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THE CARTAN CRITERION FAILS FOR TRIANGULAR SUBALGEBRAS OF A FACTOR

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It is a familiar fact that if $\mathscr{S}$ is a solvable Lie algebra of endomorphisms acting on a finite-dimensional, complex vector space V then there exists an (ordered) basis for V such that the matrix representation of each element of $\mathscr{S}$ with respect to this basis is upper triangular. Further, Cartan's Criterion states that a Lie sub-algebra, $\mathscr{S}$, of End (V) is solvable if and only if $\operatorname{tr}(A[B, C]) = 0$ for all A, B, C in $\mathscr{S}$. See, for example, [2], I.5.4. (The Lie product is given by $[B, C] = BC - CB$ and tr is the trace.) The notion of an algebra of upper triangular matrices has been generalized by Kadison and Singer [5] to the setting of von Neumann algebras. (A sub-algebra $\mathscr{T}$ of a von Neumann algebra $\mathscr{R}$ is said to be *triangular* if $\mathscr{T} \cap \mathscr{T}^*$ is a maximal abelian self-adjoint subalgebra of $\mathscr{R}$. Here, $\mathscr{T}^* = \{T^* \mid T \in \mathscr{T}\}$.) In the case of finite von Neumann algebras a trace is available, and so we may inquire if Cartan's Criterion is valid for triangular sub-algebras. The purpose of this note is to answer the question negatively; an example is given of a triangular sub-algebra of a type II_1 factor for which Cartan's Criterion does not hold.

The factor in question is the group von Neumann algebra of F_2, the free group on two generators, a and b. This factor, which we denote by $\mathscr{R}$, is a type II_1 factor which acts on the Hilbert space $\ell^2(F_2)$. For each $f \in \ell^2(F_2)$, let L_f denote the (left) convolution operator given by $L_f(g) = f * g$, for all $g \in \ell^2(F_2)$. (Recall that convolution is defined by the formula $f * g(x) = \sum_{y \in F_2} f(y)g(y^{-1}x)$.) Also, note that $(L_f)^* = L_{f*}$, where f^* is defined by $f^*(x) = \bar{f}(x^{-1})$. Then $\mathscr{R}$ is precisely the set of all those L_f which are bounded linear operators on $\ell^2(F_2)$. If, for each $x \in F_2$, we let $\delta(x)$ denote the characteristic function of $\{x\}$ and set $U_x = L_{\delta(x)}$, then $\mathscr{R}$ is the von Neumann algebra generated by the family of unitaries, $\{U_x \mid x \in F_2\}$. (The action of each unitary U_x is given by $(U_x f)(y) = f(x^{-1}y)$, for all $f \in \ell^2(F_2)$ and all $y \in F_2$. For a full exposition of the group algebra construction of Murray and von Neumann the reader may consult [4].)

Let $\mathscr{A}$ be the sub-von Neumann algebra of $\mathscr{R}$ generated by $\{U_{a^n} \mid n \in Z\}$. Note that $\mathscr{A} = \{L_f \mid L_f \in \mathscr{R}$ and supp(f) is contained in

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A, the subgroup of F_2 generated by $a\}$. $\mathscr{A}$ is clearly abelian and is, in fact, a maximal abelian *-sub-algebra of $\mathscr{R}$. Indeed, if $L_f \in \mathscr{R}$ and commutes with each U_{a^n}, then $U_{a^n} L_f U_{a^{-n}} = L_f$, whence $\delta(a^n) * f * \delta(a^{-n}) = f$ for each integer n. This implies that $f(x) = f(a^{-n}xa^n)$ for all $x \in F_2$ and all $n \in Z$. If $x \notin \{a^n \mid n \in Z\}$ then $\{a^{-n}xa^n \mid n \in Z\}$ is infinite; since f is square summable and constant of this set, it must vanish there. This proves that supp(f) is contained in the subgroup generated by a, i.e., $L_f \in \mathscr{A}$.

The unique normalized trace defined on $\mathscr{R}$ is given by the formula $\mathrm{tr}(L_f) = f(e)$, where e is the identity element in F_2. Also, observe that the mapping $\phi : \mathscr{R} \to \mathscr{A}$ given by $\phi(L_f) = L_{f|A}$ preserves the trace and is a diagonal process on $\mathscr{R}$ relative to $\mathscr{A}$. (That is, $\mathrm{tr}(S) = \mathrm{tr}(\phi(S))$ for all $S \in \mathscr{R}$, ϕ is linear, positive and $\phi(DT) = D\phi(T)$, $\phi(TD) = \phi(T)D$, for all $D \in \mathscr{A}$ and all $T \in \mathscr{R}$. See [1], 6.1.3, page 635.) This mapping is an analogue of the mapping on the set of $n \times n$ matrices which carries each matrix B to that diagonal matrix with the same diagonal entries as B. Such a mapping is multiplicative on the set of upper triangular matrices. However, the diagonal process ϕ, when restricted to the triangular algebra $\mathscr{T}$ defined below, is not multiplicative.

Let $T = U_{b^{-1}} + U_b + U_{b^2}$ and let $\mathscr{T}$ be the algebra generated by $\mathscr{A}$ and T. An arbitrary element of $\mathscr{T}$ is a finite sum of terms each of which has the form $A_0TA_1T \cdots A_{k-1}TA_k$, where $A_0, A_1, \cdots, A_k \in \mathscr{A}$ (and any, or all, of the A_i may equal the identity, I). A term of the form $A_0T \cdots TA_k$ will be said to have length k. (The possibility $k = 0$ is not excluded.) If $L_f = A_0T \cdots TA_k$, then there are constraints on the support of f. Any element $x \neq e$ in F_2 can be written in a canonical form: $x = a^{n_0}b^{m_1}a^{n_1} \cdots b^{m_j}a^{n_j}$, where only the exponents n_0 and n_j may be 0. This includes the possibility that $j = 0$, i.e., x is a power of a. The sequence of exponents is determined by x, since the group is free. For each $x \in F_2$, let $\sigma(x) = m_1 + \cdots + m_j =$ the sum of the exponents of b in the canonical form for x. (Of course, if $x \in A$ then $\sigma(x) = 0$.) If $x \in \mathrm{supp}(f)$, then $-k \leqq \sigma(x) \leqq 2k$. If it is also known that $\sigma(x) = 2k$, then x must have the form $a^{n_0}b^2a^{n_1}b^2 \cdots b^2a^{n_k}$ for a unique $(k+1)$-tuple of integers $(n_0, \cdots, n_k)$.

PROPOSITION 1. *The algebra $\mathscr{T}$ is triangular with diagonal $\mathscr{A}$.*

PROOF. It is evident that $\mathscr{A} \subset \mathscr{T} \cap \mathscr{T}^*$; for the reverse containment it suffices to show that if $S = L_f$ is a self-adjoint element of $\mathscr{T}$, then $S \in \mathscr{A}$. So, assume that S is a self-adjoint element of $\mathscr{T}$; S may be written as a finite sum of terms as above. Let k be the greatest length of any of the terms in S. We shall show that if $k > 0$ then S may be expressed as a sum of terms all of length less than k. This proves the proposition, for repeated application of the argument shows that S may be written as a sum of terms of length 0, i.e., S lies in $\mathscr{A}$.

We may write

$$S = L_f = A_0{}^1TA_1{}^1T \cdots TA_k{}^1 + \cdots + A_0{}^pTA_1{}^pT \cdots TA_k{}^p$$
$$+ \text{ a finite sum of terms of length less than } k.$$

First observe that if $x \in \operatorname{supp}(f)$, then $-k \leqq \sigma(x) \leqq k$. Indeed, it is immediate from the form of L_f and the remarks above that $-k \leqq \sigma(x) \leqq 2k$. But since L_f is self-adjoint, $f(x) = \overline{f(x^{-1})}$, whence $x^{-1} \in \operatorname{supp}(f)$. Thus $-k \leqq \sigma(x^{-1}) \leqq 2k$; the fact that $\sigma(x^{-1}) = -\sigma(x)$ now implies that $-2k \leqq \sigma(x) \leqq k$.

For each $A_i{}^j$, let $f^{(i,j)}$ be the function in $\ell^2(F_2)$ (with support in $A = \{a^n \mid n \in Z\}$) such that $A_i{}^j = L_{f^{(i,j)}}$. Consider the function

$$h = \sum_{j=1}^{p} f^{(0,j)} * \delta(b^2) * f^{(1,j)} * \delta(b^2) * \cdots * \delta(b^2) * f^{(k,j)}.$$

The support of h consists entirely of points of the form $x = a^{n_0}b^2a^{n_1}b^2 \cdots b^2a^{n_k}$, and at such points h takes the same values as f. But if $x \in \operatorname{supp}(h)$, $\sigma(x) = 2k$; hence $0 = f(x) = h(x)$. Thus

$$0 = h = \sum_{j=1}^{p} f^{(0,j)} * \delta(b^2) * \cdots * \delta(b^2) * f^{(k,j)}$$
$$= \sum_{n_0,\cdots,n_k} \left(\sum_{j=1}^{p} f^{(0,j)}(a^{n_0}) f^{(1,j)}(a^{n_1}) \cdots f^{(k,j)}(a^{n_k}) \right) \delta(a^{n_0}b^2 \cdots b^2a^{n_k}).$$

Now the $a^{n_0}b^2 \cdots b^2a^{n_k}$ are distinct for distinct $(k+1)$-tuples of integers $(n_0, \cdots, n_k)$, hence each coefficient must be zero. From this it follows that, for any k-tuple $\alpha_1, \cdots, \alpha_k$ of non-zero integers,

$$\sum_{j=1}^{p} f^{(0,j)} * \delta(b^{\alpha_1}) * \cdots * f^{(k,j)} * \delta(b^{\alpha_k})$$
$$= \sum_{n_0,\cdots,n_k} \left(\sum_{j=1}^{p} f^{(0,j)}(a^{n_0}) \cdots f^{(k,j)}(a^{n_k}) \delta(a^{n_0}b^{\alpha_1} \cdots b^{\alpha_k}a^{n_k} \right)$$
$$= \sum_{n_0,\cdots,n_k} 0 \cdot \delta(a^{n_0}b^{\alpha_1} \cdots b^{\alpha_k}a^{n_k}) = 0.$$

In particular,

$$0 = \sum_{j=1}^{p} f^{(0,j)} * (\delta(b^{-1}) + \delta(b) + \delta(b^2)) * \cdots * (\delta(b^{-1}) + \delta(b) + \delta(b^2)) * f^{(k,j)}.$$

But the function on the right hand side is just that element of $\ell^2(F_2)$ which is mapped by the left regular representation onto the operator, $A_0{}^1 T \cdots TA_k{}^1 + \cdots + A_0{}^p T \cdots TA_k{}^p$ in $\mathscr{R}$. Hence this operator is zero and S can be written as a sum of terms of length less than k.

PROPOSITION 2. *The trace-preserving diagonal process ϕ is not multiplicative when restricted to $\mathscr{T}$.*

PROOF. Since $T = U_{b^{-1}} + U_b + U_{b^2}$, $\phi(T) = 0$. But $T^2 = U_{b^{-2}} + 2U_e + 2U_b + U_{b^2} + 2U_{b^3} + U_{b^4}$, hence $\phi(T^2) = 2U_e = 2I$.

PROPOSITION 3. *The Cartan Criterion does not hold for $\mathscr{T}$.*

PROOF. Let $A = U_{a^{-1}}$, $B = U_a T$, and $C = T$. Observe that $\operatorname{tr}(ABC) = \operatorname{tr}(U_{a^{-1}} U_a TT) = \operatorname{tr}(T^2) = 2$. If $L_g = ACB = U_{a^{-1}} T U_a T$, then any element x in $\operatorname{supp}(g)$ has the form $x = a^{-1} b^n a b^m$, where $n, m \in \{-1, 1, 2\}$. In particular, $\operatorname{tr}(ACB) = 0$. Thus $\operatorname{tr}(A[B, C]) = \operatorname{tr}(ABC - ACB) = 2 \neq 0$, and Cartan's Criterion fails.

REMARK. An operator which is not in the kernel of the diagonal process ϕ may be thought of as having a nonzero diagonal part. If we take $B = U_a T$ and $C = T$ as above, then $\phi(BC) = \phi(U_a T^2) = U_a \phi(T^2) = 2U_a$ and $\phi(CB) = \phi(TU_a T) = 0$ (as can be seen from considering the support of the appropriate function). Therefore, $\phi([B, C]) = 2U_a \neq 0$ and we see that there exist commutators with non-zero diagonal part. Since $\phi(B) = \phi(C) = 0$, we see also that $\ker \phi$ is not even a Lie algebra. (Compare [3]).

We conclude with one final proposition about the algebra $\mathscr{T}$:

PROPOSITION 4. *$\mathscr{T}$ is irreducible (in $\mathscr{R}$).*

PROOF. We must show that if $E = L_f$ is a non-zero projection in $\mathscr{R}$ such that $ESE = SE$ for all $S \in \mathscr{T}$, then $E = I$. Note first that since each element of $\mathscr{A}$ leaves E invariant, E commutes with $\mathscr{A}$ and hence, $E \in \mathscr{A}$. Therefore, f has support contained in A; since $f \neq 0$, there is an integer n such that $f(a^n) \neq 0$. Let $g = \delta(b^{-1}) + \delta(b) + \delta(b^2)$. Then $ETE = TE$ implies that $f * g * f = g * f$. Let m be a non-zero integer and evaluate both $f * g * f$ and $g * f$ at $a^m b a^n$. For $g * f$ we obtain: $g * f(a^m b a^n) = \sum_x g(x) f(x^{-1} a^m b a^n) = f(b a^m b a^n) + f(b^{-1} a^m b a^n) + f(b^{-2} a^m b a^n) = 0$, since $m \neq 0$ and f has support in A. On the other hand,

$$\begin{aligned} f * g * f(a^m b a^n) &= \sum_{x,y} f(x) g(x^{-1}y) f(y^{-1}a^m b a^n) \\ &= \sum_{p,q} f(a^p) g(a^{-p} a^m b a^q) f(a^{n-q}) \\ &= f(a^m) f(a^n). \end{aligned}$$

The equalities above utilize the facts that $f(x) = 0$ unless $x = a^p$, for some p; $f(y^{-1}a^m b a^n) = 0$ unless $y = a^m b a^q$, for some q; and $g(a^{-p}a^m b a^q) = 0$ unless $m = p$ and $q = 0$. Since we know that $f * g * f = g * f$ and that $f(a^n) \neq 0$, we conclude that $f(a^m) = 0$ for all $m \neq 0$. Thus, $f = f(e)\delta(e)$ and $E = f(e)I$. Since E is a projection, we must have $f(e) = 1$ and the proposition is proven.

REMARK. Proposition 4 implies that the only elements of $\mathscr{R}$ which commute with $\mathscr{T}$ are the scalar operators.

REFERENCES

1. W. Arveson, *Analyticity in Operator Algebras,* Amer. J. Math. **89** (1967), 578–642.

2. N. Bourbaki, *Lie Groups and Lie Algebras,* Part I, Addison-Wesley, Reading, Mass., 1975.

3. J. A. Erdos, *Some Questions Concerning Triangular Operator Algebras,* preprint.

4. R. V. Kadison, *Theory of Operators,* Part II. *Operator Algebras,* Bull. A.M.S. **64** (1958), 61–85.

5. R. V. Kadison and I. M. Singer, *Triangular Operator Algebras,* Amer. J. Math. **82** (1960), 227–259.

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