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DECAY OF SOLUTIONS OF SYMMETRIC HYPERBOLIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider systems of the form $u_t + \sum_{j=1}^n A_j u_{x_j} = 0$, where the A_j's are constant $k \times k$ (hermitian) symmetric matrices, and u is a column vector of k components. We use Fourier transform to prove that non-static solutions decay in time at every point x. As a consequence, it follows that the energy of any such solution decays locally. More generally, we show that if $B(t)$ is a set which does not increase "too" fast, the energy in $B(t)$ of any non-static solution also decays.

1. **Introduction.** We consider systems of the form

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} = 0, \tag{1}$$

where the A_j's are constant $k \times k$ (hermitian) symmetric matrices, and u is a column vector of k components. These are functions of the independent variables $t \in \mathbf{R}$ and $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$. Systems of this type are the general form of a large number of equations of mathematical physics, such as Maxwell's equation, the equations of transmission lines, acoustics, elasticity (see Appendix in [7]), and even the equations of magnetogasdynamics (see [1]).

It is customary to discuss the above systems under additional assumptions on the matrices A_j. One such assumption is that the roots $\lambda = \lambda(p)$ of the characteristic equation

$$P(\lambda, p) = \det\left(\lambda I - \sum_{j=1}^{n} p_j A_j\right) = 0 \tag{2}$$

are all different from zero for $p \neq 0$, that is, the operator $\sum_{j=1}^n A_j \partial/\partial x_j$ is elliptic ([4], p. 178); or a fixed number of them never vanish for $p \neq 0$ ([3]); or the assumption contained in the definition of uniformity propagative systems of Wilcox ([7]). In our treatment we impose no restrictions on the A_j's other than those stated in the previous paragraph. This is important because there are systems, such as those of magnetogasdynamics, which possess roots $\lambda(p)$ that vanish for certain $p \neq 0$, but not identically. It has been shown that if a characteristic root $\lambda(p)$ is not identically zero then the set of those p where $\lambda(p) = 0$ is of measure zero ([1]). Since the $\lambda(p)$, for $|p| = 1$, are speeds of propagation of

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plane waves in the direction p, it is thus reasonable to expect that for a given solution u, the part of it associated with such a $\lambda(p)$ should decay in time. Our objective in this paper is to prove this result and derive from it the local decay of the energy of u, provided (of course) u is non-static. It also follows that the energy of any solution in a "slowly increasing" set is asymptotically constant.

2. **The solution of the Cauchy Problem.** We use the Fourier transform to derive an explicit formula for the solution of equation (1) with initial value $u(x, 0) = f(x)$. We assume initially that $f \in \mathscr{S}^k$, where $\mathscr{S}$ is the Schwartz space of rapidly decreasing functions. The Fourier transform of $g \in \mathscr{S}^k$ is defined by

$$\hat{g}(p) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ip\cdot x} g(x)dx,$$

and the inverse formula

$$g(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ip\cdot x} \hat{g}(p)dp$$

holds. Since, under Fourier transformation, differentiation with respect to x_j changes into multiplication by ip_j, equation (1) transforms into

$$\hat{u}_t + iA(p)\hat{u} = 0, \tag{3}$$

where $A(p) = \sum_{j=1}^n p_j A_j$.

We want to solve equation (3) with the initial condition $\hat{u}(p, 0) = \hat{f}(p)$. Let $\{e_j(p)\}$ be a complete set of orthonormal eigenvectors of $A(p)$ with corresponding eigenvalues $\lambda_j(p)$. Setting

$$\hat{u}_j(p, t) = \hat{u}(p, t)\cdot e_j(p)$$

and

$$\hat{f}_j(p) = \hat{f}(p)\cdot e_j(p),$$

equation (3) gives us, upon scalar multiplication by $e_j(p)$,

$$\frac{\partial \hat{u}_j}{\partial t} = -i\lambda_j(p)\hat{u}_j(p).$$

The solution of this equation that satisfies the initial condition $\hat{u}_j(p, 0) = \hat{f}_j(p)$ is

$$\hat{u}_j(p, t) = \hat{f}_j(p)e^{-i\lambda_j(p)t}.$$

Therefore,

$$\hat{u}(p, t) = \sum_{j=1}^{k} \hat{f}_j(p)e^{-i\lambda_j(p)t}e_j(p).$$

At this point we note that the eigenvectors $e_j(p)$ can be chosen in such way that they are measurable functions of p. That this is possible was proved by C. H. Wilcox (see [9], Theorem 2), and we assume that this choice has been made. We note also that the roots $\lambda_j(p)$ are continuous functions of p ([9], Theorem 1). Therefore, we are justified in taking the inverse Fourier transform of the above expression for $\hat{u}$. We obtain

$$u(x, t) = (2\pi)^{-n/2} \sum_{j=1}^{k} \int_{\mathbf{R}^n} \hat{f}_j(p) e^{i[p\cdot x - \lambda_j(p)t]} e_j(p)\,dp. \tag{4}$$

This formula shows that u is the superposition of k waves: each of these waves, in turn, is the superposition—given by the integration over p-space—of the plane waves

$$(2\pi)^{-n/2} \hat{f}_j(p) e^{i[p\cdot x - \lambda_j(p)t]} e_j(p).$$

This wave is a signal that propagates in the direction p with speed $v_j(p) = \lambda_j(p)/|p|$. Each one of the k terms of the sum in (4) is referred to as a normal mode of propagation. We see that each mode is associated with an eigenvalue (counting multiplicity) of $A(p)$, that is, with a speed of plane wave propagation. A j^{th} mode is excited or not, according to whether the initial value f is such that $\hat{f}_j(p)$ is different from zero or not, respectively. If $\lambda_j(p) \equiv 0$, then the associated mode does not depend on the time: it is *static*. Thus, a smooth solution u is the superposition of its static and non-static parts,

$$u(x, t) = u_s(x, t) + u_{ns}(x, t), \tag{5}$$

and it is clear what a *non-static solution* means in this case. Considering the Hilbert space $H = L^2(\mathbf{R}^n)^k$ with its usual norm

$$\|f\|^2 = \int_{\mathbf{R}^n} |f(x)|^2 dx = \sum_{j=1}^{k} \int_{\mathbf{R}^n} |f_j(x)|^2 dx,$$

we notice that for any $f \in H$, (4) defines a function $u(\cdot, t) = U(t)f \in H$ for each $t \in \mathbf{R}$: $u(\cdot, t) = U(t)f$ is called a solution with finite energy (clearly, $u(\cdot, t)$ is not, in general, a classical solution of (1)). It can be shown that $U(t) = \exp(-tA)$, $t \in \mathbf{R}$, the group of unitary operators generated by the skew selfadjoint operator

$$A = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}, \qquad D(A) = \{f \in H \mid A(p)\hat{f} \in H\}.$$

A solution with finite energy $u(\cdot, t) = U(t)f$ is said to be *non-static* if f belongs to the orthogonal complement $N(A)^{\perp}$ of the null space of A.

3. **Decay of solution.** Let us enumerate, in decreasing order of magnitude, the solutions of equation (2) that are not identically zero, say, a total of r:

$$\lambda_1(p) \geqq \cdots \geqq \lambda_r(p). \tag{6}$$

The remaining $k - r$ roots are then identically zero:

$$\lambda_{r+1}(p) = \cdots = \lambda_k(p) \equiv 0. \tag{7}$$

It follows that $u(\cdot, t) = U(t)f$ is a non-static solution if and only if

$$\hat{f}_j = 0 \text{ for } j = r + 1, \cdots, k. \tag{8}$$

THEOREM 1. *For each $m = 1, 2, \cdots$ and f in a dense subset S of $N(A)^{\perp}$, there exists a constant $C_{m,f} > 0$ such that*

$$|u(x, t)| \leqq C_{m,f}(1 + |x| + \cdots + |x|^m)|t|^{-m},$$
$$\textit{for all } t \neq 0, \textit{ for all } x \in \mathbf{R}^n.$$

PROOF. Let us introduce polar coordinates ρ, ω with $\rho > 0$, $|\omega| = 1$, through the relation $p = \rho\omega$. Making use of the (easy to check) facts that $\lambda_j(\rho\omega) = \rho\lambda_j(\omega)$ and $e_j(\rho\omega) = e_j(\omega)$, a non-static solution with initial value $f \in \mathscr{S}^k$ is given by (see (4) and (8))

$$u(x, t) = (2\pi)^{-n/2} \sum_{j=1}^{r} \int_{|\omega|=1} \left\{ \int_0^{\infty} e^{i\rho\omega\cdot x} e^{-it\rho\lambda_j(\omega)} \hat{f}_j(\rho\omega) e_j(\omega) \rho^{n-1} d\rho \right\} dS_\omega. \tag{9}$$

We further restrict f in the following way: each $\hat{f}_j(\rho\omega)$ is assumed to be of the form

$$\hat{f}_j(\rho\omega) = \varphi_j(\rho)\psi_j(\omega),$$

where $\varphi_j \in C_0^{\infty}(0, \infty)$ and ψ_j is taken from the set of C^∞ functions on the unit sphere S^{n-1} which vanish on some neighborhood of

$$N_j = \{\omega \in S^{n-1} \mid \lambda_j(\omega) = 0\}.$$

We denote by S this set of data f. It was proved in [1] that if a root $\lambda_j(\omega)$ does not vanish identically then the corresponding set N_j is of measure zero in S^{n-1}. Also, N_j is closed, for $\lambda_j(p)$ is a continuous function. It thus follows easily that S is dense in $N(A)^{\perp}$.

We are now in a position to estimate the expression in (9). Each term $u_j(x, t)$ there can be written

$$u_j(x, t) = (2\pi)^{-n/2} \int_{S^{n-1}\setminus V_j} \left(\int_\alpha^\beta e^{-it\rho\lambda_j(\omega)} F_j d\rho \right) dS_\omega,$$

where

$$F_j = e^{i\rho\omega\cdot x} \varphi_j(\rho)\psi_j(\omega)e_j(\omega)\rho^{n-1},$$

$\varphi_j \in C_0^\infty(\alpha, \beta)$ with $0 < \alpha < \beta < \infty$, and V_j is some open neighborhood of N_j. If we integrate by parts once, we obtain

$$u_j(x, t) = (2\pi)^{-n/2} \int_{S^{n-1}\setminus V_j} \left(\int_\alpha^\beta \frac{e^{-it\rho\lambda_j(\omega)} F_j'}{it\lambda_j(\omega)} d\rho \right) dS_\omega,$$

where $F_j' = \partial F_j / \partial \rho$. We can repeat this procedure any number of times, say m times, thus obtaining

$$u_j(x, t) = (2\pi)^{-n/2} \int_{S^{n-1}\setminus V_j} \left(\int_\alpha^\beta \frac{e^{-it\rho\lambda_j(\omega)} F_j^{(m)}}{[it\lambda_j(\omega)]^m} d\rho \right) dS_\omega,$$

Now, since each $\lambda_j(\omega)$ is continuous, there exists $a_j > 0$ such that $|\lambda_j(\omega)| \geqq a_j > 0$ on the set $S^{n-1}\setminus V_j$. Thus we can estimate $u_j(x, t)$ above by

$$\frac{C(1 + |x| + \cdots + |x|^m)}{|t|^m}$$

where C is a constant that depends on φ_j, ψ_j, V_j, α, β and m. The proof is complete.

Now, if $K \subset \mathbf{R}^n$ is any compact set and $u(\cdot, 0) = f \in S$, we obtain the fact that the energy of $u(\cdot, t)$ over K decays faster than any power of $1/t$:

COROLLARY. *For $u(\cdot, 0) = f \in S$ and $m = 1, 2, \cdots$, we have*

$$\|u(\cdot, t)\|_K^2 = \int_K |u(x, t)|^2 dx \leqq C_{m,f}(K)|t|^{-2m}, \text{ for all } t \neq 0.$$

The next result shows that if $B(t)$ is a set that does not increase "too" fast as $t \to \infty$, then the energy in $B(t)$ of non-static solution decays to zero.

THEOREM 2. *Let $\{B(t) \mid t > 0\}$ be a family of bounded measurable sets in $\mathbf{R}^n$ such that, for some $\alpha < 1$, $\Theta(t) = 0(t^\alpha)$ as $t \to \infty$, where $\Theta(t) = \sup\{|x| \mid x \in B(t)\}$. Then,*

$$\lim_{t\to\infty} \|v(\cdot, t)\|_{B(t)}^2 = \lim_{t\to\infty} \int_{B(t)} |v(x, t)|^2 dx = 0,$$

for any non-static solution $v(\cdot, t)$ with finite energy.

PROOF. First, let us assume that $v(\cdot, t)$ is a non-static solution with initial value in S. Also, since $\alpha < 1$, we can choose a positive integer m such that

$$\alpha(2m + n) < 2m. \tag{10}$$

Therefore, by Theorem 1, for some constant $C > 0$, we have that

$$|v(x, t)|^2 \leqq Ct^{-2m}(1 + |x|^{2m}) \leqq Ct^{-2m}(1 + \Theta(t)^{2m})$$

for any $x \in B(t)$. It follows that

$$\int_{B(t)} |v(x, t)|^2 dx \leqq Ct^{-2m}(1 + \Theta(t)^{2m}) \text{meas}(B(t))$$
$$\leqq C' t^{-2m}(1 + \Theta(t)^{2m})\Theta(t)^n,$$

and, since $\Theta(t) = 0(t^\alpha)$, we obtain

$$\int_{B(t)} |v(x, t)|^2 dx \leqq C'' t^{-2m}(t^{\alpha n} + t^{\alpha(2m+n)})$$

for t large. Hence, in view of (10), the theorem is proved in this case.

Now, let $v(\cdot, t)$ be an arbitrary non-static solution with finite energy, that is, $v(\cdot, 0) = g \in N(A)^\perp$. Since S is dense in $N(A)^\perp$, given any $\epsilon > 0$, there exists $g_\epsilon \in S$ such that $\|g - g_\epsilon\| < \epsilon$. And, since $\|U(t)\| = 1$, we obtain

$$\|v(\cdot, t)\|_{B(t)} = \|U(t)g\|_{B(t)} \leqq \|U(t)(g - g_\epsilon)\|_{B(t)} + \|U(t)g_\epsilon\|_{B(t)}$$
$$\leqq \|U(t)(g - g_\epsilon)\| + \|U(t)g_\epsilon\|_{B(t)}$$
$$= \|g - g_\epsilon\| + \|U(t)g_\epsilon\|_{B(t)} < \epsilon + \|U(t)g_\epsilon\|_{B(t)}.$$

But, by what we just proved, $\lim_{t\to\infty} \|U(t)g_\epsilon\|_{B(t)} = 0$. Hence, $\lim\sup_{t\to\infty} \|v(\cdot, t)\|_{B(t)} \leqq \epsilon$, and the proof is complete since $\epsilon > 0$ is arbitrary.

If we take $B(t)$ to be a fixed (bounded, measurable) set B for all $t > 0$, Theorem 2 yields the usual local energy decay, that is, the decay of the energy in a fixed bounded measurable set:

COROLLARY. *Given a bounded measurable set $B \subset \mathbf{R}^n$,*

$$\lim_{t\to\infty} \|v(\cdot, t)\|_B^2 = \lim_{t\to\infty} \int_B |v(x, t)|^2 dx = 0,$$

for any non-static solution $v(\cdot, t)$ with finite energy.

Now, let us say that a family $\{B(t) \mid t > 0\}$ of sets in $\mathbf{R}^n$ converges to a set B, $B(t) \to B$, if the characteristic function $\chi_{B(t)}$ of $B(t)$ converges almost everywhere to the characteristic function χ_B of B, as $t \to \infty$. In this case, it follows that the energy in $B(t)$ of any solution $u(\cdot, t)$ is asymptotically constant.

(We remark that the requirement that $\chi_{B(t)}(x) \to \chi_B(x)$ for all $x \in \mathbf{R}^n$ is equivalent to the set-theoretical notion of limit $\bigcup_{s>0} \bigcap_{t \geqq s} B(t) = \bigcap_{s>0} \bigcup_{t \geqq s} B(t) = B$.)

THEOREM 3. *In addition to the hypotheses of Theorem 2, assume that $B(t) \to B$, where B is any measurable set, not necessarily bounded. Then, for any solution $u(\cdot, t) = U(t)f$ with finite energy,*

$$\lim_{t \to \infty} \|u(\cdot, t)\|^2_{B(t)} = E_B{}^o[f]$$

exists.

PROOF. Let $u(\cdot, t) = U(t)f$ be a solution with finite energy. We can decompose $f \in H$ uniquely as a sum $f = h + g$, with $h \in N(A)$ and $g \in N(A)^\perp$. We observe that if $u(\cdot, t)$ has no static part (that is, $h = 0$), the proof is that of Theorem 2 and does not depend on the additional assumption that $B(t) \to B$. In any case, $U(t)h = h$ for all t so that

$$U(t)f = h + U(t)g.$$

And, since $U(t)f - h$ is non-static, Theorem 2 gives $\lim_{t \to \infty} \|U(t)f - h\|_{B(t)} = 0$, hence

$$\lim_{t \to \infty} (\|u(\cdot, t)\|_{B(t)} - \|h\|_{B(t)}) = 0.$$

Now, the assumption $B(t) \to B$ implies (by the Lebesgue dominated convergence theorem) that

$$\lim_{t \to \infty} \|h\|_{B(t)} = \|h\|_B \equiv (E_B{}^o[f])^{1/2}.$$

The proof is complete.

4. **Final remarks.** 1) As already mentioned in the introduction, symmetric hyperbolic systems (1) have been studied by other authors under additional conditions on the matrices A_j. Results on decay of solutions can be found in [4], [7], for example. In [8], the case of the wave equation is thoroughly investigated. Uniform (over all x-space) decay results can be found in [2], where a further assumption of convexity on the connected sheets of the "wave surface" $P(1, p) = 0$ is imposed (see also [6], where the wave equation and the Klein-Gordon equation are considered).

One approach to studying decay of solutions is clearly through the so called Riemann matrix of (1), that is, the distribution matrix-valued solution $R(x, t)$ of (1) with initial value $R(x, 0) = \delta(x)I_n$: any smooth solution $u(x, t)$ with initial value $f(x)$ can then be written as $u(x, t) = [R(\cdot, t) * f](x)$. Since $R(x, t)$ is homogeneous of degree $-n$, $R(x, t) = t^{-n}R(x/t, 1)$, and it could be thought that any (non-static) smooth solution would decay at every point x at least like t^{-n}. This is certainly not the case, due to the singularities of $R(x, t)$. In fact, concrete examples in two and three dimensions have been given where solutions decay only as $t^{-1/2}$([5]). We conjecture that there is no rate of decay for the general symmetric hyperbolic systems considered in this paper. In other words, we conjecture that there does not exist a function $\varphi(t) \longrightarrow 0$ as $t \longrightarrow \infty$, with the property that for each non-static solution u, the inequality $\sup_x|u(x, t)| \leqq C\varphi(t)$, $t > 0$, holds with some constant $C = C(u)$.

2) Given a symmetric hyperbolic system for which the null space of $A(p)$ has constant dimension for all $p \neq 0$, we can naturally associate a minimum positive speed of propagation, $\lambda_{\min} = \inf\{|\lambda_j(\omega)| \mid \omega \in S^{n-1}, j = 1, \cdots, r\}$ (see (6), (7)). In this case, it is not hard to show that the energy of any non-static solution $u(\cdot, t)$ inside a cone $|x| = Ct$ tends to zero as $t \longrightarrow \infty$, provided that $C < \lambda_{\min}$. On the other hand, if $A(\omega)$ possesses characteristic roots that vanish for certain ω; but not identically, then there is no minimum positive speed of propagation and the above result does not apply. However, since any paraboloid $|x| = Ct^{\alpha}$, $\alpha < 1$, has slope $d|x|/dt$ that approaches zero as $t \longrightarrow \infty$, we see that Theorem 2 is natural counterpart of the above mentioned result on energy decay in a cone. We contend that we cannot allow $\alpha = 1$.

3) Analogous results can be easily obtained for systems which are "perturbations" of "free" systems (1), as long as their solutions behave asymptotically as "free" solutions. For example, in [1] systems of the form $E(x)u_t + \sum_{j=1}^{n} A_j u_{x_j} = 0$ are considered. It is shown there that if $\int (1 + |x|^2)^2 \, |E(x) - I|^2 \, dx < \infty$ then, for any solution $u(\cdot, t)$ in a "certain class", there exists a (free) solution $u_+(\cdot, t)$ of $u_t + \sum_{j=1}^{n} A_j u_{x_j} = 0$ such that

$$\lim_{t\to\infty} \|u(\cdot, t) - u_+(\cdot, t)\| = 0.$$

For any such u and for $B(t) \longrightarrow B$, it clearly follows that

$$\lim_{t\to\infty} \|u(\cdot, t)\|_{B(t)} = \|h_+\|_B,$$

where h_+ is the orthogonal projection of $f_+ = u_+(\cdot, 0)$ on $N(A)$.

BIBLIOGRAPHY

1. G. S. S. Ávila, *Spectral Resolution of Differential Operators Associated with Symmetric Hyperbolic Systems*, Applicable Analysis 1 (1972), 283–299.

2. D. G. Costa, *L^p-decay rates, p big ($\leqq \infty$), and energy decay in nonbicharacteristic cones for first order hyperbolic systems*, Lecture Notes in Mathematics **446** (1975), 123–143. Springer, Berlin, New York, Heidelberg.

3. T. Ikebe, *Scattering for uniformly propagative systems*, Proc. Int. Conf. on Func. Anal. and Related Topics, Tokyo (1969), 225–230.

4. P. D. Lax and R. S. Phillips, *Scattering Theory*, Academic Press, New York, 1967.

5. D. Ludwig, *Examples of the behavior of hyperbolic equations for large times*, J. Math. Mech. **12** (1963), 557–566.

6. W. v. Wahl, *L^p-decay rates for homogeneous wave equations*, Math. Z. **120** (1971), 93–106.

7. C. H. Wilcox, *Wave operators and asymptotic solutions of wave propagation problems of classical physics*, Arch. Rat. Mech. Anal. **22** (1966), 37–78.

8. ———, *Scattering theory for the d'Alembert equation in exterior domains*, Lecture Notes in Mathematics **442** (1975). Springer, Berlin, New York, Heidelberg.

9. ———, *Measurable eigenvectors for Hermitian matrix-valued polynomials*, J. Math. Anal. Applications **40** (1972), 12–19.

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