_1, \cdots, M_h)$ be a semi-local domain, fix $k \geqq 2$, and assume that height $M_i \geqq k$, for all $i = 1, \cdots, h$. Also, assume that $R_{(g)}$* (3.4) *is a GB-ring, for some g $(0 \leqq g \leqq k - 2)$. Then the following statements hold*:

(3.5.1) *For each maximal ideal M' in R', height $M' \geqq g + 2$.*

(3.5.2) *For each minimal prime ideal z in R^*, depth $z \geqq g + 2$.*

PROOF. Since each $(R_{M_i})'$ is a quotient ring of R', since $(R_{M_i})^*$ is a direct summand of R^*, and since $(R_{M_i})_{(g)}$ is a GB-ring, if $R_{(g)}$ is (2.5.2), it suffices to prove the theorem in the case that (R, M) is a local domain. For this, the proof that both statements hold will be by induction on $g \geqq 0$. Fix a maximal ideal M' in R'. Then, since R is a GB-ring (2.5.5), height $M' > 1$ (by hypothesis and (2.4.3)). Therefore, if $g = 0$, *then* (3.5.1) holds, and so (3.5.2) holds, by (3.1.1). Thus assume that $g > 0$ (so $k > 2$) and that the conclusions hold for $g - 1$.

Since there are no height one maximal ideals in R', let $b, c \in M$ be such that height $(b, c)R' = 2$. Then $Q = MR[c/b]$ is a depth one prime ideal such that height $Q = \text{height } M - 1 \geqq k - 1$ [**8**, Lemma 4.3]. Likewise, with $x \in M'$ and not in any other maximal ideal in R', and with $N = M' \cap R[x]$, $Q' = NR[x][c/b]$ is a depth one prime ideal such that height $Q' = \text{height } N - 1 = \text{height } M' - 1$ (by [**8**, Lemmas 4.2 and 4.3], since height $(b, c)R[x] = 2$(2.4.3)). Let $S = R[c/b] - Q$, and let $R(c/b) = R[c/b]_S$. Then it is readily seen that $R(c/b) \cong R(X)/K$, for some height one prime ideal K in $R(X)$. Therefore $R(c/b)_{(g-1)} \cong R_{(g)}/KR_{(g)}$ is a GB-ring (2.5.1), and so, by induction, the conclusions hold for $R(c/b)$. Thus, since $R[x][c/b]_S \subseteq R(c/b)'$, height $Q' \geqq (g - 1) + 2$, and so height $M' \geqq g + 2$, hence (3.5.1) holds.

Now fix a minimal prime ideal z in R^*, let $w = zR^*[1/b] \cap R^*[c/b]$, let $S^* = R^*[c/b] - M^*R^*[c/b]$, where $M^* = MR^*$, and let $R^*(c/b) = R^*[c/b]_{S^*}$. Then $R(c/b)$ is a dense subspace of $R^*(c/b)$ (by [**8**, Lemma 3.2]), so if $w \subseteq M^*R^*[c/b]$, then depth $wR^*(c/b) \geqq (g - 1) + 2$, by induction. Therefore depth $w \geqq g + 2$ (since $M^*R^*[c/b]$ is a depth one prime ideal (by [**8**, Lemma 4.3])), so depth $z \geqq g + 2$ (since $R^*/z \subseteq R^*[c/b]/w \subseteq R^*[1/b]/zR^*[1/b] \cong (R^*/z)[1/(b + z)]$). Thus it remains to show that b, c can be chosen such that $b, c \in M$, height $(b, c)R' = 2$, and $zR^*[1/b] \cap R^*[c/b] \subseteq M^*R^*[c/b]$.

For this, if depth $z \geqq 3$, then let $b, c \in M$ be such that height $(b, c)R' = 2$ and height $(b + z, c + z)(R^*/z) = 2$. (This is possible, since, for each nonzero $b \in M$, $(z, b)R^*$ and bR' have only finitely many minimal prime divisors and no such minimal prime divisor contracts in R to M.) Then $zR^*[1/b] \cap R^*[c/b] \subset M^*R^*[c/b]$ (by [**8**, Remark 4.4(i)]). Thus it remains to show that depth $z \geqq 3$.

For this, by the first paragraph of this proof, depth $z > 1$. To see that depth $z > 2$, let $R^{(w)} = \cap\{R_p;\ p \in \operatorname{Spec} R \text{ and } p \neq M\}$. Then $R^{(w)} \subseteq R' = \cap\{R'_{p'};\ p' \in \operatorname{Spec} R' \text{ and height } p' = 1\}$, by (2.4.3) and [**8**, Corollary 5.7(2)], so $R^{(w)}$ is Noetherian [**1**, Corollary 1.4] and every maximal ideal in $R^{(w)}$ has height $\geqq g + 2 \geqq 3$, by (3.5.1). Also, $R^{(w)}(X)$ is a GB-ring (2.4.1) and, with J the Jacobson radical of $R^{(w)}$ and $0 \neq x \in J$, $xR^{(w)} : J = xR^{(w)}$, by [**8**, Lemma 5.6(2)]. Therefore every minimal prime ideal in $R^{(w)*}$ has depth $\geqq 3$ (3.3). Hence there does not exist an integral extension domain of $R^{(w)}$ which has a maximal chain of prime ideals of length < 3 [**12**, (2.14.1) $\Longleftrightarrow$ (2.14.2)], and so this also holds for R. Therefore every minimal prime ideal in R^* has depth $\geqq 3$ [**12**, (2.14.1) $\Longleftrightarrow$ (2.14.2)], hence depth $z \geqq 3$.

(3.5) has some important corollaries which are concerned with certain chain conditions on a ring. In order to state these corollaries, we recall the definitions of these chain conditions at this point.

DEFINITION (3.6). Let A be a ring.

(3.6.1) A satisfies the *first chain condition for prime ideals* (*f.c.c.*) in case each maximal chain of prime ideals in A has length equal to the altitude of A.

(3.6.2) A is *catenary* in case, for each pair of prime ideals $p \subset q$ in A, $(A/p)_{q/p}$ satisfies the f.c.c.

(3.6.3) A satisfies the *second chain condition for prime ideals* (*s.c.c.*) in case, for each minimal prime ideal z in A, every integral extension domain of A/z satisfies the f.c.c. and depth z = altitude A.

(3.6.4) A satisfies the *chain condition for prime ideals* (*c.c.*) in case, for each pair of prime ideals $p \subset q$ in A, $(A/p)_{q/p}$ satisfies the s.c.c.

Rings which satisfy one or more of the above conditions have been deeply investigated, and numerous properties of such rings are known. A summary of most of the results concerning such rings which are needed in what follows is given in [9, Remark 2.22–2.25].

With these definitions, five corollaries of (3.5) will now be given.

If R is a semi-local domain such that a = altitude $R \leqq 1$, then R is a GB-ring and satisfies the s.c.c., so attention is limited to $a \geqq 2$ in (3.7).

COROLLARY (3.7). *Let R be a semi-local domain, let altitude $R = a \geqq 2$, and assume that height $M = a$, for all maximal ideals M in R. Then R satisfies the s.c.c. if and only if $R_{(a-2)}$ is a GB-ring.*

PROOF. If $R_{(a-2)}$ is a GB-ring, then R is quasi-unmixed (3.5.2), so R satisfies the s.c.c [7, Theorem 1].

Conversely, if R satisfies the s.c.c, then $R_{(a-2)}$ satisfies the s.c.c. [7, Corollary 3.7], so $R_{(a-2)}$ is a GB-ring (2.5.6).

COROLLARY (3.8). *Let R be as in* (3.7). *If $R_{(a-2)}$ is a GB-ring, then $R_{[n]}$*(3.4) *is a GB-ring, for all $n \geqq 0$.*

PROOF. If $R_{(a-2)}$ is a GB-ring, then R satisfies the s.c.c. (3.7), so, for all $n \geqq 0$, $R_{[n]}$ satisfies the c.c. [7, Corollary 3.7]. Therefore $(R_{[n]})_Q$ satisfies the s.c.c., for all prime ideals Q in $R_{[n]}$, so each $(R_{[n]})_Q$ is a GB-ring (2.5.6), hence $R_{[n]}$ is a GB-ring (2.5.2).

Of course, the converse of (3.8) holds, by (2.5.2).

COROLLARY (3.9). *Let A be a Noetherian ring such that $A_{(g)}$ is a GB-ring, for some $g \geqq 0$. Then the following statements hold:*

(3.9.1) *A_P satisfies the c.c., for all prime ideals P in A such that height $P \leqq g + 2$.*

(3.9.2) *A/Q satisfies the c.c., for all prime ideals Q in A such that depth $Q \leqq g + 2$.*

Proof. (3.9.1) Let P be a prime ideal in A such that $h =$ height $P \leqq g + 2$. If $h < 2$, then clearly A_P satisfies the c.c., so assume that $h \geqq 2$. Then to show that A_P satisfies the c.c., it suffices to show that A_P/z satisfies the s.c.c., for all minimal prime ideals z in A_P[9, Remark 2.23(iv)]. For this, fix such z and let $d =$ depth z. Then $R = (A_P)_{(d-2)}$ is a GB-ring by (2.5.2), since $A_{(d-2)}$ is (2.5.5.) and $R \cong A_{(d-2)PA_{(d-2)}}$. Therefore R/zR is a GB-ring (2.5.1) and $R/zR \cong (A_P/z)_{(d-2)}$, hence A_P/z satisfies the s.c.c. (3.7).

(3.9.2) Let Q be a prime ideal in A such that $d =$ depth $Q \leqq g + 2$. If $d < 2$, then clearly A/Q satisfies the c.c., so assume that $d \geqq 2$. Then to show that A/Q satisfies the c.c., it suffices to show that $(A/Q)_M$ satisfies the s.c.c., for all maximal ideals M in A/Q [9, Remark 2.23(iv)], and the proof of this is quite similar to the proof of (3.9.1).

If A is a Noetherian ring such that altitude $A = a = 1$, then A satisfies the c.c. and $A_{[n]}$ is a GB-ring, for all $n \geqq 0$. Therefore we restrict attention to the case $a > 1$ in the next corollary, which generalizes (3.7) and (3.8).

Corollary (3.10). *Let A be a Noetherian ring and let $2 \leqq a =$ altitude $A < \infty$. If $A_{(a-2)}$ is a GB-ring, then A satisfies the c.c. and $A_{[n]}$ is a GB-ring, for all $n \geqq 0$.*

Proof. If $A_{(a-2)}$ is a GB-ring, then A_M satisfies the c.c., for all maximal ideals M in A, by (3.9.1), so A satisfies the c.c. [9, Remark 2.23(iii)]. Therefore, for all $n \geqq 0$, $A_{[n]}$ satisfies the c.c. [8, Theorem 2.6]. Thus, for each minimal prime ideal z in A, $D = A_{[n]}/zA_{[n]}$ satisfies the c.c. [9, Remark 2.23(iii)], so D is a GB-ring (as in the proof of (3.8)), hence $A_{[n]}$ is a GB-ring (2.5.1).

As with (3.8), the converse of (3.10) holds by (2.5.2).

We close this section by giving a sufficient (and necessary) condition for all localities over a Noetherian domain to be GB-rings.

Corollary (3.11). *Let A be as in* (3.10), *let I be an ideal in A, let B be a finitely generated ring over A/I, let C be an integral extension ring of B, and let S be a m.c. set in C. If $A_{(a-2)}$ is a GB-ring, then C_S is a GB-ring.*

Proof. B is a homomorphic image of $A_{[n]}$, for some $n \geqq 0$, and $A_{[n]}$ is a GB-ring (3.10), so B is (2.5.1). Therefore, by (2.4.1) and (2.5.2), C_S is a GB-ring.

4. **Two (equivalent) problems.** There are a number of open problems concerning GB-rings, but among the two most important are: (a) if R is a GB-local domain, is $R(X)$? and, (b) if R is a GB-local domain, is R

catenary? In this brief section we show that, in fact, these two problems are equivalent.

PROPOSITION (4.1). *The following statements are equivalent:*

(4.1.1) *Whenever R is a GB-local domain, $R(X)$ is a GB-ring.*

(4.1.2) *Whenever R is a GB-local domain, R is catenary.*

PROOF. Assume that (4.1.1) holds and let R be a GB-local domain. Then, with a = altitude R, $R_{(a-2)}$ is a GB-ring (by hypothesis), so R satisfies the s.c.c. (3.7). Therefore R is catenary, so (4.1.2) holds.

Conversely, assume that (4.1.2) holds and let R be a GB-local domain. Then R is catenary and a GB-ring, so R satisfies the s.c.c (2.5.6), hence $R(X)$ satisfies the s.c.c [7, Corollary 3.7]. Therefore $R(X)$ is a GB-ring, (2.5.6), so (4.1.1) holds.

COROLLARY (4.2). *If the equivalent statements in* (4.1) *hold, then the following statements are equivalent for a local domain* (R, M)*:*

(4.2.1) *R is a GB-ring.*

(4.2.2) *$R(X)$ is a GB-ring.*

(4.2.3) *R satisfies the s.c.c.*

PROOF. (4.2.1) $\Longleftrightarrow$ (4.2.2), by hypothesis and (2.5.5), and (4.2.1) $\Longleftrightarrow$ (4.2.3), by hypothesis and (2.5.6).

COROLLARY (4.3). *If the equivalent statements in* (4.1) *hold, then the following statements are equivalent for a Noetherian ring A:*

(4.3.1) *A is a GB-ring.*

(4.3.2) *$A(X)$ is a GB-ring.*

(4.3.3) *A satisfies the c.c.*

PROOF. By (2.5.1) (resp., [9, Remark 2.23(iii)]), a ring B is a GB-ring (satisfies the c.c.) if and only if B/z is a GB-ring (resp., satisfies the c.c.), for all minimal prime ideals z in B. Therefore it suffices to prove this corollary for Noetherian domains A.

Assume that (4.3.1) holds, let N be a maximal ideal in $A(X)$, and let $M = N \cap A$. Then A_M is a GB-ring, so $A(X)_N = A_M(X)$ is a GB-ring (4.2). Therefore $A(X)$ is a GB-ring (2.5.2).

Assume that (4.3.2) holds and let M be a maximal ideal in A. Then $A_M(X) = A(X)_{MA(X)}$ is a GB-ring, so A_M is a GB-ring (2.5.5), hence A_M satisfies the s.c.c. (4.2). Therefore A satisfies the c.c. [9, Remark 2.23(iii)].

Finally, assume that (4.3.3) holds and let M be a maximal ideal in A. Then A_M satisfies the s.c.c., so A_M is a GB-ring (2.5.6), hence A is a GB-ring (2.5.2).

As mentioned at the end of Section 2, (2.8) is a very surprising result to the author. The reason for this is that, for most other chain conditions, a local domain (R, M) has the condition if and only if $R(X)$ has it. (For example, f.c.c. [9, Theorem 4.11], s.c.c. [**12**, (2.15)], H_i[**13**, (2.7)], and, there exists a maximal chain of prime ideals of length n in the ring [**12**, (2.15)].) And usually $R[X]$ and $R[X]_{(M,X)}$ do not inherit the condition from R. But, as partly shown by the results in this section, for GB-rings the difficult part is to show that $R(X)$ is a GB-ring whenever R is, and then $R[X]$ and $R[X]_{(M,X)}$ do inherit the GB condition (2.8).

5. **$R(X)$ and $R(x)$.** As noted in Section 4, it is an open problem if $R(X)$ is a GB-ring whenever R is. The few results in this section came about from an effort to prove that this does hold. We begin with the following remark.

Remark (5.1). The following statements hold for a GB-Noetherian domain A:

(5.1.1) $A^{(1)} = \cap\{A_p;\ p \in \operatorname{Spec} A$ *and height* $p = 1\} \subseteq A'$, *so* $A^{(1)}$ *is a GB-ring.*

(5.1.2) *If* (A, M) *is local, then* $A^{(w)} = \cap\{A_P;\ P \in \operatorname{Spec} A$ *and* $P \neq M\}$ *is a GB-semi-local domain.*

Proof. (5.1.1) Since height one prime ideals in A' contract in A to height one prime ideals (2.4.3), $A^{(1)} \subseteq A'^{(1)} = A'$, so $A^{(1)}$ is a GB-ring (2.4.1).

(5.1.2) $A^{(w)} \subseteq A^{(1)} \subseteq A'$ (5.1.1), so $A^{(w)}$ is a quasi-semi-local GB-ring (2.4.1). Also, $A^{(w)}$ is Noetherian [1, Corollary 1.4], so $A^{(w)}$ is a GB-semi-local domain.

The next two results give characterizations of GB-rings which are sometimes useful. In particular, (5.3) will be used in the proof of the main result in this section.

Proposition (5.2). *A ring A is a GB-ring if and only if, for all prime ideals* $p \subset q$ *in A such that height* $q/p > 1$, *there does not exist a height one maximal ideal in* L', *where* $L = (A/p)_{q/p}$.

Proof. If A is a GB-ring, then each such L is a GB-ring, by (2.5.1) and (2.5.2), so there does not exist a height one maximal ideal in L' (2.4.3).

Conversely, assume that the condition holds, let B be an integral extension ring of A, and let $P \subset Q$ be prime ideals in B such that height $Q/P = 1$. Let $p = P \cap A$ and $q = Q \cap A$, so $C = (B/P)_{((A/p)-(q/p))}$ is integral over $L = (A/p)_{q/p}$ and $(Q/P)C$ is a height one maximal ideal in C. Therefore there exists a height one maximal ideal in C', so there exists a height one maximal ideal in L', by [**6**, (10.14)]. Thus height $q/p = 1$ (by hypothesis), and so A is a GB-ring (2.5.3).

PROPOSITION (5.3). *The following statements are equivalent for a Noetherian domain A:*

(5.3.1) *A is a GB-ring.*

(5.3.2) *A/p is a GB-ring, for all height one prime ideals p in A, and $A^{(1)} \subseteq A'$.*

PROOF. $A^{(1)} \subseteq A'$ if and only if height $q \cap A = 1$, for all height one prime ideals q in A' [8, Corollary 5.7(1)], and then height $n \cap A = 1$, for all height one prime ideals n in each integral extension domain of A[8, Corollary 5.9(3)]. From this it is readily seen, using (2.5.3), that (5.3.2) $\Rightarrow$ (5.3.1), and (5.3.1) $\Rightarrow$ (5.3.2), by (2.5.1) and (5.1.1).

There is a somewhat analogous, but less easily stated, characterization of a GB-Noetherian domain A using localizations. Namely, A is a GB-ring if and only if A_P is a GB-ring, for all depth one prime ideals P in A, and, for all principal integral extension domains B of A and for all adjacent prime ideals $Q \subset N$ in B such that N is maximal, height $(N \cap A)/(Q \cap A) = 1$. The proof follows easily from (2.5.2), (2.5.3), and (2.5.4).

The following proposition is the main result in this section. The characterization in (5.4) of when $R(X)$ is a GB-ring is not easy to apply, but, at least in comparison to (2.5.1), it reduces the number of prime ideals P that must be shown to be such that $R(X)/P$ is a GB-ring. Since local domains of altitude < 2 are GB-rings, we restrict attention to the case altitude $R \geqq 2$ in the proposition.

PROPOSITION (5.4). *The following statements are equivalent for a GB-local domain (R, M) such that altitude $R = a \geqq 2$:*

(5.4.1) *$R(X)$ is a GB-ring.*

(5.4.2) *For all elements x in an algebraic closure F^* of the quotient field of R such that* depth $MR[x] = 1$, $R(x) = R[x]_{MR[x]}$ *is a GB-ring.*

PROOF. Assume that (5.4.1) holds and let $x \in F^*$ be such that depth $MR[x] = 1$. Then $MR[x]$ is a prime ideal and the residue class of x modulo $MR[x]$ is transcendental over R/M (this is clear by the structure of $R[x]/MR[x]$), so $R[x]_{MR[x]} \cong R(X)/P$, for some prime ideal P. Therefore $R[x]_{MR[x]}$ is a GB-ring, by hypothesis and (2.5.1), hence (5.4.2) holds.

Conversely, assume that (5.4.2) holds. Then it will be shown that (5.4.1) holds by using (5.3) and induction on a. If $a = 2$, then (5.4.2) $\Rightarrow$ (5.4.1), for R satisfies the s.c.c. (2.4.2) (so $R(X)$ is a GB-ring (3.8) and (2.5.2)). Therefore assume that $a > 2$ and (5.4.2) $\Rightarrow$ (5.4.1) for local domains of altitude $< a$.

Since R is a GB-ring, height one prime ideals in R' lie over height one prime ideals in R (2.4.3), so it is readily seen that height one prime ideals in $R(X)' = R'(X)$ lie over height one prime ideals in $R(X)$, hence $R(X)^{(1)} \subseteq R(X)'$. Therefore, by (5.3), it remains to show that if P is a height one prime ideal in $R(X)$, then $R(X)/P$ is a GB-ring.

For this, if $P \cap R = (0)$, then $R(X)/P = R[x]_{MR[x]}$ with $x = X + P$ algebraic over R. Since depth $MR[X] = 1$ and $P \cap R[X] \subseteq MR[X]$, depth $MR[x] = 1$, so $R(X)/P$ is a GB-ring, by hypothesis. Therefore assume that $p = P \cap R \neq (0)$, so height $p = 1$ and $P = pR(X)$. Now $(R/p)(X)^{(1)} \subseteq (R/p)(X)'$, as above (since R/p is a GB-ring). Also, (5.4.2) holds for R/p, as will now be shown. Let ^{o} denote residue class modulo p and let y be an element in an algebraic closure of the quotient field of R^o such that depth $M^oR^o[y] = 1$. Then, since $R^o[y]$ is a homomorphic image of $R[X]$, it readily follows that there exist $x \in F^*$ and a prime ideal p' in $R[x]$ such that $p' \cap R = p$ and $R[x]/p' = R^o[y]$. Therefore, since depth $M^oR^o[y] = 1$ and $M^oR^o[y] = MR[x]/p'$, depth $MR[x] = 1$. Thus, by hypothesis, $R(x) = R[x]_{MR[x]}$ is a GB-ring, so $R^o[y]_{M^oR^o[y]} \cong R(x)/p'R(x)$ is a GB-ring. Therefore $R(X)/P \cong R^o(X)$ is a GB-ring, by induction on a, so $R(X)$ is a GB-ring (5.3).

(5.4) would be a somewhat nicer result if (5.4.2) could be replaced by: for all analytically independent elements b, c in M, $R(c/b) = R[c/b]_{MR[c/b]}$ is a GB-ring. (If b, c are analytically independent in R, then $MR[c/b]$ is a depth one prime ideal [8, Lemma 4.3].) Like (5.4.2), this condition is inherited by factor domains of R, but I have been unable to show it is strong enough to imply that $R(X)$ is a GB-ring. However, it is quite straightforward to show that (5.4.2) is equivalent to: for each principal local integral extension domain (L, N) of R and for each pair of analytically independent elements b, c in N, $L[c/b]_{NL[c/b]}$ is a GB-ring.

6. **GB-rings of altitude three.** If A is a GB-ring such that altitude $A < 3$, then A/z satisfies the c.c., for all minimal prime ideals z in A, by (2.5.1) and (2.4.2), so A satisfies the c.c. [9, Remark 2.23(iii)]. So, in this section we consider Noetherian GB-rings of altitude three. It seems that it should be possible to show that all such rings satisfy the c.c., but our results fall considerably short of this goal. Even so, some things worthy of note can be said about such rings, as will now be shown.

It is an open problem if every local UFD of altitude three satisfies the s.c.c. However, because of (3.2), we can show that they do if and only if they are GB-rings.

PROPOSITION (6.1). *Let R be a local* UFD *such that altitude $R = 3$. Then R is a GB-ring if and only if R satisfies the s.c.c.*

PROOF. If R is a GB-ring, then R is quasi-unmixed (3.2), so R satisfies the s.c.c. [7, Theorem 3.1], and the converse is given by (2.5.6).

Of course, if R is as in (6.1), then each height one prime ideal p in R is such that depth $p = 2$ (since p is principal), so every prime ideal P in R is such that height P + depth $P = 3$, hence R is catenary [**10**, Theorem 2.2.], and so (6.1) also follows from (2.5.6).

The next result is concerned with a more general type of ring than was (6.1).

PROPOSITION (6.2). *Let A be a Noetherian ring such that altitude $A = 3$. Then A satisfies the c.c. if and only if $A(X)$ is a GB-ring.*

PROOF. If A satisfies the c.c., then, as in the proof of (3.10), $A_{[n]}$ is a GB-ring, for all $n \geqq 0$, so $A(X)$ is a GB-ring (2.5.2).

The converse is clear by (3.10).

It is an open problem if a GB-local domain of altitude three satisfies the s.c.c. However, this holds for such rings of the form $R(X)$, as is shown by the following result.

PROPOSITION (6.3). *Let R be a local domain such that altitude $R = 3$. Then the following statements are equivalent*:

(6.3.1) *$R(X)$ is a GB-ring.*

(6.3.2) *$R(X)$ satisfies the s.c.c.*

(6.3.3) *R satisfies the s.c.c.*

(6.3.4) *R is catenary and a GB-ring.*

PROOF. (6.3.4) $\Longleftrightarrow$ (6.3.3) and (6.3.2) $\Rightarrow$ (6.3.1) by (2.5.6), (6.3.1) $\Longleftrightarrow$ (6.3.3) by (3.7), and (6.3.3) $\Rightarrow$ (6.3.2), by [7, Corollary 3.7].

As is partly indicated in the introduction to this section, the author believes that all GB-local domains satisfy the s.c.c, but is unable to prove this even for the altitude three case. The next result considers what can be said if a GB-local domain of altitude three is not catenary (hence does not satisfy the s.c.c.).

PROPOSITION (6.4). *Let (R, M) be a local domain such that altitude $R = 3$ and R is a GB-ring. Assume that R is not catenary. Then there exists a depth two minimal prime ideal in R^* and a finite (possibly empty) set S of height one prime ideals in R such that if $0 \neq x \in M - \cup\{p\,; p \in S\}$, then xR has a depth one prime divisor.*

PROOF. Since R is not catenary, there exists a height one depth one prime ideal in R, so there exists a depth two minimal prime ideal z in R^* [**12**, (2.14)]. Let $0 \neq x \in M$, and let q^* be a minimal prime divisor of $(z, x)R^*$. Then, by the Principal Ideal Theorem, height $q^*/z = 1$, so depth $q^* = 1$. Also, $q^*R^*_{q^*}$ is a prime divisor of $xR^*_{q^*}$ [**14**, Lemma 1, p. 394], hence q^* is a prime divisor of xR^*, and so $q = q^* \cap R$ is a prime divisor of xR and q^* is a prime divisor of qR^* [6, (18.11)]. Let $T^* = \{q^* ; q^*$ is a minimal prime divisor of $(z, x)R^*$, for some $0 \neq x \in M\}$ and let $S^* = \{q^* \in T^*$; height $q^* = 2\}$, so S^* is a finite set [5, Theorem 1], and T^* is an infinite set.

Fix $q^* \in T^*$ and let $q = q^* \cap R$. If $q^* \notin S^*$, then height $q^* = 1$, so q^* is a minimal prime divisor of qR^*. Thus, R/q is a GB-ring (2.5.1) such that $(R/q)^*$ has a depth one minimal prime ideal (isomorphic to q^*/qR^*), so depth $q = 1$, by (3.1.1) and (2.4.3). Therefore, for each nonzero $x \in q$, xR has q as a depth one prime divisor. On the other hand, if $q^* \in S^*$, then there are two possibilities: depth $q = 1$, so each $x \in q$ has q as a depth one prime divisor (possibly height $q = 2$); or, depth $q = 2$, so height $q = 1$ and q^* is an imbedded prime divisor of qR^*. Let $S = \{q^* \cap R;\ q^* \in S^*$ and depth $q^* \cap R = 2\}$. (Possibly S is empty.) Then, for each nonzero $x \in M - \cup\{p\,; p \in S\}$, xR has a depth one prime divisor.

Of course, a much nicer conclusion for (6.4) would be: for each nonzero $x \in M$, xR has a depth one prime divisor. This would hold (by the proof of (6.4)) if it were known that GB-local domains of altitude two are unmixed (instead of being merely quasi-unmixed (see (2.5.6) and [7, Theorem 3.1])). However, in [1, Proposition 3.3] there is given a local domain of altitude two which is not unmixed, but which is quasi-unmixed (by [6, (34.2)], since its integral closure is a regular local ring (so satisfies the s.c.c)). Therefore there exist local domains of altitude two which are GB-rings and are not unmixed.

It should also be noted that if R is as in (6.4), and if there does not exist a height two maximal ideal in R', then the Chain Conjecture (that is, the integral closure of a local domain satisfies the c.c.) fails. For, then every maximal ideal in R' has height three (2.4.3), so the Chain Conjecture would say that R' satisfies the s.c.c., hence R satisfies the s.c.c [6, (34.2)] and so must be catenary: contradiction.

7. **Three questions.** This paper will be closed with the following questions and a brief remark on one of them.

(7.1) Questions. (7.1.1) If R is a GB-local domain, is $R(X)$?

(7.1.2) If $R(X)$ is a GB-local domain, is $R(X_1, \cdots, X_n)$, for all $n \geqq 1$?

(7.1.3) If (R, M) and (S, N) are local domains such that S is integral over R, $N \cap R = M$, and S is a GB-ring, is R a GB-ring?

Concerning (7.1.3), if an integral extension domain of a local domain R is a GB-ring, R need not be a GB-ring. (For example, in [6, Example 2, pp. 203–205], the integral closure of R is a regular domain, (so satisfies the c.c. and is a GB-ring), but R is not a GB-ring.) This is why S was limited to having only one maximal ideal in (7.1.3). [Added in proof: M. Brodman has shown in *A Particular Class of Rings* (12 page preprint) that all three equations in §7 have a negative answer.]

References

1. D. Ferrand and M. Raynaud, *Fibres formelles d'un anneau local Noethérian,* Ann. Sci. Ecole Norm. Sup. **3** (1970), 295–311.

2. I. Kaplansky, *Commutative Rings,* Allyn and Bacon, Boston, 1970.

3. ———, *Adjacent prime ideals,* J. Algebra **20** (1972), 94–97.

4. W. Krull, *Zum Dimensionsbergriff der Idealtheorie* (*Beiträge zur Arithmetik kommutativer Integritätsbereiche,* III), Math. Z. **42** (1937), 745–766.

5. S. McAdam, *Saturated chains in Noetherian rings,* Indiana Univ. Math. J. **23** (1974), 719–728.

6. M. Nagata, *Local Rings,* Interscience Tracts 13, Interscience, N.Y., 1962.

7. L. J. Ratliff, Jr., *On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals* (I), Amer. J. Math. **91** (1969), 508–528.

8. ———, *On quasi-unmixed local domains the altitude formula, and the chain condition for prime ideals* (II), Amer. J. Math. **92** (1970), 99–144.

9. ———, *Characterizations of catenary rings,* Amer. J. Math. **93** (1971), 1070–1108.

10. ———, *Catenary rings and the altitude formula,* Amer. J. Math. **94** (1972), 458–466.

11. ———, *Going-between rings and contractions of saturated chains of prime ideals,* Rocky Mountain J. Math., **7** (1977), 777–787.

12. L. J. Ratliff, Jr. and S. McAdam, *Maximal chains of prime ideals in integral extension domains,* Trans. Amer. Math. Soc., **224** (1976), 103–116.

13. L. J. Ratliff, Jr. and M. E. Pettit, Jr., *Characterizations of H_i-local rings and of C_i-local rings,* Amer. J. Math., forthcoming.

14. O. Zariski and P. Samuel, *Commutative Algebra,* Vol. II, Van Nostrand, New York, 1960.

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