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TENSOR PRODUCTS AND SEMISIMPLE MODULAR REPRESENTATIONS OF FINITE GROUPS AND RESTRICTED LIE ALGEBRAS

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Chevalley has shown (Proposition 2, §5, IV of **[2]**, or see theorem 12.2 of **[4]**) that for any group G and finite dimensional semisimple G-modules V, W over a field k of characteristic 0, $V \otimes_k W$ is semisimple as a G-module (where for $g \in G$, $v \otimes w \in V \otimes W$, the action of G is given by $g\cdot(v \otimes w) = (g\cdot v) \otimes (g\cdot w)$). If G is finite, this follows trivially from Maschke's Theorem (10.8 of **[3]**). Chevalley's proof follows upon a series of steps: reduce to the connected algebraic group case, then to the Lie algebra case, and finally the proof of the corresponding theorem for Lie algebras in characteristic 0. For representations of groups and Lie algebras over fields of characteristic $p \neq 0$ the theorem fails to be true; in fact most of the steps in the reduction break down completely. The purpose of this paper is to investigate when Chevalley's result holds for finite groups and restricted Lie algebras in positive characteristic. In particular we show that Chevalley's theorem holds for a finite group over all fields if and only if the group is nilpotent. For a fixed field k of characteristic $p \neq 0$, a finite group G (resp. restricted Lie algebra $\mathscr{G}$) enjoys the tensor product property for semisimple representations provided the group (Lie algebra) can be expressed as an extension of a group (Lie algebra) all of whose representations are semisimple by a unipotent group (Lie algebra). To show this we use the fact that modules for G (resp. $\mathscr{G}$) over a field k correspond in a natural way to modules for the group algebra $k[G]$ (resp. restricted universal enveloping algebra $U_p(\mathscr{G})$) and then exploit the Hopf algebra (more precisely the bialgebra) structure of these algebras.

We review here the facts we need about Hopf algebras; more details, including proofs of any unverified assertions, may be found in **[9]**. All vector spaces, algebras, maps, tensor products, etc. are defined over a fixed groundfield k of characteristic $p \neq 0$, and everything is explicitly assumed to be finite dimensional over k. If A is a vector space we say (A, Δ, ε) (or just A for short) is a coalgebra if Δ: $A \to A \otimes A$ and ε: $A \to k$ are linear maps which satisfy the following:

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(1) $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$ (coassociativity)
(2) $(I \otimes \varepsilon) \circ \Delta = I = (\varepsilon \otimes I) \circ \Delta$ (counity)
where I denotes the appropriate identity map. If A and C are coalgebras, a linear map f: $A \to C$ is said to be a coalgebra map provided $(f \otimes f) \circ \Delta_A = \Delta_C \circ f$ and $\varepsilon_C \circ f = \varepsilon_A$. If B is an algebra with multiplication map m: $B \otimes B \to B$ and unit map η: $k \to B$, identities (1) and (2) are just dual versions of the associativity and unitary properties of m, and η.

If now A is both an algebra (with structure maps m, η) and a coalgebra (with structure maps Δ, ε), we say A is a bialgebra if Δ, ε are algebra maps; or equivalently, if m, η are coalgebra maps. If in addition there exists a map S: $A \to A$ which satisfies $m \circ (S \otimes I) \circ \Delta = \eta \circ \varepsilon = m \circ (I \otimes S) \circ \Delta$, A is called a Hopf algebra. The map S is called the antipode of the Hopf algebra, and plays a role analogous to that of the inverse map for groups; if it exists, is unique and is both an algebra and coalgebra antimorphism.

EXAMPLE 1. Let G be a group and $k[G]$ the group algebra of G. Recall $k[G]$ is a vector space with the elements of G as a basis, and multiplication the product in G extended by distributivity (see [3] for more details). It is easy to check that $k[G]$ can be made into a Hopf algebra by defining $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$. $S(g) = g^{-1}$ for all $g \in G$. and extending by linearity.

EXAMPLE 2. Let $\mathscr{G}$ be a restricted Lie algebra over k (with p-map $X \to X^{[p]}$) and $U_p(\mathscr{G})$ its restricted universal enveloping algebra. The elements of $\mathscr{G}$ generate $U_p(\mathscr{G})$ as an algebra, subject to the relations $[X, Y] = XY - YX$, $X^{[p]} = X^p$, all $X, Y, \in \mathscr{G}$. Further dim $U_p(\mathscr{G}) = p^n$ if dim $\mathscr{G} = n$ (see [1] or [5] for more details; the terminology "p-Lie algebra" is used in [1]). $U_p(\mathscr{G})$ becomes a Hopf algebra when we define $\Delta(X) = 1 \otimes X + X \otimes 1$, $\varepsilon(X) = 0$, $S(X) = -X$ for $X \in \mathscr{G}$.

For any Hopf algebra H the set $G(H) \equiv \{h \in H \mid \Delta(h) = h \otimes h$ and $\varepsilon(h) = 1\}$ is called the set of grouplike elements of H. It is routine to check that $G(H)$ is a subgroup of the group of units of H, $S(g) = g^{-1}$ for all $g \in G(H)$, and that the elements of $G(H)$ are linearly independent in H. Further $G(k[G]) = G$, and a Hopf algebra H is a group algebra if and only if it is spanned by its grouplike elements. The set $P(H) \equiv \{h \in H \mid \Delta(h) = 1 \otimes h + h \otimes 1$ and $\varepsilon(h) = 0\}$ is called the set of primitive elements of H. $P(H)$ is a restricted Lie algebra under the bracket operation $[x, y] = xy - yx$, $x, y \in P(H)$, and $P(U_p(\mathscr{G})) = \mathscr{G}$. In fact a finite dimensional Hopf algebra H is the restricted universal enveloping algebra of its space of primitive elements if and only if $P(H)$ generates H as an algebra (see chap. 13 of [9]).

If H and K are Hopf algebras a linear map $\pi: H \to K$ is a Hopf algebra map provided it is simultaneously an algebra and a coalgebra map. It then follows that π respects the antipode (i.e. $S_K \circ \pi = \pi \circ S_H$) and one may readily check that $\pi(G(H)) \subseteqq G(K)$, $\pi(P(H)) \subseteqq P(K)$, and that π restricted to $G(H)$ (resp. $P(H)$) is a group (resp. restricted Lie algebra) homomorphism. Finally group homomorphisms and Lie algebra homomorphisms induce Hopf algebra maps between the group algebras and enveloping algebras in the obvious way.

An ideal I of a Hopf algebra H is called a Hopf ideal if $\Delta(I) \subseteqq I \otimes H + H \otimes I$, $I \subseteqq \ker(\varepsilon)$, and $S(I) \subseteqq I$. The Hopf ideals of H are precisely the kernels of Hopf algebra maps with domain H; the properties they enjoy are exactly what is needed to endow the quotient H/I with a canonical Hopf algebra structure so that the projection $H \to H/I$ is a map of Hopf algebras.

For any Hopf algebra H we let $H^+ \equiv \ker(\varepsilon)$. If G is a group it is easy to see that the set $\{e - g \mid g \in G, g \neq e \equiv \text{identity of } G\}$ is a basis for $k[G]^+$. Note that if N is a normal subgroup of G we have $k[N]^+ \cdot k[G] = k[G] \cdot k[N]^+$, i.e. the right (or left) ideal of $k[G]$ generated by $k[N]^+$ is a twosided ideal. In fact it is a Hopf ideal, and the following proposition asserts that all Hopf ideals of $k[G]$ are of this form.

PROPOSITION 1. *Let G be a finite group, K a Hopf algebra, and $\pi : k[G] \to K$ a surjective Hopf algebra map. Then there exists a normal subgroup N of G such that* $\ker(\pi) = k[N]^+ \cdot k[G]$ *and* $K \cong k[G/N]$ *as Hopf algebras.*

PROOF. Since G spans $k[G]$, $\pi(G)$ spans K and by the linear independence of the grouplike elements of K we have $G(K) = \pi(G)$. Thus $K = k[\pi(G)]$. Let $N = \{g \in G \mid \pi(g) = 1\}$. Then since π restricted to G is a group homomorphism, N is normal in G. Further since π is surjective, we have $G/N \cong \pi(G)$ as groups, and so $K \cong k[G/N]$.

It only remains to show $\ker(\pi) = k[N]^+ \cdot k[G]$. Let $I = k[N]^+ \cdot k[G]$. Clearly $I \subseteqq \ker(\pi)$ because I is generated as an ideal by $\{e - n \mid n \in N\}$ If we let $|G/N| = \alpha$ and $|N| = \beta$, then $\dim k[G] = \alpha\beta$ and we have

$$\dim (\ker(\pi)) = \dim k[G] - \dim k[G/N] = \alpha(\beta - 1).$$

We next show $\dim (I) \geqq \alpha(\beta - 1)$, from which the result follows.

Let $\{g_1, g_2, \ldots, g_\alpha\}$ be a distinct set of coset representatives of N in G, $\{n_1, n_2, \ldots, n_\beta\}$ a list of the elements of N, with $g_1 = n_1 = e$. If we let $B_i = \{g_i - g_i n_j \mid 2 \leqq j \leqq \beta\}$ and $B = B_1 \cup \cdots \cup B_\alpha$, then $|B_i| = \beta - 1$ for all i and $B_i \cap B_j = \varnothing$, $i \neq j$. Moreover $B \subseteqq I$ and $|B| = \alpha(\beta - 1)$. We will be done once we show the set B is linearly independent. Now each B_i is a linearly independent set since $g_i N = \{g_i n_j \mid 2 \leqq j \leqq \beta\} \cup \{g_i\}$. Further the B_i's are linearly independent of each other since they correspond to distinct cosets of N in G. Thus B is a basis of $\ker(\pi)$ and we are done.

If A is a finite dimensional k-algebra and V a vector space, V is an A-module provided we are given a representation of A in the k-endomorphisms of V (i.e. an algebra map $\pi\colon A \to \mathrm{End}_k(V)$). The kernel of the representation is called the annihilator of the A-module V. V is said to be a simple A-module if it has no nontrivial A-submodules, and semisimple if it is a sum (necessarily direct) of simple modules. If V is a simple A-module, it follows from Wedderburn's theorem that the annihilator of V is a maximal ideal of A. We recall that the Jacobson radical $J(A)$ of A is defined to be the largest nilpotent ideal of A. $J(A)$ may be characterized as the intersection of the maximal left ideals of A, or as the intersection of the annihilators of the simple left A-modules (equivalently with "right" in place of "left"). Further $A/J(A)$ is semisimple, i.e. all its (left) modules are semisimple. See [3] for more details.

Note that if V is an A-module with $J(A) \cdot V = (0)$, then V becomes a (semisimple!) $A/J(A)$ module, and the two module structures on V "agree", so V is semisimple as an A-module. Conversely let V be semisimple, say $V = V_1 \oplus \cdots \oplus V_r$ with each V_i simple. If $\pi\colon A \to \mathrm{End}(V_i)$, $1 \leqq i \leqq r$ are the given representations, then $[\bigcap_{i=1}^r \ker(\pi_i)] \cdot V = 0$. But $J(A) \subseteqq \bigcap_{i=1}^r \ker(\pi_i)$ and thus we have the following.

(P) V is semisimple as an A-module (A finite dimensional) if and only if $J(A) \cdot V = (0)$

The motivation for considering the Hopf algebra structure on $k[G]$ and $U_p(\mathscr{G})$ is as follows. If V, W are (left) modules over an algebra A, then $V \otimes W$ is canonically an $A \otimes A$-module, with the action given by $(a \otimes b) \cdot (v \otimes w) \equiv (a \cdot v) \otimes (b \cdot w)$ for $a, b \in A$, $v \in V$, $w \in W$. Thus if A is a bialgebra, $V \otimes W$ automatically becomes an A-module by pull-back along $\Delta\colon A \to A \otimes A$. Specifically, if $\Delta(a) = \sum b_i \otimes c_i$, then $a \cdot (v \otimes w) = \sum (b_i \cdot v) \otimes (c_i \cdot w)$. Note that this construction agrees with the standard way that groups and Lie algebras act on tensor products.

It is well known (5.1.6 of [9]) that the antipode of a finite dimensional Hopf algebra is bijective. Thus in this situation the antipode is an algebra anti-isomorphism which gives a bijective correspondence between left modules and right modules. It is immediate from these remarks and any of the several characterizations of the Jacobson radical that $S(J(H)) = J(H)$ for a finite dimensional Hopf algebra H.

THEOREM 1. *Let A be a finite dimensional Hopf algebra. The tensor product of any pair of semisimple A-modules is semisimple if and only if the Jacobson radical $J(A)$ is a Hopf ideal.*

PROOF. The Jacobson radical $J \equiv J(A)$ is automatically stable under the antipode by the remarks above. Also the augmentation ideal $A^+ \equiv \ker(\varepsilon)$ is a maximal ideal and so $J \subseteqq \ker(\varepsilon)$. So we just have to show that the tensor product property is equivalent to the (coideal) condition $\Delta(J) \subseteqq J \otimes A + A \otimes J$. If this latter condition holds, then for any semisimple V, W, we have

$$J \cdot (V \otimes W) = \Delta(J) \cdot (V \otimes W) \subseteqq (J \cdot V) \otimes W + V \otimes (J \cdot W) = (0)$$

by (P). Thus, again by (P), $V \otimes W$ is semisimple.

For the converse, for any algebra A and A-module V, denote by $a_A(V)$ the annihilator of V in A. It is easy to check that if W is another A-module, then $a_{A \otimes A}(V \otimes W) = a_A(V) \otimes A + A \otimes a_A(W)$. Now let π_i: $A \to \mathrm{End}(V_i)$, $1 \leqq i \leqq k$, be a complete set of the (finitely many) distinct isomorphism classes of simple modules for the Hopf algebra A. If A has the tensor product property, then $V_i \otimes V_j$ is semisimple, $1 \leqq i, j \leqq k$. Thus $J \cdot (V_i \otimes J_j) = 0$, i.e.

$$\Delta(J) \subseteqq a_A(V_i) \otimes A + A \otimes a_A(V_j)$$

for all i, j, and so we have

$$\Delta(J) \subseteqq \bigcap_{i,j} [a_A(V_i) \otimes A + A \otimes a_A(V_j)] = J \otimes A + A \otimes J.$$

Since representations of a group G correspond in a canonical way to representations of the group algebra $k[G]$, the following theorem characterizes the situation for modular representations of finite groups.

THEOREM 2. *Let G be a finite group and k a field of characteristic $p \neq 0$. The Jacobson radical of $k[G]$ is a Hopf ideal if and only if G has a normal (i.e. unique) p-Sylow subgroup.*

PROOF. If $J = J(k[G])$ is a Hopf ideal, then by proposition 1 there is a normal subgroup N of G with the property that $S = \{e - n \mid n \in N\}$ generates J as an ideal, and $k[G]/J \cong k[G/N]$. Thus $k[G/N]$ is semisimple and so by Maschke's theorem $p \nmid |G/N|$. It follows that N must contain all elements of G having order a power of p. On the other hand, if $n \in N$, then $e - n \in J$, i.e. $e - n$ is nilpotent. So there exists a positive integer α with $0 = (e - n)^{p^\alpha} = e - n^{p^\alpha}$ from which we conclude n has order a power of p. Thus N is a subgroup of G consisting of all the elements of G having order a power of p and so is a normal p-Sylow subgroup of G.

To show the converse, let G have normal p-Sylow subgroup N. It is easy to check that $J(k[N]) = k[N]^+$, i.e. $(k[N]^+)^\alpha = (0)$ for some α. If we let I be the ideal of $k[G]$ generated by $k[N]^+$, then by the normality of N we have $I = k[N]^+ \cdot k[G] = k[G] \cdot k[N]^+$. So

$$I^{\alpha} = (k[N]^{+} \cdot k[G])^{\alpha} = (k[N]^{+})^{\alpha} \cdot k[G] = (0),$$

and so I, being a nilpotent ideal of $k[G]$, is contained in the radical. But by proposition 1 we have $k[G]/I \cong k[G/N]$ which is semisimple since $p \nmid |G/N|$. So $J(k(G]) = I$ and so is a Hopf ideal.

COROLLARY 1. *Let G be a finite group, k a field of characteristic $p \neq 0$. The tensor product of semisimple G-modules (over k) is semisimple if and only if G has a normal p-Sylow subgroup.*

If N is a normal p-Sylow subgroup of the (finite) group G, then $|N| = p^{\alpha}$ some α, and $p \nmid |G/N|$. Thus by Schur's theorem (7.5 of [3]) G is actually the semi-direct product of G/N by N. Also G is nilpotent if and only if it is the direct product of its p-Sylow subgroups, and this is equivalent to the existence of a unique (hence normal) p-Sylow subgroup for each prime p. (Theorem 6.12 of [3] and the remarks following). The following corollary is immediate.

COROLLAY 2. *Let G be a finite group. The tensor product of semisimple G-modules is semisimple (over all fields) if and only if G is nilpotent.*

Now let $\mathscr{G}$ be a restricted Lie algebra over the field k (of characteristic $p \neq 0$) and $H = U_p(\mathscr{G})$ its restricted universal enveloping algebra. A restricted representation of $\mathscr{G}$ is given by a homomorphism $\pi: \mathscr{G} \to \mathrm{End}(V)$ of restricted Lie algebras, where $\mathrm{End}(V)$ is a restricted Lie algebra under the usual bracket operation $[X, Y] = XY - YX$ and the ordinary associative pth power operation. As usual the vector space V is called the $\mathscr{G}$-module associated with the representation π.

As with the group algebra of a group, the restricted universal enveloping algebra of a Lie algebra $\mathscr{G}$ is cooked up to have the following property: restricted $\mathscr{G}$-modules correspond in a natural way to modules over the associative algebra $U_p(\mathscr{G})$. Explicitly $\mathscr{G}$ sits inside $U_p(\mathscr{G})$ as its space of primitive elements, and any restricted Lie map $\mathscr{G} \to \mathrm{End}(V)$ induces an algebra map $U_p(\mathscr{G}) \to \mathrm{End}\,(V)$ and conversely. Since $U_p(\mathscr{G})$ is finite dimensional, it (and hence $\mathscr{G}$) has only finitely many distinct equivalence classes (under isomorphism) of simple modules. If we let $\Lambda = \{\pi_1, \ldots, \pi_k\}$ be a set of representatives of these distinct classes of simple restricted $\mathscr{G}$-modules, then the corresponding set $\bar{\Lambda} = \{\bar{\pi}_1, \ldots, \pi_k\}$ ($\bar{\pi}_i$ the extension of π_i to $U_p(\mathscr{G})$) is a complete set of representatives of the simple $U_p(\mathscr{G})$-modules. Following Bourbaki (§5, no. 3 of [1] we define the p-nilpotent radical $\mathfrak{S}$ of $\mathscr{G}$ by $\mathfrak{S} = \bigcap_{\pi \in \Lambda} \ker \pi$. $\mathfrak{S}$ is a nilpotent ideal of $\mathscr{G}$, the smallest restricted ideal of $\mathscr{G}$ among the sets of kernels of semisimple restricted representations of $\mathscr{G}$. It is immediate that the Jacobson radical J of $U_p(\mathscr{G})$ is given by $J = \bigcap_{\bar{\pi} \in \bar{\Lambda}} \ker(\bar{\pi})$. We note that it is tempting, but incorrect, to assume that $\ker(\bar{\pi})$ is generated as an ideal in $U_p(\mathscr{G})$ by $\ker \pi$, π a representation of $\mathcal{G}$ (see example 4 below). But we have the following proposition.

PROPOSITION 2. *Let $\mathcal{G}$ be a restricted Lie algebra with p-nilpotent radical $\mathcal{S}$ and $H = U_p(\mathcal{G})$. Then $\mathcal{S}$ is a nilpotent Lie algebra. In addition the Jacobson radical J of H is a Hopf ideal if and only if J is generated by $\mathcal{S}$, i.e. $J = U_p(\mathcal{S})^+ \cdot H$.*

PROOF. To show $\mathcal{S}$ is nilpotent we note that $\mathcal{S} \subseteqq J$ and so the elements of $\mathcal{S}$ are nilpotent in the associative algebra H, hence nilpotent in $U_p(\mathcal{S}) \subseteqq H$. Since the adjoint representation ad: $\mathcal{S} \to \mathrm{End}(\mathcal{S})(\mathrm{ad}(X)(Y) = [X, Y])$ is the restriction of an algebra map from $U_p(\mathcal{S})$ to $\mathrm{End}(\mathcal{S})$, it follows that $\mathrm{ad}(X)$ is nilpotent for all $X \in \mathcal{S}$ and so $\mathcal{S}$ is nilpotent by Engel's theorem.

If I is any Hopf ideal of H it follows from standard coalgebra theory and 13.2.3 of [9] that $H/I \cong U_p(\mathcal{K})$ as Hopf algebras, where the canonical map $q: H \to H/I$ induces a homomorphism of restricted Lie algebras of $\mathcal{G}$ onto $\mathcal{K}$. Thus $\mathcal{K} \cong \mathcal{G}/\mathcal{N}$ for a restricted ideal $\mathcal{N}$ of $\mathcal{G}$. So if J is a Hopf ideal of H, then with J in place of I above we have $\mathcal{N} \subseteqq J$. Since $\bar{\pi}(J) = 0$ for all $\bar{\pi} \in \bar{\Lambda}$, we have $\pi(\mathcal{N}) = 0$ for all $\pi \in \Lambda$ and so $\mathcal{N} \subseteqq \mathcal{S}$. On the other hand

$$\mathcal{S} = \bigcap_{\pi \in \Lambda} \ker(\pi) \subseteqq \bigcap_{\bar{\pi} \in \bar{\Lambda}} \ker(\bar{\pi}) = J$$

so $\mathcal{S} \subseteqq J$. But then $\mathcal{S} \subseteqq J \cap P(H) = \mathcal{N}$ and so $\mathcal{S} = \mathcal{N}$, i.e. $\mathcal{S}$ generates J. The other direction is immediate, so we are done.

DEFINITION. (Exercise 23, I.4 of [1]) Let k be a perfect field of characteristic $p \neq 0$. $\mathcal{L}$ a restricted Lie Algebra over k. $\mathcal{L}$ is called p-unipotent if for all $X \in \mathcal{L}$ there exists n such that $X^{p^n} = 0$.

LEMMA. *$\mathcal{L}$ is p-unipotent if and only if $J(U_p(\mathcal{L}) = U_p(\mathcal{L})^+$.*

PROOF. If $J(U_p(\mathcal{L}) = U_p(\mathcal{L})^+$, then since $\mathcal{L} \subseteqq U_p(\mathcal{L})^+$, every element of $\mathcal{L}$ is nilpotent in $U_p(\mathcal{L})$. $\mathcal{L}$ is then p-unipotent because the p-power operation in $\mathcal{L}$ is just the associative p-th power in $U_p(\mathcal{L})$. The converse is an exercise in Bourbaki (ex. 23, I.4 of [1]), and is left to the reader.

Note that it follows from the proof of proposition 2 that the p-nilpotent radical of a restricted Lie algebra is actually p-unipotent.

DEFINITION. Let $\mathcal{G}$ be a restricted Lie algebra over the field k of characteristic $p \neq 0$. $\mathcal{G}$ is said to be toral if all the restricted representations of $\mathcal{G}$ are semisimple.

This definition is motivated by the fact that such Lie algebras appear as the Lie algebras of algebraic tori in the theory of algebraic groups (c.f. [4]).

The following is a well known theorem of Hochschild (Theorem 14 and the remark following, V.8 of [5]).

THEOREM 3. *Let $\mathscr{G}$ be a restricted Lie algebra over the field k of characteristic $p \neq 0$. Then $\mathscr{G}$ is toral if and only if $\mathscr{G}$ is Abelian with injective p-th power map.*

THEOREM 4. *Let k be a perfect field of characteristic $p \neq 0$, $\mathscr{G}$ is a finite dimensional restricted Lie algebra over k. Then the tensor product of semisimple $\mathscr{G}$-modules is semisimple if and only if $\mathscr{G}$ is an extension of a toral Lie algebra $\mathscr{K}$ by a p-unipotent Lie algebra $\mathfrak{S}$, i.e. there exists an exact sequence.*

$$(*) \qquad 0 \to \mathfrak{S} \to \mathscr{G} \to \mathscr{K} \to 0,$$

where $\mathfrak{S}$ is the p-nilpotent radical of $\mathscr{G}$ and $\mathscr{K}$ is toral. Further, if k is algebraically closed, the extension splits, i.e. $\mathscr{G}$ is the semi-direct product of $\mathscr{K}$ by $\mathfrak{S}$.

PROOF. By theorem 1 the tensor product property is equivalent to the radical $J = J(U_p(\mathscr{G}))$ being a Hopf ideal. From the proposition we get an exact sequence of the form $(*)$ where $\mathfrak{S}$ is the p-nilpotent radical of $\mathscr{G}$. Further, since $\mathfrak{S}$ generates a nilpotent ideal in $U_p(\mathscr{G})$, it follows that it does so in $U_p(\mathfrak{S})$ and so $\mathfrak{S}$ is p-unipotent by the lemma. Now the restricted universal enveloping algebra functor $U_p(\ )$ takes the exact sequence $(*)$ into an exact sequence of Hopf algebras.

$$(**) \qquad k \to U_p(\mathfrak{S}) \xrightarrow{i} U_p(\mathscr{G}) \xrightarrow{p} U_p(\mathscr{K}) \to k$$

(exactness for $(**)$ is equivalent to i being injective, and $\ker(p) = U_p(\mathfrak{S})^+ \cdot U_p(\mathscr{G})$; see [7] for more details). Since $U_p(\mathscr{K}) \cong U_p(\mathscr{G})/J$ (recall $J = \ker(p) = U_p(\mathfrak{S})^+ \cdot U_p(\mathscr{G})$) and this latter algebra is semisimple, all the restricted representations of $\mathscr{K}$ are semisimple; i.e. $\mathscr{K}$ is toral.

For the converse, let $\mathscr{G}$ be expressed as an extension of a toral Lie algebra by a p-unipotent as in $(*)$, and π_1, π_2 simple representation of $\mathscr{G}$ in V_1, V_2 respectively. Since $\mathfrak{S}$ is p-unipotent, the elements of $\mathfrak{S}$ are nilpotent in $U_p(\mathfrak{S})$, hence in $U_p(\mathscr{G})$. So $\pi_i(\mathfrak{S}) = 0$, $i = 1, 2$, by lemma 2, §4, no.3 of [1]. If $\pi_1 \hat{\otimes} \pi_2$ denotes the representation of $\mathscr{G}$ on $V_1 \otimes V_2$, it is then clear from the way this representation is defined that $\pi_1 \hat{\otimes} \pi_2(\mathfrak{S}) = 0$. So $V_1 \otimes V_2$ has a canonical $\mathscr{K} \cong \mathscr{G}/\mathfrak{S}$-module structure which is "the same" as its $\mathscr{G}$-module structure. In particular $V_1 \otimes V_2$ is semismimple as a $\mathscr{G}$ module because $\mathscr{K}$ is toral.

If the field is algebraically closed and J is a Hopf ideal of $U_p(\mathscr{G})$, then by the main result of [6] the exact sequence $(**)$ splits, i.e. there exists a Hopf algebra map j: $U_p(\mathscr{K}) \to U_p(\mathscr{G})$ with $p \circ j = I$. Since Hopf algebra maps take primitives to primitives, it is easy to check that j induces a Lie algebra splitting for $(*)$, which gives $\mathscr{G}$ the requisite semi-direct product structure.

We note that in the theorem $\mathfrak{S}$ and $\mathscr{K}$ are both nilpotent Lie algebras. $\mathscr{G}$ is not necessarily nilpotent, however, unless the given extension is central or trivial. Thus the restricted Lie algebras enjoying the tensor product property is not identical with the class of nilpotent Lie algebras. The following examples illustrate the situation.

EXAMPLE 3. Let $\mathscr{G}_m$ and $\mathscr{G}_a$ be the one dimensional Lie algebras with bases $\{X\}$, $\{Y\}$ respectively and p-th power maps $X^p = X$, $Y^p = 0$ (these are the Lie algebras of the multiplicative and additive groups of the field k, c.f. [4]). It is immediate that the map $\mathfrak{a}$: $\mathscr{G}_m \to \mathrm{End}(\mathscr{G}_a)$, $\mathfrak{a}(X) = I$, gives a representation of $\mathscr{G}_m$ as derivations of $\mathscr{G}_a$, and if we let $\mathscr{G} = \mathscr{G}_a \times_{\mathfrak{a}} \mathscr{G}_m$ (semidirect product), we see that $\mathscr{G}$ has basis $\{X, Y\}$ with $Y^p = 0$, $X^p = X$, $[X, Y] = Y$. Clearly $\mathscr{G}_a$ is p-unipotent and $\mathscr{G}_m$ is toral, so by the theorem $\mathscr{G}$ has the tensor product property. Yet $\mathscr{G}$ is definitely not nilpotent.

EXAMPLE 4. Let $\mathscr{G}$ be the restricted Lie algebra over k with basis $\{T, D, I\}$ where $[D, T] = I$, $[D, I] = 0 = [T, I]$, $I^p = I$, $D^p = T^p = 0$. If we let A be the truncated polynomial algebra $A = k[t]/(t^p)$, then there is a natural faithful restricted representation of $\mathscr{G}$ as derivations of A, where $D = d/dt$, $I =$ identity, and $T =$ multiplication by t. We claim first that A is a simple $\mathscr{G}$-module. For if $a = \sum \alpha_i t^i \in A$ with k the largest index for which $\alpha_k \neq 0$, we see that $\alpha_k(k!)\cdot 1 = D^k\cdot a \in \mathscr{G}\cdot a$ ($\equiv \mathscr{G}$ submodule of A generated by a). It follows that $1 \in \mathscr{G}\cdot a$ and hence $t^j = T^j\cdot 1$, $\varepsilon\, \mathscr{G}\cdot a$, $0 \leqq j \leqq p-1$, and so $\mathscr{G}\cdot a = A$.

Next we note that the existence of a faithful simple restricted representation for $\mathscr{G}$ forces the p-nilpotent radical $\mathfrak{S}$ of $\mathscr{G}$ to be zero. Thus if $\mathscr{G}$ were to have the tensor product property for semisimple representations, by theorem 4 we would have $\mathscr{G} = \mathscr{G}/(0) = \mathscr{G}/\mathfrak{S}$ a toral Lie algebra, which is clearly not the case as $\mathscr{G}$ is neither Abelian nor has injective p-th power map. But $\mathscr{G}$ is a nilpotent Lie algebra since $\mathscr{G}^2 \equiv [\mathscr{G}, \mathscr{G}] = k\cdot I$ and $\mathscr{G}^3 \equiv [\mathscr{G}, \mathscr{G}^2] = (0)$

For modular representations of infinite groups and non-restricted representations of Lie algebras the situation is more complicated. This is mainly due to the facts that the group algebra and the ordinary universal enveloping algebra are infinite dimensional, and the Jacobson radical of such algebras does not contain enough information about even the finite dimensional semisimple representations, in the sense that property (P) fails to hold. For example, if $\mathscr{G}$ is an n-dimensional Abelian Lie algebra with basis $\{X_1, \ldots, X_n\}$, then the ordinary university enveloping algebra $U(\mathscr{G})$ is isomorphic to the polynomial albegra $k[X_1, \ldots, X_n]$. The Jacobson radical of this algebra is (0), yet it has plenty of non-semisimple finite dimensional modules (e.g. $V = k[X_1, \ldots, X_n]/(f^r)$, f any non constant polynomial, $r > 1$, with the canonical module structure). All is not lost, however, Given any

Hopf algebra H (not necessarily finite dimensional) there exists a "dual" Hopf algebra H^0, and the finite dimensional H-modules in fact are the comodules for H^0. A certain subcoalgebra, called the coradical, of the dual Hopf algebra, contains all the information about the finite dimensional semisimple comodules, and it turns out there is a condition on the coradical (that it be a sub Hopf algebra), dual to that in theorem 1, which is equivalent to the tensor product property. This will be investigated in greater detail in **[8]**.

As a final application of the ideas studied in this paper, we characterize, via $U_p(\mathscr{G})$, those restricted Lie algebras $\mathscr{G}$ having finite dimensional faithful semisimple restricted representations. Such a Lie algebra is said to be p-reductive.

THEOREM 5. *Let $\mathscr{G}$ be a restricted Lie algebra over the field k of characteristic $p \neq 0$, $\mathfrak{S}$ the p-nilpotent radical of $\mathscr{G}$, and $H = U_p(\mathscr{G})$. The following are equivalent.*

1) *$\mathscr{G}$ is p-reductive.*
2) *H contains no proper nilpotent Hopf ideals.*
3) *$\mathfrak{S} = (0)$.*

If k is perfect, the above are equivalent to the following.

4) *$\mathscr{G}$ contains no p-unipotent restricted ideals.*

PROOF. The second condition is equivalent to the statement that the Jacobson radical J of H contains no non-zero Hopf ideals. We recall (c.f. the proof of proposition 2) that every Hopf ideal I of H is of the form $I = U_p(\mathscr{K})^+ \cdot H$ for some restricted ideal $\mathscr{K}$ of $\mathscr{G}$. If I is a nilpotent ideal of H, then $\mathscr{K} \subseteqq I \subseteqq J$ and so $\mathscr{K}$ must be contained in the kernel of any semisimple restricted representation of $\mathscr{G}$. So if $\mathscr{G}$ is p-reductive, $\mathscr{K}$ must be zero, from which (2) follows.

To prove that (2) implies (3) it is only necessary to note that $\mathfrak{S}$ generates a Hopf ideal of H which is necessarily contained in the Jacobson radical of H.

$\mathfrak{S}$ is the smallest restricted ideal of $\mathscr{G}$ among the kernels of the semisimple restricted representations of $\mathscr{G}$, thus $\mathscr{G}$ must have a faithful semisimple restricted representation if $\mathfrak{S} = (0)$.

Finally if k is perfect, (4) implies (3) because $\mathfrak{S}$ is p-unipotent. On the other hand, any p-unipotent ideal of $\mathscr{G}$ generates a nilpotent Hopf ideal of H, so (4) follows from (2).

Our definition and condition (3) of the theorem are equivalent conditions for reductivity for Lie algebras in characteristic 0. Other equivalent conditions for a Lie algebra $\mathscr{G}$ to be reductive in characteristic 0 are the following: (a) The adjoint representation of $\mathscr{G}$ is semisimple; (b) $\mathscr{G}$ is the product of an Abelian and a semisimple Lie algebra; and (c) the radical (largest solvable ideal) of $\mathscr{G}$ is the center of $\mathscr{G}$ (Proposition 5, no. 4, §6, I of [**1**]). Example 4 above is both nilpotent and p-reductive and hence shows that (b) and (c) fail in characteristic $p \neq 0$. Any Abelian restricted Lie algebra with non-injective p-th power map (e.g., $\mathscr{G}_a$) furnishes a counter-example for (a).

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Note added in proof. It has come to the author's attention that a proof of the equivalence of parts 1 and 4 of Theorem 5 is contained in R. L. Wilson's earlier paper, *A characterization of p-reductive Lie algebras*, Proc. Amer. Math. Soc. **32** (1972), 89-90.