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ANALOGOUS FUNCTION THEORIES FOR CERTAIN SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Transmutation operators are used to establish analogous function theories for the heat equation, the equation of generalized axially symmetric potential theory, and the Euler-Poisson-Darboux equation. Under these transformations correspondences are established relating fundamental solutions, polynomial solutions, associated functions, generating functions, and expansion theorems including Fourier transform criteria. In some cases, the transmutation operators must be interpreted in the generalized sense as acting on distributions.

1. Introduction. In [**14**], D.V. Widder pointed out numerous analogies between classical function theory and expansion theory for solutions of the heat equation. He did this by comparing, by means of a table, the underlying function sets, the generating functions for these sets, the orthogonality properties of these sets, etc. More recently E.G. Dunne and D.G. Mugler [7] extended these analogies by examining the corresponding functions and generalized function sets for the one-dimensional wave equation. Again, comparisons were made by means of a table. In [5], the authors made use of integral transformations connecting the solutions of the heat equation with solutions of the Laplace and wave equation to relate isomorphically various aspects of these function theories as exhibited in these tables. When taken in the classical sense, these integral transformations impose restrictions on the growth of the functions to be transformed. However, when interpreted in the generalized sense they permit almost complete reconcilliation with the tables of [7] and [**14**].

The purpose of this paper is to extend the methods and results of [5] to function theories corresponding to the equation of generalized axially symmetric potential theory (GASPT) and the Euler-Poisson-Darboux

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(EPD) equation. The GASPT equation assumes a role in pseudo-analytic function theory similar to Laplace's equation for the classical theory. Hence, a treatment of the connections between solutions of the heat equation and GASPT will provide an extension of these types of analogies noted by Widder to another important function theory. The basic functions used in the theory for GASPT involve the Jacobi polynomials and are closely related to the polynomial sets employed by B. Muckenhoupt and E.M. Stein [9]. As expected, the function theory for EPD involves distributions and is not as rich as that for GASPT. Nevertheless, many of the analogies are quite striking even in the case of this singular hyperbolic equation.

The basic integral transformations connecting solutions of the heat equation with solutions of the GASPT and EPD equations will be given in §2. These will be used along with the heat polynomials and the associated heat functions to construct the GASPT polynomials, the associated GASPT functions, and their generating functions in §3. The generalized Cauchy-Riemann equations will also be given for corresponding functions in these sets. The EPD polynomials, associated EPD functions, and their generating functions will be given in §4. In §5, we give expansion theorems corresponding to the various function sets developed in §3 and §4. Except for the bounds and asymptotic estimates for the Jacobi polynomials, the proofs of these are similar to the ones carried out in [4] and will generally be omitted for the sake of brevity. Finally, in §6 we provide Fourier transform criteria for expansions in terms of associated functions.

2. Transformations. Let $w(x, t)$ denote a solution of the heat problem:

$$w_t(x, t) = w_{xx}(x, t),\ t > 0;\ w(x, 0+) = \phi(x). \tag{2.1}$$

For $\mu < 1/2$, it follows from [2] that the function

$$u^\mu(x, y) = \frac{y^{1-2\mu}}{\Gamma(1/2-\mu)} \int_0^\infty e^{-\sigma y^2}\, \sigma^{-\mu-1/2}\, w(x, 1/4\sigma)d\sigma \tag{2.2}$$

satisfies the GASPT problem

$$\frac{\partial^2 u(x, y)}{\partial y^2} + \frac{2\mu}{y}\frac{\partial u(x, y)}{\partial y} + \frac{\partial^2 u(x, y)}{\partial x^2} = 0,\ y > 0;\ u(x, 0+) = \phi(x). \tag{2.3}$$

With the change of variable $\sigma = 1/4s$, (2.2) defines a transformation T_5^μ from $w(x, t)$ to $u^\mu(x, y)$ as follows: (see [5] for transforms T_1, T_2, T_3, T_4)

$$\begin{aligned} u^\mu(x, y) &= T_5^\mu\, w(x, t) \\ &= \frac{4^{\mu-1/2}\, y^{1-2\mu}}{\Gamma(1/2-\mu)} \int_0^\infty e^{-y^2/4s} s^{\mu-3/2}\, w(x, s)\, ds. \end{aligned} \tag{2.4}$$

Similarly, we find that the function

(2.5) $$v^{\mu}(x,y) = T_6^{\mu} w(x,t) = \frac{-1}{2\Gamma(1/2-\mu)} \int_0^{\infty} e^{-\sigma y^2} \sigma^{-\mu-3/2} w(x, 1/4\sigma) d\sigma$$

satisfies the equation

(2.6) $$\frac{\partial^2 v(x,y)}{\partial y^2} - \frac{2\mu}{y} \frac{\partial v(x,y)}{\partial y} + \frac{\partial^2 v(x,y)}{\partial x^2} = 0.$$

In later sections, we shall be concerned with solutions $u(x, y)$ and $v(x, y)$ of (2.3) and (2.6) which satisfy the following generalization of the Cauchy-Riemann equations:

(2.7) $$y^{2\mu} \frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}; \quad - y^{2\mu} \frac{\partial u(x,y)}{\partial y} = \frac{\partial v(x,y)}{\partial x}.$$

These will be used to define a pseudo-analytic function $f(z) = u(x, y) + iv(x, y)$. (see **[1]** and **[13]**).

Finally, we note from **[3]** that the transformation T_7^{μ} defined by

(2.8) $$e^{\mu}(x,t) = T_7^{\mu} w(x,t) = t^{1-2\mu} \Gamma(1/2+\mu) \mathscr{L}_s^{-1}\{s^{-\mu-1/2} w(x, 1/4s)\}_{s\to t^2}$$

defines a solution of the EPD problem:

(2.9) $$\frac{\partial^2 e(x,t)}{\partial t^2} + \frac{2\mu}{t} \frac{\partial e(x,t)}{\partial t} = \frac{\partial^2 e(x,t)}{\partial x^2}, \; t > 0,$$
$$e(x, 0+) = \phi(x), \; e_t(x, 0+) = 0.$$

3. Basic function sets for GASPT.

(i) **GASPT polynomial sets.** Suppose we select the function $w(x, t)$ above to be one of the heat polynomials **[11]**

$$h_n(x,t) = n! \sum_{j=0}^{[n/2]} \frac{x^{n-2j} t^j}{j!(n-2j)!}, \; n = 0, 1, 2, \ldots.$$

If we introduce this into (2.2) and evaluate the integral in the sense of generalized functions, we obtain

(3.1) $$\begin{aligned} u_n^{\mu}(x,y) &= T_5^{\mu} h_n(x,t) \\ &= n! \Gamma(\mu+1/2) \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j} y^{2j}}{4^j j! \Gamma(\mu+j+1/2)} \end{aligned}$$

which is defined for all μ except $\mu = -(2k + 1)/2$, $k = 0, 1, 2, \ldots$ The generating function $U^{\mu}(x, y, a)$ for the collection of all such polynomials can be obtained directly from the generating function for the heat polynomials,

$$e^{ax+a^2t} = \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n(x, t). \tag{3.2}$$

Hence we obtain

$$\begin{aligned} U^\mu(x, y, a) &= \sum_{n=0}^{\infty} \frac{a^n}{n!} u_n^\mu(x, y) \\ &= T_5^\mu e^{ax+a^2t} \\ &= (2/a)^{\mu-1/2}\Gamma(\mu+1/2)\, e^{ax}\, y^{1/2-\mu}\, J_{\mu-1/2}(ay). \end{aligned} \tag{3.3}$$

Making use of formula (4.10.6) of **[12]**, we have

$$U^\mu(x, y, a) = \sum_{n=0}^{\infty} \frac{\Gamma(2\mu)}{\Gamma(n + 2\mu)} r^n P_n^{(\mu)}(\cos\theta) a^n \tag{3.4}$$

where $x = r\cos\theta$ and $y = r\sin\theta$ and $P_n^{(\mu)}(z)$ is an ultraspherical polynomial. Comparing (3.3) with (3.4), it follows that

$$u_n^\mu(x, y) = \frac{\Gamma(2\mu)n!}{\Gamma(n + 2\mu)} r^n P_n^{(\mu)}(\cos\theta). \tag{3.5}$$

These polynomials differ from the polynomials $r^n P_n^{(\mu)}(\cos\theta)$ used in **[9]** by constant multipliers, which will not alter the regions of convergence in the expansion theorems.

From (3.5) and the properties of the Jacobi polynomials (see **[8]** and **[12]**, we have

$$u_n^\mu(x, y) = \frac{\Gamma(\mu + 1/2)n!}{\Gamma(n + \mu + 1/2)} r^n P_n^{(\mu-1/2,\mu-1/2)}(\cos\theta) \tag{3.6}$$

and, in rectangular coordinates,

$$\begin{aligned} u_{2n}^\mu(x, y) &= \frac{n!\Gamma(\mu + 1/2)}{\Gamma(n + \mu + 1/2)} (x^2 + y^2)^n P_n^{(\mu-1/2,-1/2)}\left(\frac{x^2 - y^2}{x^2 + y^2}\right), \\ u_{2n+1}^\mu(x, y) &= \frac{xn!\Gamma(\mu + 1/2)}{\Gamma(n + \mu + 1/2)} (x^2 + y^2)^n P_n^{(\mu-1/2,1/2)}\left(\frac{x^2 - y^2}{x^2 + y^2}\right). \end{aligned} \tag{3.7}$$

Next, suppose we introduce the heat polynomials into (2.5). By interpreting the resulting integrals in the generalized sense, we obtain

$$\begin{aligned} v_n^\mu(x, y) &= \frac{-n!}{2\Gamma(1/2-\mu)} \sum_{j=0}^{[n/2]} \frac{x^{n-2j}}{4^j j!(n-2j)!} \int_0^\infty e^{-y^2\sigma} \sigma^{-\mu-j-3/2} d\sigma \\ &= (1/2)\Gamma(\mu+1/2)n!\, y^{2\mu+1} \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j} y^{2j}}{4^j j!(n-2j)!\, \Gamma(\mu+j+3/2)} \\ &= \frac{y^{2\mu+1}}{2\mu + 1} u_n^{\mu+1}(x, y). \end{aligned} \tag{3.8}$$

Using this along with (3.6), we obtain

$$
\begin{aligned}
&v_{2n}^{\mu}(x, y) \\
&\quad = \frac{n!\Gamma(\mu+1/2)}{2\Gamma(n+\mu+3/2)} y^{2\mu+1} (x^2+y^2)^n P_n^{(\mu+1/2,-1/2)} ((x^2-y^2)/(x^2+y^2)),
\end{aligned}
$$

(3.9)

$$
\begin{aligned}
&v_{2n+1}^{\mu}(x, y) \\
&\quad = \frac{xn!\Gamma(\mu+1/2)}{2\Gamma(n+\mu+3/2)} y^{2\mu+1}(x^2+y^2)^n P_n^{(\mu+1/2,1/2)} ((x^2-y^2)/(x^2+y^2)).
\end{aligned}
$$

These can be expressed in polar form and compared with the ultraspherical polynomials to yield

$$
(3.10) \qquad v_n^{\mu}(x, y) = \frac{n!\Gamma(2\mu+2)}{(2\mu+1)\Gamma(n+2\mu+2)} y^{2\mu+1} r^n P_n^{(\mu+1)} (\cos\theta).
$$

Using the relation (3.8), we obtain the generating function $V^{\mu}(x, y, a)$ for the set $\{v_n^{\mu}(x, y)\}$ as follows:

$$
(3.11) \qquad
\begin{aligned}
V^{\mu}(x, y, a) &= \sum_{n=0}^{\infty} (a^n/n!)\, v_n^{\mu}(x, y) \\
&= \frac{y^{2\mu+1}}{2\mu+1} \sum_{n=0}^{\infty} (a^n/n!)\, u_n^{\mu+1}(x, y) \\
&= (2/a)^{\mu-1/2}\, \Gamma(\mu+1/2)\, e^{ax} y^{\mu+1/2} J_{\mu+1/2}(ay).
\end{aligned}
$$

Finally, it is an elementary exercise to show that the polynomials $\{u_n^{\mu}(x, y)\}$ and $\{v_n^{\mu}(x, y)\}$ satisfy the following generalized Cauchy-Riemann equations:

$$
(3.12) \qquad
\begin{aligned}
\frac{\partial}{\partial y}(nv_{n-1}^{\mu}(x, y)) &= y^{2\mu} \frac{\partial u_n^{\mu}(x, y)}{\partial x} \\
\frac{\partial}{\partial x}(nv_{n-1}^{\mu}(x, y)) &= -\, y^{2\mu} \frac{\partial u_n^{\mu}(x, y)}{\partial y}.
\end{aligned}
$$

(ii) **Associated GASPT functions.** Let $\{\tilde{h}_n(x, t)\}$ denote the set of associated heat functions [11], which are the Appell transforms of the heat polynomials $h_n(x, t)$. We define the sets of functions $\{U_n^{\mu}(x, y)\}$ and $\{V_n^{\mu}(x, y)\}$ by means of the relations

$$
(3.13) \qquad U_n^{\mu}(x, y) = T_5^{\mu}\tilde{h}_n(x, t), \quad V_n^{\mu}(x, y) = T_6^{\mu}\tilde{h}_n(x, t).
$$

By applying the Appell transform to the generating function for the $\{h_n(x, t)\}$, we obtain the generating function $\tilde{H}(x, t, a)$ for the associated heat functions:

$$
(3.14) \qquad \tilde{H}(x, t, a) = (4\pi t)^{-1/2} e^{-(x-2a)^2/(4t)}.
$$

It is an easy calculation to show that

$$
(3.15)\qquad
\begin{aligned}
T_5^{\mu}\tilde{H}(x,t,a) &= \frac{\Gamma(1-\mu)}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\,y^{1-2\mu}\,[y^2+(x-2a)^2]^{\mu-1},\\
T_6^{\mu}\tilde{H}(x,t,a) &= \frac{\Gamma(1-\mu)}{2\sqrt{\pi}\,\mu\Gamma(1/2-\mu)}\,[y^2+(x-2a)^2]^{\mu}.
\end{aligned}
$$

Since $y^2 + (x - 2a)^2 = r^2[1 - 2(x/r)(2a/r) + (2a/r)^2]^2$, with $r^2 = x^2 + y^2$, it will be observed that the right-hand members of (3.15) involve the generating function for the ultraspherical polynomials. Using the expansions of these generating functions and the relations between the ultraspherical and Jacobi polynomials, we obtain

$$
(3.16)\qquad
\begin{aligned}
&U_{2n}^{\mu}(x,y)\\
&\quad= \frac{2^{4n}n!\,\Gamma(n+1-\mu)}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\,y^{1-2\mu}(x^2+y^2)^{\mu-1-n}\,P_n^{(1/2-\mu,\,-1/2)}\left(\frac{x^2-y^2}{x^2+y^2}\right),\\
&U_{2n+1}^{\mu}(x,y)\\
&\quad= \frac{2^{4n+2}n!\,\Gamma(n+2-\mu)}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\,xy^{1-2\mu}(x^2+y^2)^{\mu-2-n}P_n^{(1/2-\mu,\,1/2)}\left(\frac{x^2-y^2}{x^2+y^2}\right),\\
&V_{2n}^{\mu}(x\;y)\\
&\quad= \frac{-2^{4n-1}\,n!\,\Gamma(n-\mu)}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\,(x^2+y^2)^{\mu-n}\,P_n^{(-1/2-\mu,\,-1/2)}\left(\frac{x^2-y^2}{x^2+y^2}\right),\\
&V_{2n+1}^{\mu}(x,y)\\
&\quad= \frac{-2^{4n+1}n!\,\Gamma(n+1-\mu)}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\,x(x^2+y^2)^{\mu-n-1}\,P_n^{(-1/2-u,\,1/2)}\left(\frac{x^2-y^2}{x^2+y^2}\right).
\end{aligned}
$$

These functions exist for all values of μ except $0,1,2,3,\ldots$ and $1/2$. The exceptional value of $1/2$ can be eliminated by dropping the factor $\Gamma(1/2-\mu)$, in which case

$$
(3.17)\qquad U_n^{1/2} = (2^n n!/r^{n+1})P_n(\cos\theta),\quad V_n^{1/2} = (2^n n!/r^{n-1})\,P_n^{(-1/2)}(\cos\theta),
$$

where $P_n(z)$ denotes a Legendre polynomial.

From the fact that $\tilde{h}_n(x,t) = (-2)^n\, D_x^n k(x,t)$ with $k(x,t) = (4\pi t)^{-1/2}e^{-x^2/4t}$, it follows that

$$
(3.18)\qquad
\begin{aligned}
U_n^{\mu}(x,y) &= T_5^{\mu}\tilde{h}_n(x,t) = (-2)^n\, D_x^n\, T_5^{\mu}\, k(x,t)\\
&= ((-2)^n/\sqrt{\pi}\,\Gamma(1/2-\mu))\,y^{1-2\mu}D_x^n\int_0^{\infty} e^{-(x^2+y^2)\sigma}\,\sigma^{-\mu}d\sigma\\
&= (-2)^n\,\lambda\, y^{1-2\mu}\,D_x r^{2\mu-2}
\end{aligned}
$$

with $\lambda = \Gamma(1-\mu)/\sqrt{\pi}\,\Gamma(1/2-\mu)$. Similarly,

$$
(3.19)\qquad V_n^{\mu}(x,y) = (-2)^n\lambda\, D_x^{n-1}(x\, r^{2\mu-2}).
$$

From these "Rodrigues" type formulas, it is easy to verify the generalized Cauchy-Riemann equations:

$$\frac{\partial}{\partial y}\left(\frac{-V^{\mu}_{n+1}x, y)}{2}\right) = y^{2\mu}\frac{\partial}{\partial x}U^{\mu}_{n}(x, y)$$

$$\frac{\partial}{\partial x}\left(\frac{-V^{\mu}_{n+1}(x, y)}{2}\right) = -y^{2\mu}\frac{\partial}{\partial y}U^{\mu}_{n}(x, y).$$

4. Basic functions sets for EPD.

(i) **EPD polynomial sets.** If we apply the transformation T^{μ}_{t}, defined in equation (2.8), to the heat polynomials, we obtain

$$\begin{aligned} e^{\mu}_{n}(x, t) &= T^{\mu}_{t}h_{n}(x, t) \\ &= n!\,\Gamma(\mu + 1/2)\sum_{j=0}^{[n/2]}\frac{x^{n-2j}t^{2j}}{4^{j}j!(n-2j)!\Gamma(j+\mu+1/2)} \end{aligned} \tag{4.1}$$

which satisfy the Euler-Poisson-Darboux equation and $e^{\mu}_{n}(x, 0) = x^{n}$, $(\partial e^{\mu}_{n}/\partial t)(x, 0) = 0$. It is clear that

$$e^{\mu}_{n}(x, iy) = u^{\mu}_{n}(x, y) \tag{4.2}$$

and, therefore, that the generating function $\mathscr{E}^{\mu}(x, t, a)$ for the EPD polynomials is

$$\mathscr{E}^{\mu}(x, t, a) = (2/a)^{\mu-1/2}\,\Gamma(\mu + 1/2)\,e^{ax}t^{1/2-\mu}I_{\mu-1/2}(at). \tag{4.3}$$

This can also be obtained directly from the generating function for the heat polynomials as follows:

$$\mathscr{E}^{\mu}(x, t, a) = T^{\mu}_{t}e^{ax+a^{2}t}. \tag{4.4}$$

It follows from (4.2) and (3.7) that $e^{\mu}_{n}(x, t)$ can be expressed in terms of Jacobi polynomials:

$$\begin{aligned} &e^{\mu}_{2n}(x, t) \\ &= (n!\Gamma(\mu+1/2)/\Gamma(n+\mu+1/2))(t^{2}-x^{2})^{n}P_{n}^{(-1/2,\,\mu-1/2)}((t^{2}+x^{2})/(t^{2}-x^{2})) \\ &e^{\mu}_{2n+1}(x, t) \\ &= (xn!\Gamma(\mu+1/2)/\Gamma(n+\mu+1/2))(t^{2}-x^{2})^{n}P_{n}^{(1/2,\,\mu-1/2)}((t^{2}+x^{2})/(t^{2}-x^{2})). \end{aligned} \tag{4.5}$$

These polynomials are defined for all $\mu \neq -(2k + 1)/2$, $k = 0, 1, 2, \ldots$

(ii) **Associated EPD functions.** We define the associated EPD functions by transforming the associated heat functions $\tilde{h}_{n}(x, t)$ as follows:

$$\mathscr{E}^{\mu}_{n}(x, t) = T^{\mu}_{t}\tilde{h}_{n}(x, t). \tag{4.6}$$

We go directly to the generating function for these solutions.

$$
\begin{aligned}
\tilde{\mathscr{E}}^{\mu}(x,t,a) &= T_t^{\mu}(4\pi t)^{-1/2}e^{-(x-2a)^2/4t}\\
&= (1/\sqrt{\pi})t^{1-2\mu}\Gamma(\mu+1/2)\mathscr{L}_s^{-1}\{s^{-\mu}e^{-(x-2a)^2 s}\}_{s\to t^2} \qquad (4.7)\\
&= (t^{1-2\mu}\Gamma(\mu+1/2)/\sqrt{\pi}\,\Gamma(\mu))[t^2-(x-2a)^2]_+^{\mu-1}.
\end{aligned}
$$

Here the subscript $+$ denotes a function defined by the formula when $t^2 > (x-2a)^2$ and by zero otherwise. We relate the generating function to the ultraspherical polynomials by

$$
\begin{aligned}
\tilde{\mathscr{E}}^{\mu}(x,t,a) &= \frac{t^{1-2\mu}\Gamma(\mu+1/2)\rho^{2\mu-2}}{\sqrt{\pi}\,\Gamma(\mu)}[1-2(ix/\rho)(2ia/\rho)+(2ia/\rho)^2]_+^{\mu-1} \qquad (4.8)\\
&= \frac{t^{1-2\mu}\Gamma(\mu+1/2)\rho^{2\mu-2}}{\sqrt{\pi}\,\Gamma(\mu)}\sum_{n=0}^{\infty}P_n^{(1-\mu)}(ix/\rho)(2ia/\rho)^n
\end{aligned}
$$

where $\rho = \sqrt{t^2-x^2}$. Hence,

$$
\mathscr{E}_n^{\mu}(x,t) = \frac{t^{1-2\mu}\Gamma(\mu+1/2)n!\,2^n i^n}{\sqrt{\pi}\,\Gamma(\mu)(t^2-x^2)^{n/2+1-\mu}}P_n^{(1-\mu)}(ix/\rho) \qquad (4.9)
$$

for $t^2 > x^2$, and zero otherwise. Using formulas (4.1.5) and (4.7.1) of **[12]**, we can express these functions in terms of Jacobi polynomials:

$$
\begin{aligned}
&\mathscr{E}_{2n}^{\mu}(x,t)\\
&\quad= \frac{(-1)^n n!4^{2n}t^{1-2\mu}\Gamma(\mu+1/2)}{\sqrt{\pi}\,\Gamma(\mu-n)\,(t^2-x^2)^{n+1-\mu}}P_n^{(-1/2,1/2-\mu)}((t^2+x^2)/(t^2-x^2)) \qquad (4.10)\\
&\mathscr{E}_{2n+1}^{\mu}(x,t)\\
&\quad= \frac{(-1)^n n!\,4^{2n+1}x\,t^{1-2\mu}\Gamma(\mu+1/2)}{\sqrt{\pi}\,\Gamma(\mu-n-1)(t^2-x^2)^{n+2-\mu}}P_n^{(1/2,1/2-\mu)}((t^2+x^2)/(t^2-x^2)).
\end{aligned}
$$

These functions exist for all values of μ except $0, 1, 2, \ldots$ and $\mu = -1/2$. The exceptional value of $-1/2$ can be eliminated by dropping the factor $\Gamma(\mu+1/2)$.

If μ is a positive integer the associated EPD functions involve distributions. To see this we recall that $\tilde{h}_n(x,t) = (-2)^n D_x^n k(x,t)$. Hence, if $\mu = m$, a positive integer,

$$
\begin{aligned}
\mathscr{E}_n^{m}(x,t) &= ((t^{1-2m}\Gamma(m+1/2)(-2)^n/\sqrt{\pi})D_x^n\mathscr{L}_s^{-1}\{s^{-m}e^{-x^2 s}\}_{s\to t^2} \qquad (4.11)\\
&= ((t^{1-2m}\Gamma(m+1/2)(-2)^n/\sqrt{\pi}\,\Gamma(m))\,D_x^n(t^2-x^2)_+^{m-1}.
\end{aligned}
$$

If $n \leqq m-1$ this is an ordinary function, while if $n \geqq m$ it involves distributions.

5. Expansion theorems. In **[4]** we showed that when $\phi(x) = \sum_{n=0}^{\infty} a_n x^n$ is analytic for $|x| < \sigma$, then the series of GASPT polynomials $u^{\mu}(x,y) = \sum_{n=0}^{\infty} a_n u_n^{\mu}(x,y)$, for $\mu > -1/2$, converges to a solution of the GASPT equation for $|z| < \sigma$, with $u^{\mu}(x,0) = \phi(x)$. Similarly we proved that the series of EPD polynomials $e^\mu(x, t) = \sum_{n=0}^\infty a_n e_n^\mu(x, t)$, for $\mu > -1/2$, converges to a solution of the EPD equation for $|x| + |t| < \sigma$, with $e^\mu(x, 0) = \phi(x)$, $e_t^\mu(x, 0) = 0$. In this paper, we have extended the definitions of these polynomials to all values of $\mu \neq -1/2 - k$, $k = 0, 1, 2, \ldots$ However, the theorems alluded to above are still valid.

The proofs of these theorems begin with a proof of convergence of the given series in some region K of the plane, which depends on the asymptotic bounds on the coefficients a_n. Having established the existence of the functions $u^\mu(x, y)$ or $e^\mu(x, t)$ in K, it remains to prove that the given function satisfies the differential equation and the boundary conditions. This generally involves the use of standard identities for the Jacobi polynomials and asymptotic bounds for these polynomials to show the uniform convergence of the series of derivatives on appropriate compact subsets of K (see **[4]** for examples). These arguments become repetitious and lengthy. For this reason, in this paper we will concentrate on the geometry of the regions of convergence (and divergence) of the given series, omitting the details of the proofs that the series satisfy a certain partial differential equation.

For the proofs of these theorems we shall need the following asymptotic bounds stated in **[12]**. If α and β are arbitrary reals and c is a positive constant, then

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2}\, O(n^{-1/2}), & cn^{-1} \leqq \theta \leqq \pi/2 \\ O(n^\alpha), & 0 \leqq \theta \leqq cn^{-1}. \end{cases} \tag{5.1}$$

THEOREM 5.1. *Let a_n be real, $n = 0, 1, 2, \ldots,$ and suppose* $\limsup_{n\to\infty} |a_n|^{1/n} = \sigma^{-1}$, $\sigma > 0$. *Then the series $u^\mu(x, y) = \sum_{n=0}^\infty a_n u_n^\mu(x, y)$ converges to a solution of* (2.3) *when $r < \sigma$, $\mu \neq -1/2 - k$, $k = 0, 1, 2, \ldots,$ but does not converge everywhere for $r < \sigma + \varepsilon$, $\varepsilon > 0$.*

PROOF. Using (3.6) the given series is

$$u^\mu(x, y) = \Gamma(\mu + 1/2) \sum_{n=0}^\infty \frac{a_n\, n!\, r^n P_n^{(\mu-1/2,\mu-1/2)}(\cos\theta)}{\Gamma(n + \mu + 1/2)}.$$

The restrictions on μ are obvious from this form. The case $\mu > -1/2$ was covered in **[4]**, which we shall not repeat. For the case $\mu < -1/2$, we use the bounds (5.1). If $\mu < -1/2$, $\theta = 0$. then $P_n^{(\mu-1/2,\mu-1/2)}(\cos\theta) = O(n^{\mu-1/2})$. If $0 < \delta \leqq \theta \leqq \pi/2$, then $P_n^{(\mu-1/2,\mu-1/2)}(\cos\theta) = \theta^{-\mu}\, O(n^{-1/2})$. Since $\mu < 0$, $P_n^{(\mu-1/2,\mu-1/2)}(\cos\theta) = O(n^{-1/2})$ for $0 \leqq \theta \leqq \pi/2$. By hypothesis, for $r < R < \sigma$ there exists a constant $M > 0$ such that $|a_n| \leqq M/R^n$. Hence for N sufficiently large

$$\left|\sum_{n=N}^\infty a_n u_n^\mu(x, y)\right| \leqq K \sum_{n=N}^\infty \frac{n!\, n^{-1/2}}{\Gamma(n + \mu + 1/2)} (r/R)^n.$$

Since the comparison series converges by the ratio test, the given series converges for $0 \leqq \theta \leqq \pi/2$, $r < \sigma$. For $\pi/2 \leqq \theta \leqq \pi$, let $\varphi = \pi - \theta$ and use the fact that $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ to reduce this case to the one just treated. The cases $\pi \leqq \theta \leqq 3\pi/2$ and $3\pi/2 \leqq \theta \leqq 2\pi$ are treated similarly. When $y = 0$, $u^\mu(x, 0) = \sum_{n=0}^\infty a_n x^n$ which diverges for $|x| > \sigma$.

THEOREM 5.2. *Let c_n be real, $n = 0, 1, 2, \ldots$ and suppose* $\limsup_{n\to\infty} |c_n|^{1/n} = \sigma^{-1}$, $\sigma > 0$. *Then the series $v^\mu(x, y) = \sum_{n=0}^\infty c_n v_n^\mu(x, y)$ converges to a solution of* (2.6) *when $r < \sigma$, $y \neq 0$, $\mu \neq -1/2 - k$, $k =$* 0, 1, 2, ... *but does not converge everywhere for $r < \sigma + \varepsilon$, $\varepsilon > 0$. The restriction $y \neq 0$ can be dropped if $\mu \geqq 1/2$.*

PROOF. Using (3.6) and (3.8), the given series is

$$v^\mu(x, y) = \frac{y^{2\mu+1}\Gamma(\mu + 1/2)}{2} \sum_{n=0}^\infty \frac{c_n n!\, r^n P_n^{(\mu+1/2,\mu+1/2)}(\cos\theta)}{\Gamma(n + \mu + 3/2)}.$$

The restrictions on μ are obvious from this form. The restriction $y \neq 0$ is required so that $y^{2\mu+1}$ will have continuous first and second derivatives when $\mu < 1/2$. The convergence proof is similar to that of theorem 5.1. For the divergence proof, we consider first the case $\mu > -3/2$. Then by formula (7.32.2) of **[12]** $\max |P_n^{(\mu+1/2,\mu+1/2)}(\cos\theta)| \sim n^q$ where $q = \max[-1/2, \mu + 1/2]$. Since $\sum_{n=0}^\infty c_n r^n$ diverges if $r > \sigma$, there is a constant $\rho > 1$ and a subsequence of integers such that $|c_n r^n| \geqq \rho^n$. Then on this subsequence

$$|c_n n!\, r^n P_n^{(\mu+1/2,\mu+1/2)}(\cos\theta)/\Gamma(n+\mu+3/2)| \sim |c_n r^n|\, n!\, n^q/\Gamma(n + \mu + 3/2).$$

By Stirling's formula, this behaves as $Kn^{q-\mu-1/2}|c_n r^n| \geqq Kn^{q-\mu-1/2}\rho^n$, where K is a positive constant. The failure of this term to tend to zero on the subsequence shows that the series diverges for $r > \sigma$. For $\mu < -3/2$, we use formula (8.21.10) of **[12]** to show that there is an angle θ, $0 < \theta < \pi$, where $P_n^{(\mu+1/2,\mu+1/2)}(\cos\theta) \sim n^{-1/2}$. Then by the argument above the general term of the series does not tend to zero for this angle and $r > \sigma$.

Theorems 5.1 and 5.2 can be used to give a representation theory for certain pseudo-analytic functions. Let a_n, $n = 0, 1, 2, \ldots$, be real and $\limsup_{n\to\infty} |a_n|^{1/n} = \sigma^{-1}$. Let $c_{n-1} = na_n$, $n = 1, 2, 3, \ldots$ and let

$$u^\mu(x, y) = \sum_{n=0}^\infty a_n u_n^\mu(x, y),$$

$$(5.2)\qquad \begin{aligned} v^\mu(x, y) &= \sum_{n=1}^\infty n a_n v_{n-1}^\mu(x, y) \\ &= \sum_{n=1}^\infty c_{n-1} v_{n-1}^\mu(x, y) \\ &= \sum_{n=0}^\infty c_n v_n^\mu(x, y). \end{aligned}$$

When $\mu > -1/2$, by (3.12) the function

$$f(z) = u^{\mu}(x, y) + iv^{\mu}(x, y) \tag{5.3}$$

satisfies the pseudo-analytic conditions (2.7) for $r < \sigma$. If $\mu < -1/2$ and $\mu \neq -1/2 - k$, $k = 1, 2, 3, \ldots$, (5.3) satisfies the pseudo-analytic conditions in any part of the convergence region where $y \neq 0$. This result can be generalized to the case where a_n is complex.

THEOREM 5.3. *Suppose that* $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e = \sigma$. *Then the series*

$$U(x, y) = \sum_{n=0}^{\infty} b_n U_n^{\mu}(x, y), \quad V(x, y) = \sum_{n=0}^{\infty} b_n V_n^{\mu}(x, y) \tag{5.4}$$

converge for $r > \sigma$ *to solutions of* (2.3) *and* (2.6) *respectively for* $\mu \neq 0$, $1, 2, \ldots, y \neq 0$, *but do not converge everywhere for* $r > \sigma - \varepsilon$, $\varepsilon > 0$.

PROOF. The proof is similar to those of theorems 5.1 and 5.2.

If $\limsup_{n\to\infty} 2n\,|b_n|^{1/n}/e = \sigma$, we can represent certain pseudo-analytic functions for $r > \sigma$. Let

$$U(x, y) = \sum_{n=0}^{\infty} b_n\, U_n^{\mu}(x, y), \quad V(x, y) = \sum_{n=0}^{\infty} b_n(-1/2\ V_{n+1}^{\mu}(x\ y)). \tag{5.5}$$

Then, using (3.20), the function

$$F(z) = U(x, y) + iV(x, y) \tag{5.6}$$

satisfies the pseudo-analytic conditions for $r > \sigma$, $y \neq 0$ and $\mu \neq 0, 1, 2, \ldots$.

Next we turn to some expansion theorems for the EPD functions.

THEOREM 5.4. *Let* $\limsup |a_n|^{1/n} = \sigma^{-1}$, $\sigma > 0$. *Then* $\sum_{n=0}^{\infty} a_n e_n^{\mu}(x, t)$ *converges for* $|x| + |t| < \sigma$ *to a solution of* (2.9) *but does not converge everywhere for* $|x| + |t| < \sigma + \varepsilon$, $\varepsilon > 0$.

PROOF. Using formula (8.23.1) of **[12]**, it is easy to show that $|e_n^{\mu}(x, t)|^{1/n} \cong |x| + |t|$. Then using the root test, it follows that $\sum_{n=0}^{\infty} |a_n e_n^{\mu}(x, t)|$ converges for $|x| + |t| < \sigma$. If $t = 0$, $\sum_{n=0}^{\infty} a_n e_n^{\mu}(x, 0) = \sum_{n=0}^{\infty} a_n x^n$ wnich diverges for $|x| > \sigma$.

THEOREM 5.5. *Let* $\limsup 2n|b_n|^{1/n}/e = \sigma > 0$. *Then* $\sum_{n=0}^{\infty} b_n \mathscr{E}_n^{\mu}(x, t)$ *converges for* $|t| > |x| + \sigma$ *to a solution of the* EPD *equation, but does not converge everywhere for* $|t| > |x| + \sigma - \varepsilon$, $\varepsilon > 0$.

PROOF. Using Stirling's formula and formula (8.23.1) of **[12]**, it is easy to show when $t^2 - x^2 > 0$ that $|\mathscr{E}_n^{\mu}(x, t)|^{1/n} \cong (2n/e)(|x| + |t|)/(t^2 - x^2)$. Then using the root test, it follows that $\sum_{n=0}^{\infty} |b_n \mathscr{E}_n^{\mu}(x, t)|$ converges for $(|x| + |t|)/(t^2 - x^2) < 1/\sigma$ or $|t| > |x| + \sigma$. On the other hand for every

$\varepsilon > 0$ there is a subsequence such that $|b_n|^{1/n} > e(\sigma - \varepsilon/2)/2n$. If $|t| - |x| < \sigma - \varepsilon$, then $(|x| + |t|)/(t^2 - x^2) > 1/(\sigma - \varepsilon)$ and $|b_n \mathscr{E}_n^\mu(x, t)|^{1/n} > (\sigma - \varepsilon/2)/(\sigma - \varepsilon) > 1$ and the series diverges, provided $x \neq 0$.

THEOREM 5.6. *Let* $\limsup_{n\to\infty} n\, 2^\rho\, |b_n|^{\rho/n}/\rho e \leqq \sigma$, *where* $\rho < 1$. *Then the series* $\sum_{n=0}^\infty b_n \mathscr{E}_n^\mu(x, t)$ *converges for* $|t| > |x|$.

PROOF. Using formula (8.23.1) of **[12]**, it can be shown for $t^2 - x^2 > 0$, that $|\mathscr{E}_n^\mu(x, t)| \sim n! 2^n (|x| + |t|)^n/(t^2 - x^2)^n$. Also for arbitrary $\varepsilon > 0$, $|b_n| \leqq [\rho e(\sigma + \varepsilon)/n\, 2^\rho]^{n/\rho}$ for $n \geqq N$, N sufficiently large. Hence

$$\sum_{n=N}^\infty |b_n \mathscr{E}_n^\mu(x, t)| \leqq M \sum_{n=N}^\infty \left[\frac{\rho e(\sigma + \varepsilon)}{n}\right]^{n/\rho} \frac{n!(|x| + |t|)^n}{(t^2 - x^2)^n}$$

for some constant M. A ratio test shows that the comparison series converges for all x and t such that $t^2 - x^2 > 0$.

6. Fourier transform criteria. In [11] Widder and Rosenbloom proved the following theorem concerning the expansion of heat functions in terms of the associated heat functions $\tilde{h}_n(x, t)$.

THEOREM 6.1. *The series* $\sum_{n=0}^\infty b_n \tilde{h}_n(x, t)$ *converges for* $t > \sigma \geqq 0$ *if and only if*

$$\sum_{n=0}^\infty b_n \tilde{h}_n(x, t) = (1/2\pi) \int_{-\infty}^\infty e^{ixs - ts^2} \phi(s) ds \tag{6.1}$$

where $\phi(s) \in \mathfrak{A}(2, \sigma)$ *and* $b_n = \phi^{(n)}(0)/[n!(-2i)^n]$. *[Here* $\mathfrak{A}(2, \sigma)$ *denotes the class of entire functions of growth* $(2, \sigma)$*].*

In this section, we show that there are analogous theorems for the associated GASPT functions and the associated EPD functions.

THEOREM 6.2. *Let* $\mu < 1$. *Then the series* $\sum_{n=0}^\infty b_n \tilde{U}_n^\mu(x, y)$ *converges for* $y > \sigma \geqq 0$ *if and only if*

$$\sum_{n=0}^\infty b_n \tilde{U}_n^\mu(x, y) = (1/2\pi) \int_{-\infty}^\infty e^{ixs} \phi(s)\, W_\mu(ys) ds \tag{6.2}$$

where $\tilde{U}_n^\mu(x, y) = \Gamma(1/2 - \mu)\, U_n^\mu(x, y)$, $W_\mu(ys) = \int_{-\infty}^\infty \exp(-t - y^2 s^2/4t) (dt/t^{\mu+1/2})$, $\phi(x) \in \mathfrak{A}(1, \sigma)$, *and* $b_n = \phi^{(n)}(0)/[n!(-2i)^n]$.

PROOF. If the series converges as stated and $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e = \sigma' > \sigma$, then the series diverges for $|z| < \sigma'$, $x \neq 0$, contrary to assumption. Hence, $\phi(z) = \sum_{n=0}^\infty (-2i)^n b_n z^n \in \mathfrak{A}(1, \sigma)$. Conversely, if

$$\begin{aligned} K^\mu(x, y) &= \Gamma(1/2 - \mu)\, T_\xi^\mu \{(4\pi t)^{-1/2} e^{-x^2/4t}\} \\ &= (y^{1-2\mu}\, \Gamma(1 - \mu)/\sqrt{\pi})(1/(x^2 + y^2)^{1-\mu}) \\ &= (1/2\pi) \int_{-\infty}^\infty e^{ixs}\, W_\mu(ys) ds \end{aligned}$$

then

$$\tilde{U}_n(x, y) = (-2)^n D_x^n K^\mu(x, y) = (1/2\pi)\int_{-\infty}^{\infty} (-2is)^n e^{ixs} W_\mu(ys)ds$$

and

$$\sum_{n=0}^{\infty} b_n \tilde{U}_n(x, y) = (1/2\pi)\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (-2is)^n b_n e^{ixs} W_\mu(ys)ds$$
$$= (1/2\pi)\int_{-\infty}^{\infty} e^{ixs}\psi(s) W_\mu(ys)ds$$

provided the term-by-term integration is valid for $y > \sigma$. It is because $W_\mu(ys) = 2(2/ys)^{\mu-1/2} K_{\mu-1/2}(ys)$ and $K_{\mu-1/2}(z) \sim (\pi/2z)^{1/2} e^{-z}$ as $z \to \infty$. Note that $\mu < 1$ insures that the Fourier transform exists in the classical sense.

THEOREM 6.3. *The series* $\sum_{n=0}^{\infty} b_n \tilde{\mathscr{E}}_n^\mu(x, t)$ *converges in the space of ultra-distributions* Z' *if and only if*

$$\sum_{n=0}^{\infty} b_n \tilde{\mathscr{E}}_n^\mu(x, t) = (1/2\pi)\int_{-\infty}^{\infty} e^{ixs} \psi(s) M_\mu(ts)ds \tag{6.3}$$

where $\tilde{\mathscr{E}}_n^\mu(x, t) = [\Gamma(\mu+1/2)]^{-1} \mathscr{E}_n^\mu(x, t)$, $M_\mu(ts) = 2(2/ts)^{\mu-1/2} J_{\mu-1/2}(ts)$, *and* $\psi(s) = \sum_{n=0}^{\infty} (-2is)^n b_n$ *is a distribution in* D'.

PROOF. Since $M_\mu(ts)$ is a multiplier in D' the integral is an ultradistribution in Z' if $\psi(s)$ is in D'. We have

$$L^\mu(x, t) = T_t^\mu\{(4\pi t)^{-1/2} e^{-x^2/4t}\}/\Gamma(\mu + 1/2)$$
$$= t^{1-2\mu}(t^2 - x^2)_+^{\mu-1}/\sqrt{\pi}\,\Gamma(\mu)$$
$$= (1/2\pi)\int_{-\infty}^{\infty} e^{ixs} M_\mu(ts)ds$$

where the Fourier transform is defined in the generalized sense, (see [**15**]). Also

$$\tilde{\mathscr{E}}_n^\mu(x, t) = (-2)^n D_x^n L^\mu(x, t) = (1/2\pi)\int_{-\infty}^{\infty} (-2is)^n e^{ixs} M_\mu(ts)ds$$

and

$$\sum_{n=0}^{\infty} b_n \tilde{\mathscr{E}}_n^\mu(x, t) = (1/2\pi)\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (-2is)^n b_n e^{ixs} M_\mu(ts)ds$$
$$= (1/2\pi)\int_{-\infty}^{\infty} e^{ixs} \psi(s) M_\mu(ts)ds$$

provided the term-by-term integration is valid. It is by the continuity of the Fourier transform in Z'.

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