ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 3, Summer 1979

ON SPACES OF MEROMORPHIC FUNCTIONS

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1. **Introduction.** The set $H(D)$ of all analytic functions on a plane domain D provides a beautiful setting for topics in function theory, functional analysis, harmonic analysis, and many other areas. Endowed with the topology ρ of uniform convergence on compact subsets of D, the set $H(D)$ becomes a locally convex, topological vector space (TVS), even a topological algebra, with the usual operations.

The set $M(D)$ of all meromorphic functions on D is also a vector space, even an algebra, with the usual operations. One customarily associates with $M(D)$ the topology σ of uniform convergence on compact subsets of D in the spherical metric. It is our purpose in this discussion to consider $M(D)$ with various topologies from the point of view of topological vector spaces.

A characterization of the topological dual space $H'(D)$ of $H(D)$, as a TVS, is well known:

PROPOSITION 1. *The dual space $H'(D)$ of $H(D)$ is isomorphic to the space of all analytic functions on $\overline{\mathbf{C}} - D$ ($\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$), that vanish at ∞. In particular, each $L \in H'(D)$ may be represented in the form*

$$L(f) = \int_C f(z)g(z)\,dz \qquad \text{for all } f \in H(D)$$

for some function g analytic in $\overline{\mathbf{C}} - D$ with $g(\infty) = 0$, and some finite (rectifiable) curve system C in D and in the region of analyticity of g.

REMARK 1. In this form Proposition 1 appears first perhaps in a note of O. Teichmüller [5] (more convenient references are [3, § 27] and [4, ch. 4]). Earlier, R. Caccioppoli [1] and L. Fantappiè [2] had given very similar representations for, respectively, continuous and analytic linear functionals on spaces of analytic functions over closed connected sets.

In the following section we shall observe that $M(D)$ with the topology σ is not a TVS. In fact, there is no topology even comparable to σ that makes $M(D)$ into a locally convex TVS. It is therefore not surprising, as we shall see, that $M(D)$ with the topology σ does not have any nontrivial continuous linear functionals.

In spite of these negative results, we shall exhibit in Section 3 a topology υ, which is not comparable to σ, that makes $M(D)$ into a locally convex TVS, and we shall represent its dual space.

Received by the editors on October 12, 1977.

2. **Negative Results.** Let $s_p(z) = 1/(z - p)$, $s_\infty(z) \equiv 0$, and $S = \{s_p{}^n : p \in \mathbf{C}, n = 1, 2, 3, \cdots\}$. Assume for the moment that $M(D)$ is endowed with the topology σ. We single out three properties that we shall use.

(i) It is a consequence of Runge's theorem that the linear span of S is dense in $M(D)$.

(ii) The mapping $p \longrightarrow s_p$ is continuous on $\overline{\mathbf{C}}$.

(iii) The multiplication operation $(s_p{}^m, s_q{}^n) \longrightarrow s_p{}^m s_q{}^n$ is continuous on S.

PROPOSITION 2. *Suppose τ is a topology on $M(D)$ that is weaker than (or equal to) σ. Then $M(D)$ with the topology τ admits no nontrivial continuous linear functionals.*

PROOF. Let L be a linear functional on $M(D)$ that is continuous in the topology τ. Define $\phi(p) = L(s_p)$. Since τ is weaker than σ, properties (i)–(iii) remain valid, and

$$\lim_{\Delta p \to 0} \frac{\phi(p + \Delta p) - \phi(p)}{\Delta p} = \lim_{\Delta p \to 0} L(s_p s_{p+\Delta p}) = L(s_p{}^2)$$

by (ii) and (iii). Therefore ϕ is an analytic function of p in $\mathbf{C}$. In addition, $\phi(p) \longrightarrow \phi(\infty) = 0$ as $p \longrightarrow \infty$ by property (ii). Consequently, $\phi \equiv 0$ by Liouville's theorem. In a similar fashion

$$0 = \phi^{(n)}(p) = n!L(s_p{}^{n+1}) \qquad \text{for all } n \geqq 1.$$

Now it follows from property (i) that $L \equiv 0$ on $M(D)$.

A TVS is a vector space with a topology for which the addition and scalar multiplication operations are continuous. With the topology σ we shall see that $M(D)$ is not a TVS. Fix $p \in D$. An example which shows that addition fails to be continuous is given by the sequences $\{s_{p+1/n}\}$ and $\{s_{p-1/n}\}$. Both sequences converge to s_p. However, the difference fails to converge to zero since it assumes the value ∞ on each compact neighborhood of p for all n sufficiently large. Similarly, an example which shows that scalar multiplication fails to be continuous is given by the sequence $\{(1/n)s_p\}$. Here the scalars $1/n$ tend to zero as $n \longrightarrow \infty$, but $(1/n)s_p$ does not converge to zero for the same reason as before. Since convergence in a topology stronger than σ would imply convergence in the topology σ, the same example furnishes slightly more:

PROPOSITION 3. *Suppose τ is a topology on $M(D)$ that is stronger than (or equal to) σ. Then $M(D)$ with the topology τ is not a* TVS. *In particular, scalar multiplication is not continuous.*

We summarize our negative results.

PROPOSITION 4. *There is no topology τ that is comparable to σ such that $M(D)$ with the topology τ is a locally convex* TVS.

PROOF. If τ is weaker than σ and $M(D)$ with the topology τ were a locally convex TVS, then by an extension of the Hahn-Banach theorem ([3, § 20]) there would be a wealth of linear functionals, contradicting Proposition 2. If τ is stronger than σ, *then Proposition* 3 applies.

Proposition 4 indicates that one must depart essentially from the familiar topology σ in order to obtain a useful TVS structure for $M(D)$. It leaves open the possibility that there are topologies that are not comparable with σ which make $M(D)$ into a locally convex TVS. We shall provide such an example and represent its dual space.

3. **An Example.** Let $\{K_n\}$ be a canonical exhaustion of D by compact sets whose interiors are connected and whose boundaries are finite systems C_n of rectifiable Jordan curves. Corresponding to $f \in M(D)$ let

$$\sum_{k=1}^{n_j(f)} \frac{a_{jk}(f)}{(z - p_j(f))^k}$$

be the principal parts of f at its poles $p_j(f)$, and define

$$P_n(z; f) = \sum_{p_j(f) \in K_n} \sum_{k=1}^{n_j(f)} \frac{a_{jk}(f)}{(z - p_j(f))^k},$$

$$Q_n(z; f) = \sum_{p_j(f) \in K_n} \sum_{k=1}^{n_j(f)} a_{jk}(f)(z - p_j(f))^k,$$

$$R_n(z; f) = f(z) - P_n(z; f).$$

Define also the seminorms

$$\|f\|_n = \max_{z \in K_n} |Q_n(z; f)| + \max_{z \in K_n} |R_n(z; f)|.$$

Actually, $\|\cdot\|_n$ defines a norm on $M(D)$, but unfortunately the sequence $\{\|f\|_n\}$ is not necessarily increasing as a function of n. It therefore is more natural to use the translation invariant metric d defined on $M(D)$ in the usual way by setting

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n}\|f - g\|_n/(1 + \|f - g\|_n).$$

This metric defines a locally convex topology v on $M(D)$ (see [3, § 18]), which agrees on the subset $H(D)$ with the usual topology ρ. Observe that f_k converges to f in the topology v as $k \to \infty$ if and only if

$$\lim_{k\to\infty} \|f_k - f\|_n = 0 \qquad \text{for each } n = 1, 2, 3, \cdots.$$

Note that $P_n(z; f)$, $Q_n(z; f)$, and $R_n(z; f)$ are linear functions of f. It is then a consequence of the triangle inequality and the fact that norm convergent sequences are bounded that addition and scalar multiplication in $M(D)$ are continuous operations in the topology v. We summarize some pertinent points of the discussion so far in the following proposition.

PROPOSITION 5. *The set of meromorphic functions $M(D)$ with the topology v is a locally convex TVS.*

The topology v has the undesirable property that it depends on the exhaustion. Indeed, if $\{K_n\}$ and $\{K_n'\}$ are two exhaustions of D and $p_0 \in \partial K_n$ for some n, but $p_0 \notin \partial K_n'$ for all n, then the functions s_p converge to s_{p_0} as p approaches p_0 from outside K_n, but diverge relative to the exhaustion $\{K_n\}$.

It is a consequence of the Hahn-Banach theorem that a locally convex TVS has a separating family of continuous linear functionals. For the space $M(D)$ with the topology v the point evaluations $f \to Q_n(z_0; f)$ and $f \to R_n(z_0; f)$ for $z_0 \in K_n$ already provide a separating family of continuous linear funtionals. The following proposition gives a more complete representation for the dual space of $M(D)$ with the topology v.

PROPOSITION 6. *Let L be a continuous linear functional of $M(D)$ with the topology v. Then there exist a positive integer n and a function g defined in $\overline{\mathbf{C}}$ such that*

$$L(f) = \sum_{p_j(f)\in K_n} \sum_{k=1}^{n_j(f)} \frac{-1}{(k-1)!} a_{jk}(f) g^{(k-1)}(p_j(f)) + \frac{1}{2\pi i} \int_{C_m} R_m(z; f) g(z)\, dz$$

for all $f \in M(D)$ and any $m > n$. The function g is analytic on $\overline{\mathbf{C}} - K_n$ and $g(\infty) = 0$. In addition, for $\nu = 1, \cdots, n$ the restriction of g to the interior of $K_\nu - K_{\nu-1}$ (define $K_0 = \phi$) is analytic and its derivatives (denoted by $g^{(k-1)}$) extend continuously to $K_\nu - K_{\nu-1}$.

Conversely, every such representation defines a continuous linear functional.

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PROOF. Let L be a continuous linear functional. Suppose for the purpose of contradiction that there exists a sequence $\{f_m\}$ in $M(D)$ such that f_m is analytic on K_m, $\|f_m\|_m = 1$, and $|L(f_m)| \to \infty$ as $m \to \infty$. Since the sequence $\{f_m\}$ is locally uniformly bounded there exists a subsequence $f_{m_k} \to f_0 \in H(D)$ as $K \to \infty$. Now on the one hand, $|L(f_{m_k})| \to |L(f_0)| < \infty$ as $k \to \infty$ since L is continuous; on the other hand, $|L(f_{m_k})| \to \infty$ by assumption. This contradiction leads to the assertion that there exists an integer n such that L is a bounded linear functional relative to the norm $\| \cdot \|_n$ on the subspace $A_n = \{f \in M(D) : f$ is analytic on $K_n\}$.

Observe that $\| \cdot \|_n$ is just the familiar sup norm on A_n. It follows that L is continuous on A_n relative to the norm $\| \cdot \|_n$.

For fixed $\zeta \in \mathbf{C}$ let $s_\zeta(z) = 1/(z - \zeta)$ and define $g(\zeta) = -L(s_\zeta)$. If $\zeta_0 \notin \bigcup_{\nu=1}^{\infty} C_\nu$, then

$$-\frac{g(\zeta) - g(\zeta_0)}{\zeta - \zeta_0} = L(s_\zeta s_{\zeta_0}) \to L(s_{\zeta_0}^2)$$

as $\zeta \to \zeta_0$ since L is continuous in the topology v. The same limit exists for $\zeta_0 \in \bigcup_{\nu=n+1}^{\infty} C_\nu$ since L is continuous on A_n relative to the norm $\| \cdot \|_n$. Therefore g is analytic on $\mathbf{C} - \bigcup_{\nu=1}^{n} C_\nu$ and clearly vanishes at ∞. One easily verifies in a similar fashion that the derivatives satisfy

$$R_m(z; f) = \frac{-1}{2\pi i} \int_{C_m} R_m(\zeta; f) s_\zeta(z)\, d\zeta$$

for $z \in K_n$. If we consider $R_m(\,\cdot\,; f)$ as belonging to A_n with the norm $\| \cdot \|_n$ and recall that L is continuous on A_n relative to the norm $\| \cdot \|_n$, then we may write

$$\begin{aligned} L(R_m(\,\cdot\,; f)) &= \frac{-1}{2\pi i} \int_{C_m} R_m(\zeta; f) L(s_\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \int_{C_m} R_m(\zeta; f) g(\zeta)\, d\zeta. \end{aligned}$$

The interchange of L and $\int_{C_m}$ is permitted since the integral is the limit in the norm $\| \cdot \|_n$ of functions in A_n.

Since

$$f(z) = \sum_{p_j(f) \in K_m} \sum_{k=1}^{n_j(f)} a_{jk}(f) s_{p_j(f)}^k + R_m(z; f),$$

the desired representation follows by applying L to both sides.

Conversely, it is evident that each such representation defines a continuous linear functional on $M(D)$ with the topology v.

Observe that the representation of Proposition 6 reduces to the representation of Proposition 1 on the subspace $H(D)$ of $M(D)$. However, for $M(D)$ it becomes important that the function g is defined further.

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