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CONVERGENCE OF APPROXIMATION METHODS FOR EIGENVALUES OF COMPLETELY CONTINUOUS QUADRATIC FORMS

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DEDICATED TO N. ARONSZAJN

0. **Introduction.** In 1951 Aronszajn [5] outlined a general procedure for approximating the eigenvalues of a real quadratic form $\mathfrak{B}$ relative to a positive definite quadratic form $\mathfrak{A}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are defined on a vector space V and $\mathfrak{B}$ is completely continuous with respect to $\mathfrak{A}$. This procedure encompassed all the then known variational approximation techniques, including the Rayleigh-Ritz, Weinstein, and Aronszajn methods. Briefly stated, the procedure is as follows:

The original eigenvalue problem is replaced by an auxiliary problem determined by a pair of forms $(\mathfrak{B}_0, \mathfrak{A}_0)$. The auxiliary problem is so chosen as to be explicitly solvable and such that its eigenvalues give approximations, however bad, to the corresponding eigenvalues of the original problem. These initial approximations are improved by introducing intermediate problems, whose eigenvalues give successively better (or at least no worse) approximations to those of the original problem. The auxiliary problem and the intermediate problems are chosen using the so-called monotony principles, and the eigenvalues of the intermediate problems are computed using those of the auxiliary problem and certain perturbation determinants (often called Weinstein-Aronszajn determinants). Depending on the method used, one gets either upper or lower approximations to the eigenvalues of the original problem. By using different methods, one can obtain both upper and lower approximations (and therefore a posteriori error estimates).

In [5] Aronszajn described how the above procedure could be applied in the case of four basic approximation methods, derived the corresponding Weinstein-Aronszajn determinants, and investigated the convergence of the approximating eigenvalues to the corresponding eigenvalues of the original problem. For three of the methods considered he was able to show the convergence under relatively mild and "natural" hypotheses. For the fourth method (Aronszajn's method) he was able to show it only under the additional hypothesis that $\mathfrak{A}_0$ be equivalent to $\mathfrak{A}$. He left open the questions of whether this hypothesis is necessary or whether conver-

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gence holds for this method under milder hypotheses than sufficed for the other methods.

Since 1951, the theory of variational approximation methods has been greatly enlarged. Many generalizations, extensions, variations, and applications have been discovered (see **[12]** **[23]** **[24]** and the references cited there), most still within the general framework outlined above. The theory of Weinstein-Aronszajn determinants has also been much generalized **[6]** **[9]** **[10]** **[13]** **[15]**, and many useful and important convergence results for variational approximation methods have been obtained (e.g., **[11]** **[16]** **[17]** **[18]** **[19]**). Most of these convergence results, however, have confined themselves to a particular method and often to special settings; e.g., when $\mathfrak{A}_0$ is equivalent to $\mathfrak{A}$.

The aims of the present paper are to investigate the questions of convergence in the general framework proposed by Aronszajn in **[5]**, to obtain convergence criteria which will be useful in formulation of the intermediate problems, and to provide answers to some of the questions raised by Aronszajn in **[5]**. These investigations are carried out using the theory of *discrete convergence* of Banach spaces in the form developed by Stummel **[20]** **[21]** **[22]**.

In §1 it is shown how this theory can be adapted to the study of eigenvalue approximations, and a general convergence theorem is derived using the notion of *discrete compactness.* In §2 this result is used to obtain convergence criteria in a form applicable to variational methods. Necessity and sufficiency of these conditions is investigated in this section and in §3.

In §4 the convergence criteria are applied to the four basic types of approximation methods considered by Aronszajn, and the convergence results of **[5]** are recovered. In §5 the special case of Aronszajn's method is investigated in detail and some of Aronszajn's original questions answered.

Finally, in §§ 6 and 7, some special cases and variations of Aronszajn's method are discussed, and applications of the convergence criteria of §2 are illustrated.

1. **Discrete convergence and the basic approximation theorem.** Let V be a (non-trivial) complex vector space, and $\mathfrak{A}, \mathfrak{A}_0, \mathfrak{A}_1, \ldots$ be positive definite quadratic forms on V. Assume also that V is separable with respect to the norm $\mathfrak{A}^{1/2}$. We shall say that $\{\mathfrak{A}_n\}$ forms a *discrete approximation* to $\mathfrak{A}$ on V provided $\mathfrak{A}_n(v) \to \mathfrak{A}(v)$ for every $v \in V$. Let $X, X_0, X_1, \ldots$ be the Hilbert space completions of V with respect to the norms $\mathfrak{A}^{1/2}, \mathfrak{A}_0^{1/2}, \mathfrak{A}_1^{1/2}, \ldots$ respectively.

DEFINITION 1.1. Let $x \in X$ and $\{x_n\}$ be a sequence such that $x_n \in X_n$, $n = 0, 1, \ldots$. Then

1. $\{x_n\}$ *converges weakly discretely* to x (written $x_n \rightharpoonup_d x$) if and only if the sequence $\{\mathfrak{A}_n(x_n)\}$ is bounded and $\mathfrak{A}_n(x_n, v) \to \mathfrak{A}(x, v)\ \forall v \in V$.

2. $\{x_n\}$ *converges discretely* to x (written $x_n \to_d x$) if and only if $x_n \rightharpoonup_d x$ and $\mathfrak{A}_n(x_n) \to \mathfrak{A}(x)$.

LEMMA 1.1. 1. *$x_n \to_d 0$ if and only if $\mathfrak{A}_n(x_n) \to 0$.*

2. *$v \to_d v$ for every $v \in V$.*

3. *If $\mathfrak{A}_n(x_n) \leqq M$ for $n = 0, 1, \ldots$ then*

   a. *$x_n \to_d x$ if and only if $\mathfrak{A}_n(x_n, y_n) \to \mathfrak{A}(x, y)$ whenever $y_n \rightharpoonup_d y$.*

   b. *$x_n \rightharpoonup_d x$ if and only if $\mathfrak{A}_n(x_n, y_n) \to \mathfrak{A}(x, y)$ whenever $y_n \to_d y$.*

PROOF. Parts 1 and 2 are obvious. To prove 3:

a. Assume $\mathfrak{A}_n(x_n, y_n) \to \mathfrak{A}(x, y)$ whenever $y_n \rightharpoonup_d y$. Then $\mathfrak{A}_n(x_n, v) \to \mathfrak{A}(x, v)$ for all $v \in V$, so $x_n \rightharpoonup_d x$. But then also $\mathfrak{A}_n(x_n) = \mathfrak{A}_n(x_n, x_n) \to \mathfrak{A}(x, x) = \mathfrak{A}(x)$ by hypothesis; i.e., $x_n \to_d x$.

Conversely, if $x_n \to_d x$, let $y_n \rightharpoonup_d y$. Given $\varepsilon > 0$, choose $v \in V$ such that $\mathfrak{A}(x - v) < \varepsilon^2$. Then

$$\begin{aligned}\mathfrak{A}_n(x_n - v) &= \mathfrak{A}_n(x_n) - 2Re\,\mathfrak{A}_n(x_n, v) + \mathfrak{A}_n(v)\\ &\to \mathfrak{A}(x) - 2Re\,\mathfrak{A}(x, v) + \mathfrak{A}(v) = \mathfrak{A}(x - v) < \varepsilon^2.\end{aligned}$$

Let N be such that $\mathfrak{A}(y), \mathfrak{A}_n(y_n) \leqq N^2$. Then

$$\begin{aligned}|\mathfrak{A}_n(x_n, y_n) - \mathfrak{A}(x, y)| &\leqq |\mathfrak{A}_n(x_n, y_n) - \mathfrak{A}_n(v, y_n)| + |\mathfrak{A}_n(v, y_n) - \mathfrak{A}(v, y)|\\ &\quad + |\mathfrak{A}(v, y) - \mathfrak{A}(x, y)|\\ &\leqq \mathfrak{A}_n(x_n - v)^{1/2}\mathfrak{A}_n(y_n)^{1/2} + |\mathfrak{A}_n(v, y_n) - \mathfrak{A}(v, y)|\\ &\quad + \mathfrak{A}(v - x)^{1/2}\,\mathfrak{A}(y)^{1/2},\end{aligned}$$

so $\limsup |\mathfrak{A}_n(x_n, y_n) - \mathfrak{A}(x, y)| \leqq 2\varepsilon N$. Therefore $\mathfrak{A}_n(x_n, y_n) \to \mathfrak{A}(x, y)$.

b. Assume $\mathfrak{A}_n(x_n, y_n) \to \mathfrak{A}(x, y)$ whenever $y_n \rightharpoonup_d y$. Then in particular $\mathfrak{A}_n(x_n, v) \to \mathfrak{A}(x, v)$ for all $v \in V$, so that $x_n \rightharpoonup_d x$. The converse follows as in the proof of *a*.

THEOREM 1.1. *$\{X_n\}$ forms a discrete approximation to X in the sense of Stummel* [**20**, section 1.1], i.e.,

1. *For every $x \in X$, there is a sequence $\{x_n\}$ with $x_n \in X_n$ such that $x_n \to_d x$.*
2. If *$x_n \to_d x$, then $\mathfrak{A}_n(x_n) \to \mathfrak{A}(x)$.*
3. If *$x_n \to_d x$ and $y_n \to_d y$, then $\alpha x_n + \beta y_n \to_d \alpha x + \beta y$ for all $\alpha, \beta \in \mathbf{C}$.*

PROOF. Parts 2 and 3 follows immediately form Definition 1.1 and Lemma 1.1.3. To prove part 1 note that all three of the above properties hold for V in place of X; i.e., $\{X_n\}$ is a discrete approximation to the dense subspace V of X. But then, given $x \in X$ and $\varepsilon > 0$, there exists by the proof of [**20**, Lemma 4.1.6], $x_n \in X_n$, $u_n \in V$, $u \in V$ such that $u_n \to_d u$, $\mathfrak{A}(x - u) \leqq \varepsilon^2$, and $\limsup \mathfrak{A}_n(x_n - u_n) \leqq \varepsilon^2$. Then for all $v \in V$,

$$\limsup |\mathfrak{A}_n(x_n, v) - \mathfrak{A}(x, v)|$$
$$\leqq \limsup [|\mathfrak{A}_n(x_n, v) - \mathfrak{A}_n(u_n, v)| + |\mathfrak{A}_n(u_n, v) - \mathfrak{A}(u, v)| + |\mathfrak{A}(u, v) - \mathfrak{A}(x, v)|] \leqq 2\varepsilon M,$$

where $\mathfrak{A}_n(v)^{1/2}, \mathfrak{A}(v)^{1/2} \leqq M$, and

$$\limsup [\mathfrak{A}_n(x_n)^{1/2} - \mathfrak{A}(x)^{1/2}|$$
$$\leqq \limsup [\mathfrak{A}_n(x_n - u_n)^{1/2} + |\mathfrak{A}_n(u_n)^{1/2} - \mathfrak{A}_n(u)^{1/2}| + \mathfrak{A}_n(u - x)^{1/2}]$$
$$\leqq 2\,\varepsilon.$$

Therefore $\mathfrak{A}_n(x_n, v) \to \mathfrak{A}(x, v)$ $v \in V$ and $\mathfrak{A}_n(x_n) \to \mathfrak{A}(x)$; i.e., $x_n \to_d x$. The theorem is proved.

Given a subsequence $\{\mathfrak{A}_{n_j}\}$ of $\{\mathfrak{A}_n\}$, note that $\{X_{n_j}\}$ is still a discrete approximation to X. Therefore the notions of discrete and weakly discrete convergence of sequences $\{x_{n_j}\}$ with $x_{n_j} \in X_{n_j}$ are well defined. In view of the separability of X we have [**20**, Theorem 2.3.1]:

PROPOSITION 1.1. Let $\{\mathfrak{A}_n(x_n)\}$ *be a bounded sequence,* $x_n \in X_n$, $n = 0, 1, \ldots$. *Then there exists a subsequence* $\{x_{n_j}\}$ of $\{x_n\}$ *and an element* $x \in X$ *such that* $x_{n_j} \rightharpoonup_d x$.

Next, let $\mathfrak{B}$, $\mathfrak{B}_0$, $\mathfrak{B}_1$, ... be real quadratic forms on V which are completely continuous with respect to $\mathfrak{A}$, $\mathfrak{A}_0$, $\mathfrak{A}_1$,... respectively. Then $\mathfrak{B}$, $\mathfrak{B}_0$, $\mathfrak{B}_1$, ... can be continuously extended to X, X_0, X_1, ... respectively. We introduce the following condition:

$$\text{(C)} \qquad x_n \rightharpoonup_d x \Rightarrow \mathfrak{B}_n(x_n) \to \mathfrak{B}(x).$$

REMARK 1.1. Let $\|\mathfrak{B}_n\|_n$ be the bound of $\mathfrak{B}_n$ with respect to $\mathfrak{A}_n$, so that $|\mathfrak{B}_n(v)| \leqq \|\mathfrak{B}_n\|_n \mathfrak{A}_n(v)$ for all $v \in V$.

If the sequence $\{\|\mathfrak{B}_n\|_n\}$ is bounded, then condition (C) is equivalent to the condition

$$v_n \in V,\ v_n \rightharpoonup_d x \Rightarrow \mathfrak{B}_n(v_n) \to \mathfrak{B}(x).$$

REMARK 1.2. The forms $\mathfrak{B}$, $\mathfrak{B}_n$ define compact operators B, B_n on X, X_n respectively according to the formula

$$\mathfrak{B}(x, y) = \mathfrak{A}(Bx, y),\ \mathfrak{B}_n(x, y) = \mathfrak{A}_n(B_n x, y).$$

Condition (C) then says that $(\{B_n\}, B)$ is weakly discretely compact in the sense of Stummel [**20**]. In other contexts [**4**] [**17**] this notion appears under the name of "collective compactness."

The following theorem is simply a restatement of [**21**, Theorem 3.1.8]:

THEOREM 1.2. *Let the positive definite quadratic forms* $\mathfrak{A}_n$, $n = 0, 1, \ldots$ *form a discrete approximation to* $\mathfrak{A}$ on V. *Let* $\mathfrak{B}$, $\mathfrak{B}_n$ *be real quadratic* *forms on V completely continuous with respect to $\mathfrak{A}$, $\mathfrak{A}_n$ respectively, $n = 0, 1, \ldots$. Let λ be an eigenvalue of $\mathfrak{B}$ with respect to $\mathfrak{A}$ of mulitplicity $m > 0$, and P be the orthogonal projection of X onto the eigenspace corresponding to λ. If $\{\|\mathfrak{B}_n\|_n\}$ is bounded and if condition* (C) *holds, then for all sufficiently large n there exist eigenvalues $\lambda_1^n, \ldots, \lambda_m^n$* of $\mathfrak{B}_n$ *with respect to $\mathfrak{A}_n$ such that $\lambda_j^n \to \lambda$ for every $j = 1, \ldots, m$. Moreover, if P_n is the orthogonal projection of X_n onto the direct sum of the eigenspaces corresponding to $\lambda_1^n, \ldots, \lambda_m^n$* then *$P_n v \to_d Pv$ for every $v \in V$.*

For variational approximation methods the conditions of Theorem 1.2 suffice to insure the convergence of the eigenvalues of the intermediate problems to those of the original problem. However, condition (C), while simple to state, is difficult to verify. Our purpose in the remaining sections will be to obtain conditions on the forms $\mathfrak{B}_n$, $\mathfrak{A}_n$ which will imply condition (C) but will be easier to verify.

2. **The general setting.** From now on $\mathfrak{A}$, $\mathfrak{A}_0$, $\mathfrak{A}_1, \ldots$ will always denote positive definite quadratic forms on the complex vector space V and $\mathfrak{B}$, $\mathfrak{B}_0$, $\mathfrak{B}_1, \ldots$ will denote real quadratic forms on V which are completely continuous with respect to $\mathfrak{A}$, $\mathfrak{A}_0$, $\mathfrak{A}_1, \ldots$ respectively. In addition, we shall always assume that V is separable with respect to $\mathfrak{A}^{1/2}$.

DEFINITION 2.1. The sequence $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ is an *allowable approximation* to $(\mathfrak{B}, \mathfrak{A})$ on V if and only if the following conditions are satisfied:

(A1) $\mathfrak{A}_n(v) \to \mathfrak{A}(v)$ for all $v \in V$.

(A2) There is a positive definite form $\mathfrak{A}_{LB}$ on V such that $\mathfrak{A}_{LB} \leqq \mathfrak{A}$ and $\mathfrak{A}_{LB} \leqq \mathfrak{A}_n$ for $n = 0, 1, 2, \ldots$.

(A3) $\mathfrak{B}$ is completely continuous with respect to $\mathfrak{A}_{LB}$ (and therefore is completely continuous with respect to each $\mathfrak{A}_n$).

(A4) $\|\mathfrak{B} - \mathfrak{B}_n\|_n \to 0$.

REMARK 2.1. The conditions (A1) through (A4) are quite similar to but somewhat weaker than the conditions formulated by Aronszajn in [5]. For variational approximation methods (A3) is always satisfied with either $\mathfrak{A}_{LB} = \mathfrak{A}_0$ or $\mathfrak{A}_{LB} = \mathfrak{A}$. For specific methods it is reasonably simple to choose the approximating problems so that the remaining conditions are satisfied (see, e.g., [5]),

REMARK 2.2. It follows from (A4) that the eigenvalues of $\mathfrak{B}_n$ relative $\mathfrak{A}_n$ converge to those of $\mathfrak{B}$ relative $\mathfrak{A}$ if and only if those of $\mathfrak{B}$ relative $\mathfrak{A}_n$ do. Thus in proofs we will often assume that $\mathfrak{B}_n = \mathfrak{B}$, $n = 0, 1, \ldots$.

Let $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ be an allowable approximation to $(\mathfrak{B}, \mathfrak{A})$ in V. Then $\{\mathfrak{A}_n\}$ is a discrete approximation to $\mathfrak{A}$, and the notions of discrete and weakly discrete convergence are defined. Also, conditions (A1)–(A4) insure that $\{\|\mathfrak{B}_n\|_n\}$ is bounded. Therefore convergence of the eigenvalues of $\mathfrak{B}_n$ relative $\mathfrak{A}_n$ to those of $\mathfrak{B}$ relative $\mathfrak{A}$ is insured provided condition (C) holds. In this connection note also that Remark 1.1 applies.

Let X_{LB} be the Hilbert space completion of V with respect to $\mathfrak{A}_{LB}{}^{1/2}$ and ι: $X \to X_{LB}$ be the unique bounded linear transformation such that $\iota v = v$ for all $v \in V$. Let $\mathfrak{B}_{LB}$ be the continuous extension of $\mathfrak{B}$ to X_{LB}. Then clearly

$$\mathfrak{B}(x, y) = \mathfrak{B}_{LB}(\iota x, \iota y) \text{ for all } x, y \in X. \tag{2.1}$$

[Choose sequences $\{u_n\}$, $\{v_n\}$ in V which converge to x, y respectively in X. They then converge to ιx, ιy respectively in X_{LB} and $\mathfrak{B}(x, y) = \lim \mathfrak{B}(u_n, v_n) = \mathfrak{B}_{LB}(\iota x, \iota y)$.]

PROPOSITION 2.1. Let $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ be an allowable approximation for $(\mathfrak{B}, \mathfrak{A})$ on V. Then (C) is equivalent to each of the following

$$\text{(C}'\text{)} \qquad v_n \in V,\ v_n \rightharpoonup_d x \Rightarrow \mathfrak{B}_n(v_n, v) \to \mathfrak{B}(x, v) \text{ for all } v \in V,$$

$$\text{(C}''\text{)} \qquad v_n \in V,\ v_n \rightharpoonup_d x,\ v_n \rightharpoonup_{\mathfrak{A}_{LB}} x' \Rightarrow \mathfrak{B}(x) = \mathfrak{B}_{LB}(x'),$$

and (C) is implied by the condition

$$\text{(B)} \qquad v_n \in V,\ v_n \rightharpoonup_d x,\ v_n \rightharpoonup_{\mathfrak{A}_{LB}} x' \Rightarrow x' = \iota x.$$

PROOF. By property (A4) no generality is lost if we assume $\mathfrak{B}_n = \mathfrak{B}$. Property (A3) insures that if $v_n \rightharpoonup_{\mathfrak{A}_{LB}} x'$, then $\mathfrak{B}(v_n) \to \mathfrak{B}_{LB}(x')$. The equivalence of (C) and (C″) follows easily from this fact, Remark 1.1, and Proposition 1.1. Clearly, also, (C) implies (C′).

If (C′) holds and if $v_n \rightharpoonup_d x$, $v_n \rightharpoonup_{\mathfrak{A}_{LB}} x'$ then by (2.1), (C′), and (A3),

$$\mathfrak{B}_{LB}(\iota x, v) = \mathfrak{B}(x, v) = \lim \mathfrak{B}(v_n, v) = \mathfrak{B}_{LB}(x', v) \text{ for all } v \in V.$$

Hence by continuity $\mathfrak{B}_{LB}(\iota x, y) = \mathfrak{B}_{LB}(x', y)$ for all $y \in X_{LB}$ and

$$\mathfrak{B}_{LB}(x') = \mathfrak{B}_{LB}(x', x') = \mathfrak{B}_{LB}(\iota x, x') = \mathfrak{B}_{LB}(\iota x, \iota x) = \mathfrak{B}(x, x) = \mathfrak{B}(x).$$

Thus (C′) implies (C″).

Since clearly (B) implies (C″), the theorem is proved.

The advantage of condition (B) is that it is expressed entirely in terms of $\mathfrak{A}$ and its approximating forms $\mathfrak{A}_n$. Thus we have:

THEOREM 2.1. Let $\mathfrak{A}$, $\{\mathfrak{A}_n\}_{n=0,1,\dots}$ be so chosen that (A1) and (A2) hold. If also (B) is satisfied then the eigenvalues of $\mathfrak{B}_n$ relative $\mathfrak{A}_n$ converge to those of $\mathfrak{B}$ relative $\mathfrak{A}$ for every $\mathfrak{B}$ which satisfies (A3) and every sequence $\{\mathfrak{B}_n\}$ which satisfies (A4).

Although (B) is not in general equivalent to (C), in some cases it is. In other cases, condition (B) follows automatically from (A1) and (A2).

PROPOSITION 2.2. *Let* $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ *be an allowable approximation to* $(\mathfrak{B}, \mathfrak{A})$

on V, and let $\mathfrak{B}_{LB}$ be the continuous extension of $\mathfrak{B}$ to X_{LB}. If $\mathfrak{B}_{LB}$ is positive definite on X_{LB}, then (B) *is equivalent to* (C).

PROOF. Assume (C′) holds, and let $v_n \rightharpoonup_d x$, $v_n \rightharpoonup_{\mathfrak{A}_{LB}} x'$. Then $\mathfrak{B}(v, \iota x) = \mathfrak{B}_{LB}(v, x')$ or $\mathfrak{B}(v, x' - \iota x) = 0$ for all $v \in V$. Choose a sequence $\{u_n\}$ in V such that $u_n \rightharpoonup_{\mathfrak{A}_{LB}} x' - x$. Then $\mathfrak{B}_{LB}(x' - \iota x) = 0$, so $x' = \iota x$.

PROPOSITION 2.3. *Let $\mathfrak{A}$, $\{\mathfrak{A}_n\}$ satisfy* (A1) *and* (A2) *with $\mathfrak{A}_{LB} = \mathfrak{A}$. Then* (B) *is satisfied.*

PROOF. For each $n = 0, 1, \ldots, \mathfrak{A}(v) \leqq \mathfrak{A}_n(v)$ for all $v \in V$. Hence there is a bounded self-adjoint linear operator A_n on X_n such that

$$\mathfrak{A}_n(A_n u, v) = \mathfrak{A}(u, v) \text{ for all } v \in V. \tag{2.1}$$

Let $u \in V$. Then

$$\begin{aligned} \mathfrak{A}_n(A_n u) &= \sup_{v \in V} \frac{|\mathfrak{A}_n(A_n u, v)|^2}{\mathfrak{A}_n(v)} = \sup_{v \in V} \frac{|\mathfrak{A}(u, v)|^2}{\mathfrak{A}_n(v)} \\ &\leqq \mathfrak{A}(u) \sup_{v \in V} \frac{\mathfrak{A}(v)}{\mathfrak{A}_n(v)} \leqq \mathfrak{A}(u). \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), $A_n u \rightharpoonup_d u$. Moreover, since $\mathfrak{A}_n(u) \to \mathfrak{A}(u)$, for every $\varepsilon > 0$ there exists an N such that for $n \geqq N$

$$\mathfrak{A}_n(A_n u) \geqq \frac{|\mathfrak{A}_n(A_n u, u)|^2}{\mathfrak{A}_n(u)} = \frac{\mathfrak{A}(u)^2}{\mathfrak{A}_n(u)} = \mathfrak{A}(u) \frac{\mathfrak{A}(u)}{\mathfrak{A}_n(u)} \geqq \mathfrak{A}(u)(1 - \varepsilon).$$

Thus $(1 - \varepsilon)\mathfrak{A}(u) \leqq \mathfrak{A}_n(A_n u) \leqq \mathfrak{A}(u)$ for $n \geqq N$; i.e., $\mathfrak{A}_n(A_n u) \to \mathfrak{A}(u)$. Thus $A_n u \to_d Au$, for all $u \in V$.

Finally, let for all $v_n \in V$, $v_n \rightharpoonup_d x$. By Lemma 1.1.3 $\mathfrak{A}(v_n, u) = \mathfrak{A}_n(v_n, A_n u) \to \mathfrak{A}(x, u)$ for all $u \in V$, so (B) clearly holds.

PROPOSITION 2.4. *Let $\mathfrak{A}$, $\{\mathfrak{A}_n\}$ satisfy* (A1) *and* (A2) *with $\mathfrak{A}_{LB} \leqq \mathfrak{A}_n \leqq \mathfrak{A}$, $n = 0, 1, \ldots$. If $\mathfrak{A}$ is equivalent to $\mathfrak{A}_{LB}$ then* (B) *is satisfied.*

PROOF. Since $\mathfrak{A}$ and $\mathfrak{A}_{LB}$ are equivalent we may take $X_{LB} = X$ and ι the identity operator on X. For $n = 0, 1, \ldots$ let L_n be the bounded self-adjoint linear operator on X such that

$$\mathfrak{A}(L_n u, v) = \mathfrak{A}_n(u, v) \text{ for all } u, v \in V. \tag{2.3}$$

Then $|\mathfrak{A}(L_n u, u)| = |\mathfrak{A}_n(u)| \leqq \mathfrak{A}(u)$ for all $u \in V$, so $\|L_n\| \leqq 1$. Using this fact and (2.3) we see easily that $L_n v \rightharpoonup_{\mathfrak{A}} v$ for all $v \in V$. (The operators $M_n = 1 - L_n$ are non-negative bounded self-adjoint operators which converge weakly to zero in X and hence converge strongly to zero [14, Cor. 3.2, page 452]).

Let $v_n \rightharpoonup_d x$, $v_n \rightharpoonup_{\mathfrak{A}_{LB}} x'$. Then $v_n \rightharpoonup_{\mathfrak{A}} x'$, so for all $v \in V$

$$\mathfrak{A}(x, v) = \lim_n \mathfrak{A}_n(v_n, v) = \lim_n \mathfrak{A}(v_n, L_n v) = \mathfrak{A}(x', v).$$

Therefore $x = x'$ and (B) is satisfied.

3. **The necessity of condition (B).** Let $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ be an allowable approximation to $(\mathfrak{B}, \mathfrak{A})$ on V. Then condition (B) is a sufficient, but not in general a necessary, condition for the convergence of the eigenvalues of $\mathfrak{B}_n$ relative $\mathfrak{A}_n$ to those of $\mathfrak{B}$ relative $\mathfrak{A}$. In this section, however, we shall prove a converse to Theorem 2.1; more specifically, we shall show that condition (B) is necessary if convergence of the eigenvalues is to hold for all appropriate choices of $\mathfrak{B}$ and $\{\mathfrak{B}_n\}$. First we consider a special case where condition (C) is necessary for convergence of the approximating eigenvalues.

PROPOSITION 3.1. *Let $\mathfrak{A}$, $\{\mathfrak{A}_n\}$ satisfy* (A1) *and* (A2) *and let $\mathfrak{B}$ be a real quadratic form of rank one on V, bounded with respect to $\mathfrak{A}_{LB}$. (Thus $\{(\mathfrak{B}, \mathfrak{A}_n)\}$ is an allowable approximation to $\{(\mathfrak{B}, \mathfrak{A})\}$ on V.) Let λ be the non-zero eigenvalue of $\mathfrak{B}$ relative $\mathfrak{A}$ and λ_n be the non-zero eigenvalue of $\mathfrak{B}$ relative $\mathfrak{A}_n$, $n = 0, 1, \ldots$. Then $\lambda_n \to \lambda$ if and only if condition* (C) *holds.*

PROOF. $\mathfrak{B}$ defines operators B, B_n on X, X_n respectively by the equations

$$\mathfrak{B}(u, v) = \mathfrak{A}(Bu, v) = \mathfrak{A}_n(B_n u, v) \text{ for all } u, v \in V. \tag{3.1}$$

Since $\mathfrak{B}$ has rank one, then $B = \lambda P$, $B_n = \lambda P_n$, where P, P_n are orthogonal projections in X, X_n respectively with one dimensional range. Thus

$$\mathfrak{B}(u, v) = \lambda_n \mathfrak{A}_n(P_n u, v) = \lambda \mathfrak{A}(Pu, v), \mathfrak{B}(u) = \lambda_n \mathfrak{A}_n(P_n u) = \lambda \mathfrak{A}(Pu).$$

If $\lambda_n \to \lambda$, then

$$\mathfrak{A}_n(P_n u, v) = \lambda \lambda_n^{-1} \mathfrak{A}(Pu, v) \to \mathfrak{A}(Pu, v),\ \mathfrak{A}_n(P_n u) = \lambda \lambda_n^{-1} \mathfrak{A}(Pu) \to \mathfrak{A}(Pu).$$

Thus $P_n u \to_d Pu$ for all $u \in V$. Let $v_n \in V$, $v_n \rightharpoonup_d x$. Then

$$\mathfrak{B}(u, v_n) = \lambda_n \mathfrak{A}_n(P_n u, v_n) \to \lambda \mathfrak{A}(Pu, x) = \mathfrak{B}(u, x) \text{ for all } u \in V.$$

It follows that $\lambda_n \to \lambda$ implies condition (C′) and therefore (C). The converse is clear.

LEMMA 3.1. *Let $\mathfrak{A}$, $\{\mathfrak{A}_n\}$ satisfy* (A1) *and* (A2). *If condition* (B) *is not satisfied, then there is a form $\mathfrak{B}$ of rank one on V, bounded with respect to $\mathfrak{A}$, such that condition* (C) *fails to hold.*

PROOF. Let $v_n \in V$, $v_n \rightharpoonup_d x$, $v_n \rightharpoonup_{\mathfrak{A}_{LB}} x'$ where $x' \neq \iota x$. Then $\iota x = \alpha x' + \beta y$, where $\mathfrak{A}_{LB}(x', y) = 0$, $\mathfrak{A}_{LB}(y) = 1$, and $\beta = \mathfrak{A}_{LB}(\iota x, y)$. We may assume $\beta \neq 0$ (else choose any non-zero $u_0 \in V$ and replace v_n, x, x' by $v_n + u_0$, $x + u_0$, $x' + u_0$ respectively). Let B be the bounded self-adjoint linear operator on X_{LB} such that $By = y$ and $B = 0$ on $X_{LB} - [y]$. Define

$$\mathfrak{B}_n(u, v) = \mathfrak{B}(u, v) = \mathfrak{A}_{LB}(Bu, v), u, v \in V.$$

Then $\mathfrak{B}$ is bounded with respect to $\mathfrak{A}_{LB}$(with bound one) and is ofrank one. But

$$\mathfrak{B}(x') = \mathfrak{A}_{LB}(Bx', x') = 0,$$

while

$$\mathfrak{B}(\iota x) = \mathfrak{A}_{LB}(B\iota x, \iota x) = \beta\mathfrak{A}_{LB}(y, \iota x) = |\beta|^2 \neq 0.$$

Hence (C″) and therefore (C) fails.

REMARK 3.1. The non-negative form $\mathfrak{B}$ constructed in the last example can be used to construct more complicated examples, as follows:

Since $\mathfrak{B}$ is of rank one, there is a decomposition $V = V' \dotplus [u_0]$, where $\mathfrak{B}(u_0) = 1$ and $\mathfrak{B}(v', v) = 0$ for all $v' \in V'$, $v \in V$. Then

$$X = X' \dotplus [u_0], \quad X_n = X_n' \dotplus [u_0],$$

where X', X_n' is the closure of V in X, X_n respectively. Let B, B_n be the bounded self-adjoint operator defined on X, X_n respectively by $\mathfrak{B}$:

$$\mathfrak{B}(u, v) = \mathfrak{A}_n(B_n u, v) = \mathfrak{A}(Bu, v)$$

and $Q: X \to X'$, $Q_n': X_n \to X_n'$ be the projections along $[u_0]$.

Let $\lambda > 0$ be the non-zero eigenvalue of $\mathfrak{B}$ relative $\mathfrak{A}$, and let $\mathfrak{B}'$ be any real quadratic form on V' which is completely continuous with respect to $\mathfrak{A}_{LB}$(restricted to V') and is such that $\| \mathfrak{B}' \|_{\mathfrak{A}_{LB}} \leqq \lambda/2$. Extend $\mathfrak{B}'$ to V by setting

$$\mathfrak{B}'(v) = \mathfrak{B}'(Qv) = \mathscr{B}'(Q_n v) \quad \text{for all } v \in V,$$

and define the new form $\hat{\mathfrak{B}} = \mathfrak{B} + \mathfrak{B}'$ on V. Then $\{(\hat{\mathfrak{B}}, \mathfrak{A}_n)\}$ is an allowable approximation to $(\hat{\mathfrak{B}}, \mathfrak{A})$ on V, but the eigenvalues of $\hat{\mathfrak{B}}$ relative $\mathfrak{A}_n$ do not converge to those of $\hat{\mathfrak{B}}$ relative $\mathfrak{A}$.

We combine the preceding results to obtain:

THEOREM 3.1. *Let* $\mathfrak{A}$, $\{\mathfrak{A}_n\}$ *satisfy* (A1) *and* (A2). *If* (*B*) *fails, then there are real quadratic frms* $\mathfrak{B}$ *on* V *and allowable approximations* $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ *to* $(\mathfrak{B}, \mathfrak{A})$ *on* V *such that the eigenvalues of* $\mathfrak{B}_n$ *relative* $\mathfrak{A}_n$ *do not converge to those of* $\mathfrak{B}$ *relative* $\mathfrak{A}$.

4. **Applications to standard variational methods.** In this section the results of the preceding sections are applied to obtain convergence criteria for the general types of variational approximation methods considered by Aronszajn in [5]. In each case one begins with a completely continuous problem coming from a pair $(\mathfrak{B}, \mathfrak{A})$ and constructs (completely continuous) approximating problems using pairs $(\mathfrak{B}_n, \mathfrak{A}_n)$ whose eigenvalues approximate those of the original problem.

For methods of the first kind (which includes the Rayleigh-Ritz and Weinstein methods) one can always take $\mathfrak{A}_{LB} = \mathfrak{A}_n = \mathfrak{A}$, $n = 0, 1, 2, \ldots$ and either $\mathfrak{B}_0 \leqq \mathfrak{B}_1 \leqq \ldots \leqq \mathfrak{B}$ or $\mathfrak{B}_0 \geqq \mathfrak{B}_1 \geqq \ldots \geqq \mathfrak{B}$. Thus conditions (A1), (A2), (A3) and (B) are automatically satisfied, and (A4) will hold provided only [14, Theorem 8.3.5]:

(A4)′ $$\mathfrak{B}_n(v) \to \mathfrak{B}(v) \quad \text{for all } v \in V.$$

This condition is therefore all that is needed to insure convergence of the approximating eigenvalues to the actual eigenvalues.

For methods of the second kind, two cases are considered. In the first case

$$\mathfrak{A}_0 \geqq \mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{A}, \quad \mathfrak{B}_0 \leqq \mathfrak{B}_1 \leqq \cdots \leqq \mathfrak{B}.$$

One may therefore take $\mathfrak{A}_{LB} = \mathfrak{A}$, and conditions (A2) and (A3) are automatically satisfied. $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ will be an allowable approximation provided only (A1) and (A4) hold, in which case (B) follows from Proposition 2.3. Convergence of the approximating eigenvalues to the actual eigenvalues is thus assured if one only chooses $(\mathfrak{B}_n, \mathfrak{A}_n)$ to satisfy:

(A1) $$\mathfrak{A}_n(v) \to \mathfrak{A}(v) \quad \text{for all } v \in V,$$

(A4) $$\| \mathfrak{B} - \mathfrak{B}_n \|_n \to 0 \text{ as } n \to \infty.$$

In the second case of methods of the second kind (which includes Aronszajn's method) one has

$$\mathfrak{A}_0 \leqq \mathfrak{A}_1 \leqq \cdots \leqq \mathfrak{A}, \quad \mathfrak{B}_0 \geqq \mathfrak{B}_1 \geqq \cdots \geqq \mathfrak{B}.$$

One need only take $\mathfrak{A}_{LB} = \mathfrak{A}_0$ to satisfy (A2). Then $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ will be an allowable approximation provided

(A1) $$\mathfrak{A}_n(v) \to \mathfrak{A}(v) \text{ for all } v \in V.$$

(A3) $\mathfrak{B}$ is completely continuous with respect to $\mathfrak{A}_0$ on V.

(A4) $$\|\mathfrak{B} - \mathfrak{B}_n\|_n \to 0.$$

If, in addition, $\mathfrak{A}_0$ is equivalent to $\mathfrak{A}$, condition (B) follows from Proposition 2.4, and convergence of the approximating eigenvalues to the actual ones is assured.

At this point we have recovered the convergence criteria established by Aronszajn in [5]. How the criteria can be satisfied for each method is further discussed there. In this paper Aronszajn asks whether, in the last case considered above, the condition that $\mathfrak{A}_0$ be equivalent to $\mathfrak{A}$ is necessary. We shall show in the next section that it is not, but that some condition (e.g., condition (B)) in addition to (A1), (A3), and (A4)) is necessary if the desired convergence of the eigenvalues is to hold in general.

5. **Further investigation of Aronszajn's method.** In Aronszajn's method, given the pair $(\mathfrak{B}, \mathfrak{A})$ on V, a base pair $(\mathfrak{B}_0, \mathfrak{A}_0)$ is chosen so that $\mathfrak{A}_0 \leqq \mathfrak{A}$, $\mathfrak{B} \leqq \mathfrak{B}_0$. In addition, $\mathfrak{A}_0$ is chosen so that $\mathfrak{B}$ is completely continuous with respect to $\mathfrak{A}_0$ and so that $\mathfrak{A}$ is quasi-bounded with respect to $\mathfrak{A}_0$. Thus

$$\mathfrak{A}(u, v) = \mathfrak{A}_0(Lu, v) \text{ for all } u, v \in V,$$

where L is a symmetric operator in X_0 with dense domain V.

To construct the intermediate problems one defines $\mathfrak{A}' = \mathfrak{A} - \mathfrak{A}_0$, and chooses a sequence $\{p_j\}$ in V whose elements are linearly independent modulo the null space N of $\mathfrak{A}'$ in V. Let P_n be the projection, orthogonal with respect to $\mathfrak{A}'$, of V onto $[p_1, \ldots, p_n]$.
Thus

$$V = N_n \dotplus [p_1, \ldots, p_n],$$

where $N_n = \{v \in V: \mathfrak{A}'(u, p_j) = 0, j = 1, \ldots, n\}$, and P_n is the projection of V along N_n.

Define

$$\mathfrak{A}_n(u) = \mathfrak{A}_0(u) + \mathfrak{A}'(P_n u), \; n = 1, 2, \ldots$$

Then $\mathfrak{A}_0 \leqq \mathfrak{A}_1 \leqq \ldots \leqq \mathfrak{A}$, (A2) holds with $\mathfrak{A}_{LB} = \mathfrak{A}_0$, and (A3) holds. (A1) will hold provided the sequence $\{p_j\}$ is so chosen that

(A1′) $$V' = N + \operatorname{span} \langle p_j \rangle \text{ is dense in } X.$$

Analogously one defines $\mathfrak{B}' = \mathfrak{B}_0 - \mathfrak{B}$ and the null space N_1 of $\mathfrak{B}'$ in V, then chooses a sequence $\{q_n\}$ in V linearly independent modulo N_1. Let

$$\mathfrak{B}_n(u) = \mathfrak{B}_0(u) - \mathfrak{B}'(Q_n u),$$

where Q_n is the projection, orthogonal with respect to $\mathfrak{B}'$, of V onto $[q_1, \ldots, q_n]$. Then $\mathfrak{B}_0 \geqq \mathfrak{B}_1 \geqq \ldots \geqq \mathfrak{B}$, and (A4) holds provided

(A4′) $$M + \operatorname{span} \langle q_j \rangle \text{ is dense in } X.$$

If $\{p_j\}$, $\{q_j\}$ are chosen to satisfy (A1′) and (A4′), then $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ is an allowable approximation to $(\mathfrak{B}, \mathfrak{A})$ on V. We wish to know when (B) is satisfied.

First, note that since $\mathscr{A}$ is quasi-bounded with respect to $\mathscr{A}_0$, it is closable in the Hilbert space X_0 **[14]**, and the mapping ι of X into X_0 is injective. Thus we may consider X as a subset of X_0 and ι as an embedding. Moreover, L has a positive self-adjoint extension L_0 with square root $L_0^{1/2}$ and domain $\mathscr{D}(L_0)$ such that

$$V' \subset V = \mathscr{D}(L) \subset \mathscr{D}(L_0) \subset \mathscr{D}(L_0^{1/2}) = X \subset X_0,$$
$$\mathfrak{A}(x, y) = \mathfrak{A}_0(L_0 x, y) \text{ for all } x \in \mathscr{D}(L_0), y \in X,$$
$$\mathfrak{A}(x, y) = \mathfrak{A}_0(L_0^{1/2} x, L_0^{1/2} y) \text{ for all } x, y \in X.$$

Note that, in view of (A1′), V' is a core of $L_0^{1/2}$.

Next note that the eigenvalues of $\mathfrak{B}$ relative to $\mathfrak{A}$ and of $\mathfrak{B}_n$ relative to $\mathfrak{A}_n$ do not change if we replace V by the dense subspace V' of X, nor does the discrete approximation of X by $\{X_n\}$. Thus condition (B) may be replaced by:

$$\text{(B}'\text{)} \qquad v_n \in V',\ v_n \rightharpoonup_d x,\ v_n \rightharpoonup_{\mathfrak{A}_0} x_0 \Rightarrow x = x_0.$$

Moreover,

LEMMA 5.1. *Let $v_n \in V'$. Then $v_n \rightharpoonup_d x$ if and only if $\{\mathfrak{A}_n(v_n)\}$ is bounded and* $\mathfrak{A}(v_n, v) \to \mathfrak{A}(x, v)$ for all $v \in V'$.

(If $v \in V'$, then for all sufficiently large n, $P_n\ v = v$ and $\mathfrak{A}_n(v_n, v) = \mathfrak{A}(v_n, v)$.) Finally,

THEOREM 5.1. *If $L(V')$ is dense in X_0, then condition* (B′) *is satisfied.*

PROOF. Let $v_n \in V'$, $v_n \rightharpoonup_d x$, $v_n \rightharpoonup_{\mathfrak{A}_0} x_0$. Then for all $v \in V'$

$$\mathfrak{A}_0(Lv, x) = \mathfrak{A}(v, x) = \lim_n \mathfrak{A}(v, v_n) = \lim_n \mathfrak{A}_0(Lv, v_n) = \mathfrak{A}_0(Lv, x_0),$$

so that $\mathfrak{A}_0(Lv, x - x_0) = 0$ for all $v \in V'$. Then $x = x_0$, and the theorem is proved.

Since $\mathfrak{A}_0(L_0x, x) = \mathfrak{A}(x) \geqq \mathfrak{A}_0(x)$, $x \in \mathscr{D}(L_0)$, L_0 has a bounded inverse in X_0. Therefore $L(V') = L_0(V')$ is dense if and only if V' is a core of L_0. Thus, if we only choose V' to be a core of L_0, both (A1′) and (B′) are satisfied, and the approximate eigenvalues obtained using Aronszajn's method converge to those of the original problem. One of Aronszajn's questions is now answered; i.e., it is not necessary to choose $\mathfrak{A}_0$ equivalent to $\mathfrak{A}$ in order to insure the desired convergence. However, the following example shows that some condition is needed in addition to the allowability of $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ with respect to $(\mathfrak{B}, \mathfrak{A})$.

EXAMPLE. Let X_0 be a separable Hilbert space with norm $\|\cdot\|_0$, and L_0 be an unbounded self-adjoint operator with dense domain $\mathscr{D}(L_0)$ in X_0 and such that $(L_0x, x)_0 \geqq (1 + k)\|x\|_0^2$ for all $x \in X_0$. Let V be any subset of $\mathscr{D}(L_0)$ which is a core of $L_0^{1/2}$ but not of L_0, and L be the restriction of L_0 to V.

Define $\mathfrak{A}_0(u) = \|u\|_0^2$, $\mathfrak{A}(u, v) = \mathfrak{A}_0(Lu, v)$, and $\mathfrak{A}'(u) = \mathfrak{A}(u) - \mathfrak{A}_0(u)$ for all v, $u \in V$. Then $\mathfrak{A} \geqq (1 + k)\mathfrak{A}_0$, $\mathfrak{A}'$ is equivalent to $\mathfrak{A}$ ($\mathfrak{A}' \leqq \mathfrak{A} \leqq (1 + 1/k)\mathfrak{A}'$), and $X = \mathscr{D}(L_0^{1/2})$ is a Hilbert space with norm $\mathfrak{A}(x)^{1/2} = \mathfrak{A}_0(L_0^{1/2}x)^{1/2}$, $x \in X$.

Let $\{p_j\}$ be any sequence in V which is orthonormal with respect to $\mathfrak{A}'$ and which is complete in X with respect to $\mathfrak{A}$ (and $\mathfrak{A}'$). Let $V' = \operatorname{span}\langle p_j\rangle$ and construct the intermediate forms $\mathfrak{A}_n(u) = \mathfrak{A}_0(u) + \mathfrak{A}'(P_nu)$ according to Aronszajn's procedure.

Since $L(V)$ is not dense in X_0, there exists $y_0 \neq 0$ such that $\mathfrak{A}_0(y_0, Lv) = 0$ for all $v \in V$. Let $x_0 = M_0^{-1}y_0$, where $M_0 = L_0 - 1$. (Note that $\mathfrak{A}'(x) = \mathfrak{A}_0(M_0x, x) \geqq k\mathfrak{A}_0(x)$ for all $x \in \mathscr{D}(M_0) = \mathscr{D}(L_0) \subset X$.) Then $x_0 \in X$ and

$$\mathfrak{A}_0(y_0, x) = \mathfrak{A}_0(M_0x_0, x) = \mathfrak{A}'(x_0, x) \text{ for all } x \in X. \tag{5.1}$$

Choose a sequence $\{u_n\}$ in V' such that $u_n \rightharpoonup_{\mathfrak{A}_0} y_0$ in X_0. Then for all $v \in V'$, (5.1) implies that

$$\begin{aligned}\mathfrak{A}'(u_n, v) &= \mathfrak{A}_0(u_n, M_0v) = \mathfrak{A}_0(u_n, L_0v) - \mathfrak{A}_0(u_n, v)\\ &\to \mathfrak{A}_0(y_0, Lv) - \mathfrak{A}_0(y_0, v) = -\mathfrak{A}'(x_0, v).\end{aligned}$$

Using a standard diagonal process we choose a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$|\mathfrak{A}'(v_n, p_j)|^2 \leqq |\mathfrak{A}'(x_0, p_j)|^2 + \frac{1}{j^2}, \; n \geqq j. \tag{5.2}$$

Using (5.2) we see easily that

$$\begin{aligned}\mathfrak{A}'(P_nv_n) &= \sum_{j=1}^{n} |\mathfrak{A}'(v_n, p_j)|^2 \leqq \sum_{j=1}^{n} \left[|\mathfrak{A}'(x_0, p_j)|^2 + \frac{1}{j^2}\right]\\ &\leqq \sum_{j=1}^{\infty} |\mathfrak{A}'(x_0, p_j)|^2 + \sum_{j=1}^{\infty} \frac{1}{j^2} = \mathfrak{A}'(x_0) + \frac{\pi^2}{6}.\end{aligned}$$

Therefore $\mathfrak{A}_n(v_n) = \mathfrak{A}_0(v_n) + \mathfrak{A}'(P_nv_n)$ is bounded.

Since also

$$\mathfrak{A}(v_n, v) = \mathfrak{A}_0(v_n, Lv) \to \mathfrak{A}_0(y_0, Lv) = 0 \text{ for all } v \in V',$$

Lemma 5.1 implies that $v_n \rightharpoonup_d 0$. But $v_n \rightharpoonup_{\mathfrak{A}_0} y_0 \neq 0$. Hence condition (B′) fails.

On the other hand, the choice of V' insures that (A1′) and (A2) hold. Thus Theorem 3.1 implies that there are forms $\mathfrak{B}$ and allowable approximations $\{(\mathfrak{B}_n, \mathfrak{A}_n)\}$ to $(\mathfrak{B}, \mathfrak{A})$ such that the approximating eigenvalues do not converge to the actual ones.

REMARK 5.1. Note in the preceding example that once V was chosen, there was no way to choose V' so that (B′) would hold; i.e., the failure of (B′) did not depend on the particular choice of $\{p_j\}$.

REMARK 5.2. See Weinberger [**23**, Theorem 4.8.2] for other conditions which insure convergence of the approximating eigenvalues for Aronszajn's method. Notice that if condition (B′) holds, then Weinberger's conditions are satisfied for every choice of $\mathfrak{B}$.

6. The Bazley distinguished choice. In order to compute the eigenvalues of the intermediate nth problem from those of the auxiliary problem one investigates the zeros and poles of an associated perturbation determinant. In the case of Aronszajn's method this determinant is (with $\mathfrak{B}_n = \mathfrak{B}$ for convenience) **[5] [12] [23] [24]**:

$$W(\lambda) = \det\{\mathfrak{A}_0((B_0 - \lambda I)^{-1} Mp_i - \frac{1}{\lambda} p_i, Mp_j)\}, \tag{6.1}$$

where B_0, M are operators in X_0 defined by $\mathfrak{B}(u, v) = \mathfrak{A}_0(B_0 u, v)$, $\mathfrak{A}'(u, v) = \mathfrak{A}_0(Mu, v)$ for all $u, v \in V$, and $i, j = 1, \ldots, n$.

Although the auxiliary problem is chosen so that $(B_0 - \lambda I)^{-1} Mp_i$ is known, it may, e.g., be known only in the form of an infinite series, and finding zeros and poles of $W(\lambda)$ may be quite difficult. The idea of Bazley's distinguished choice [7] **[23]** is to choose $Mp_j = x_j$, where $\{x_j\}$ is an orthonormal sequence in X_0 of eigenvectors of B_0. Then $W(\lambda)$ can be replaced by the simpler determinant

$$W(\lambda) = \det\{\lambda_i \mathfrak{A}'(p_i, p_j) - \lambda(\delta_{ij} + \mathfrak{A}'(p_i, p_j))\}, \tag{6.2}$$

where λ_i is the eigenvector of B_0 corresponding to x_i. However, there remains the question convergence of the approximating eigenvalues. For cases where the convergence properties cannot easily be deduced using ad hoc measures, we offer the following modifications:

Choose ε, $0 < \varepsilon < 1$, and replace $\mathfrak{A}_0$ by $\hat{\mathfrak{A}}_0 = (1 - \varepsilon)\,\mathfrak{A}_0$; i.e., use $(\mathfrak{B}, \hat{\mathfrak{A}}_0)$ to define the auxiliary problem to $(\mathfrak{B}, \mathfrak{A})$ instead of $(\mathfrak{B}, \mathfrak{A}_0)$. Then $\mathfrak{A}'$, B_0, M are replaced by $\hat{\mathfrak{A}}' = \mathfrak{A}' + \varepsilon\mathfrak{A}_0$, $\hat{B}_0 = B_0/(1 - \varepsilon)$, and $\hat{M} = (M + \varepsilon)/(1 - \varepsilon)$. We assume that $\{x_j\}$ is a complete orthonormal set of eigenvectors of B_0 in X_0 (and therefore of $\hat{B}_0$). $\hat{M}$ is invertible, and we may define $\hat{p}_j = \hat{p}_j(\varepsilon) = \hat{M}^{-1} x_j$, $j = 1, 2, \ldots$, and the associated forms $\mathfrak{A}_n$ following Aronszajn's method. Then $V' = \operatorname{span} \langle \hat{p}_j \rangle$ will be a core of $\hat{M}$, and conditions (A1′) and (B′) will hold. The associated Weinstein-Aronszajn determinant for the nth approximation is

$$W_\varepsilon(\lambda) = \det\left\{\frac{\lambda_i}{1-\varepsilon}\,[\mathfrak{A}'(\hat{p}_i, \hat{p}_j) + \varepsilon\,\mathfrak{A}_0(\hat{p}_i, \hat{p}_j)] - \lambda\,[\delta_{ij} + \mathfrak{A}'(\hat{p}_i, \hat{p}_j) + \varepsilon\mathfrak{A}_0(\hat{p}_i, \hat{p}_j)]\right\}, \quad i, j = 1, \ldots, n.$$

which, although more complicated than (6.2), is easier to handle than (6.1). Moreover it yields approximate eigenvalues (not the same as those obtained from (6.2) of course) which are sure to converge to those of $\mathfrak{B}$ relative $\mathfrak{A}$ as $n \to \infty$.

7. **Special cases and variations.** In this section we consider the special case that $\mathfrak{A}(u\ \ v) = (Au, v)_0$, where A is a (positive) self-adjoint operator in a Hilbert space $\mathscr{H}_0$ with domain $V = \mathscr{D}(A) \subset \mathscr{H}_0$. Similarly, $\mathfrak{A}_0(u, v) = (A_0 u, v)_0$, where A_0 is a self-adjoint operator with bounded inverse in $\mathscr{H}_0$,

and so chosen that $V \subset \mathscr{D}(A_0)$ and $0 \leqq \mathfrak{A}_0(u) \leqq \mathfrak{A}(u)$ for all $u \in V$. Then $\mathfrak{A}$ is quasi-bounded with respect to $\mathfrak{A}_0$ with $L = A_0^{-1}A$.

For this situation we will consider two methods—Aronszajn's method and the Bazley-Fox variation of Aronszajn's method. In each case we shall show how the intermediate forms $\mathfrak{A}_n$ may be so chosen that (A1′) and (B′) are satisfied. Thus, if $\mathfrak{B}$ satisfies (A3), and if the forms $\mathfrak{B}_n$ are chosen to satisfy (A4), then Theorem 2.1 will apply.

For Aronszajn's method we make use of the results of section 5. Specifically, we define intermediate forms $\mathfrak{A}_n$ using Aronszajn's method and a sequence $\{p_j\}$ in V, linearly independent modulo the null space N of $\mathfrak{A}'$, such that $V' = N \dotplus \operatorname{span} \langle p_j \rangle$ is dense in V with respect to the norm $\mathfrak{A}^{1/2}$, and such that $A(V')$ is dense in $\mathscr{H}_0$. Clearly, (A1′) is satisfied. Moreover, given any $x_0 \in X_0$, $y_0 = A_0 x_0 \in \mathscr{H}_0$, and there is a sequence $\{v_n\}$ in V' such that $Av_n \to_{\mathscr{H}_0} y_0$. But then $Lv_n = A_0^{-1} Av_n \to_{\mathscr{H}_0} x_0$. Thus $L(V')$ is dense in X_0, and condition (B′) is satisfied.

As an example, consider the case that A is an elliptic linear differential operator of order $2m$ on $V = P^{2m}(\Omega) \cap P_0^m(\Omega)$, where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary (see [1] [2] [3] for properties of spaces P^α of Bessel Potentials and for specific conditions on Ω and A). We assume in addition that $\mathfrak{A}(u, v) = (Au, v)_{L_2}$ defines $\mathfrak{A}$ as a positive definite quadratic form equivalent to the P^m norm on V and such that $A^{-1}: L_2(\Omega) \to V$ viewed as a mapping of $L_2(\Omega)$ into $P^{2m}(\Omega)$, is bounded.

Let A_0 be an elliptic differential operator of order $2m_0 \leqq 2m$ on $V_0 = P^{2m_0}(\Omega) \cap P_0^{m_0}(\Omega)$, such that $\mathfrak{A}_0(u, v) = (A_0 u, v)_{L_2}$ defines a positive definite quadratic form equivalent to the P^{m_0} norm on V_0 and such that A_0^{-1}: $L_2(\Omega) \to V_0 \subset P^{2m_0}(\Omega)$. Also assume that $\mathfrak{A}$ is quasi-bounded with respect to $\mathfrak{A}_0$ and $\mathfrak{A} \geqq \mathfrak{A}_0$.

Choose a sequence $\{p_j\}$ in V as in section 5 such that $V' = N \dotplus \operatorname{span} \langle p_j \rangle$ is dense in V with respect to the P^{2m} norm. Then V' is dense in V with respect to the P^m norm and therefore with respect to the norm $\mathfrak{A}^{1/2}$. Moreover, if $x \in L_2(\Omega)$, then there is a sequence $\{v_n\}$ in V' such that

$$v_n \xrightarrow[P^{2m}]{} A^{-1}x \in V.$$

Then $Av_n \to x$, so $A(V')$ is dense in $L_2(\Omega)$. It follows that the forms $\mathfrak{A}_n$, defined using $\{p_j\}$, are such that (A1′) and (B′) hold.

Finally, we consider the Bazley-Fox variation of Aronszajn's method, applied to the situation described in the first paragraph of this section [**8**] [**23**] [**24**]. This method requires that the form $\mathfrak{A}' = \mathfrak{A} - \mathfrak{A}_0$ be expressible as

$$\mathfrak{A}'(u, v) = (Cu, Cv)_1,\ u, v \in V,$$

where C is a linear transformation from $\mathscr{H}_0$ into a second Hilbert space

$\mathscr{H}_1$. It is further assumed that $V \subset \mathscr{D}(C)$, and that C has an adjoint $C^*: \mathscr{H}_1 \to \mathscr{H}_0$ defined by

$$(Cx, y)_1 = (x, C^*y)_0 \text{ for all } x \in \mathscr{D}(C),\ y \in \mathscr{D}(C^*).$$

Choose a sequence $\{p_j\}$ in V, linearly independent modulo the null space N of C, and let Q_n be the orthogonal projection of $\mathscr{H}_1$ onto $[Cp_1, \ldots, Cp_n]$. Define

$$\mathfrak{A}_n(u) = \mathfrak{A}_0(u) + \|Q_nCu\|^2_{\mathscr{H}_1} \text{ for all } u \in V.$$

Then $0 \leqq \mathfrak{A}_0(u) \leqq \mathfrak{A}_1(u) \leqq \ldots \leqq \mathfrak{A}(u)$.

Suppose we can choose $\{p_j\}$ so that $V' = N \dotplus \operatorname{span}\langle p_j\rangle$ is dense in V, that $A(V')$ is dense in X, that $C(V') \subset \mathscr{D}(C^*)$, and that $C(V')$ is dense in $\mathscr{H}_1$. (This is always possible if $C(V)$ is a subspace of $\mathscr{D}(C^*)$ dense in $\mathscr{H}_1$.) Then

$$\mathfrak{A}_n(u, v) = ((A_0 + C^*Q_nC)u, v)_0 \text{ for all } u, v \in V,$$

where $A_n = A_0 + C^*Q_nC$ is a finite dimensional perturbation of A_0 (see [**6**]).

The choice of V' insures that (A1′) holds and that Lemma 5.1 still applies. Suppose $\{v_n\}$ is a sequence in V' such that $v_n \rightharpoonup_d x$, $v_n \rightharpoonup_{\mathfrak{A}_0} x_0$. Then for all $y \in \mathscr{H}_0$,

$$(v_n, y)_0 = \mathfrak{A}_0(v_n, y_0) \to \mathfrak{A}_0(x_0, y_0) = (x_0, y)_0,$$

where $y_0 = A_0^{-1}y \in \mathscr{D}(A_0)$; i.e., $v_n \rightharpoonup_{\mathscr{H}_0} x_0$. But then for all $v \in V'$,

$$(x, Av)_0 = \mathfrak{A}(x, v) = \lim_n \mathfrak{A}(v_n, v) = \lim_n (v_n, Av)_0 = (x_0, Av)_0,$$

so $x = x_0$. Therefore (B′) and (B) hold for this approximation scheme.

It is hoped that the examples of this section and of section 4, 5, and 6 illustrate how the conditions of section 3 can be used to choose approximating problems so that the desired convergence of the approximating eigenvalues is achieved.

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