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SYSTEMS OF QUADRATIC FORMS II

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Dedicated to the memory of Gus Efroymson

0. Introduction. This paper is intended to continue the work of the papers [4] of the author and [3] of D. Leep. A system of r quadratic forms in n variables over a field F is replaced by a "quadratic map" $q: V \to W$ where V, W are F-vector spaces of dimension n resp. r. Section 1 contains some general definitions and properties of quadratic maps. In §2 I introduce the u-invariants $u_r (r = 1, 2, \ldots)$ and prove (or collect) results about these invariants. The main result of the present paper is given in §3. It concerns invariants u_r' which I hope are strongly related to u_r.

As the whole topic is still in "statu nascendi" it should be no surprise to the reader that there are more remarks, questions and problems than theorems. This paper owes a lot to discussions with D. Leep and D. Shapiro and in particular to a preprint of [3].

1. Generalities. Let F be an arbitrary (commutative) field with char $F \neq 2$, and let V, W be finite-dimensional F-vectorspaces.

DEFINITION 1. A map $q: V \to W$ is called quadratic if it has the following two properties:

a) $q(av) = a^2 q(v)$ for $a \in F$, $v \in V$

b) The map $b: V \times V \to W$ defined by

$$b(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$$

is F-bilinear. It is called the "symmetric bilinear map associated to q".

DEFINITION 2. Two quadratic maps $q: V \to W$, $q': V' \to W'$ are called equivalent if there are F-isomorphisms $\sigma: V' \to V$, $\tau: W \to W'$ with $q'(v') = \tau(q(\sigma v'))$. This implies in particular that dim $V =$ dim V' and dim $W =$ dim W'.

REMARK. For dim $W = 1$ this reduces to similarity of the quadratic forms q, q' over F, not to ordinary equivalence of q, q'.

The radical Rad $q = \{v \in V | b(v, V) = 0\}$, regularity of q (i.e., Rad $q = 0$) and the (outer) direct sum of quadratic maps $q_i: V_i \to W$ are defined in an obvious way. For $q: V \to W$ the space $\underbrace{V \oplus \cdots \oplus V}_{m}$ is denoted by mV and the induced quadratic map from mV to W is denoted by mq, i.e.,

$$mq(v_1 \oplus \cdots \oplus v_m) = q(v_1) + \cdots + q(v_m), \; v_i \in V.$$

DEFINITION 3. A quadratic map $q\colon V \to W$ is called isotropic if there exists $v \in V$, $v \neq 0$ with $q(v) = 0$.

DEFINITION 4. A quadratic map $q\colon V \to W$ is called hyperbolic if there exists a subspace $U \subset V$ with $\dim U \geqq (1/2) \dim V$ and $q(u) = 0$ for all $u \in U$. We write $q \sim 0$ for a hyperbolic quadratic map.

REMARKS. 1) For $\dim W = 1$ the above definitions coincide with the classical definitions for a quadratic form q only if q is regular.

2) The definition of hyperbolic forms is crucial. However, I do not know whether it is best possible. Other possibilities would be the following:

a) $V = U_1 + U_2$ with subspaces U_i and $q|U_i = 0$ $(i = 1, 2)$.

b) All maximal totally isotropic subspaces of V have the same dimension $\geqq (1/2) \dim V$.

c) We have b) and any two maximal totally isotropic subspaces of V can be transformed into one another by an element of the orthogonal group $0(q)$ (which is defined in an obvious way).

DEFINITION 5. A quadratic map q has order m if $mq \sim 0$. q has finite order if $mq \sim 0$ for some natural number m.

REMARKS. 1) It can happen that for instance $q \nsim 0$, $2q \nsim 0$ but $3q \sim 0$ and $4q \sim 0$.

2) Perhaps only two-powers $m = 2^\mu$ should be allowed in the definition of finite order.

PROPOSITION 1. *Let $q\colon V \to W$ be a quadratic map with radical* Rad q. *Let V_0 be any complement of* Rad q *in V. Then $q_0 = q|V_0\colon V_0 \to W$ is determined up to equivalence by q and q_0 is regular.*

This proposition allows us in many cases to consider only regular maps q.

PROPOSITION 2. *If F is infinite and if $q\colon V \to W$ is anisotropic then q is strongly regular, i.e., there exists an F-linear map $\lambda\colon W \to F$ such that the quadratic form $\lambda \circ q$ is regular.*

PROPOSITION 3. *If the maps $q_i\colon V_i \to W$ are (strongly) regular $(i = 1, \ldots, m)$ then $q = q_1 \oplus \cdots \oplus q_m$ is (strongly) regular. (Suppose F infinite)*

PROPOSITION 4. *Let F be nonreal with level $s = s(F)$ and let $m = 2s$. Then for any quadratic map $q\colon V \to W$ we have $mq \sim 0$.*

2. The invariants $u_r(F)$.

DEFINITION 6. $u_r(F) = \mathrm{Max}\{n \mid$ there exists a quadratic map $q\colon V \to W$ with the following properties: $\dim V = n$, $\dim W = r$, q anisotropic, q has finite order$\}$.

REMARKS. 1) For $r = 1$ this is the u-invariant of Elman-Lam [**2**].

2) For a nonreal field F this is the "u-invariant for systems" of Leep [**3**], since Prop. 4 implies that the condition "q has finite order" is automatically fulfilled.

3) One can ask whether the condition "q has finite order" may be replaced by weaker conditions such as "q is indefinite with respect to any ordering of F". However, the following examples show that then $\mathrm{Max}\{n \mid \cdots\}$ tends to be ∞.

4) Example with $r = 1$ (well-known): $F = \mathbf{R}(x, y)$, $q = \langle \underbrace{1, \ldots, 1}_{n-3}, x, y, -xy\rangle$ is anisotropic for any $n \geqq 3$, but totally indefinite.

5) Example with $r = 3$ [**1**]: $F = \mathbf{R}$, $x = (x_1, \ldots, x_n)$, $q_1(x) = x_1^2 - x_2^2$, $q_2(x) = x_1x_2$, $q_3(x) = x_1^2 + x_2^2 - (2/n-2)(x_3^2 + \cdots + x_n^2)$. The system $q = \{q_1, q_2, q_3\}$ is anisotropic for any $n \geqq 3$ though every form in the pencil defined by q_1, q_2, q_3 has trace zero (when considered as a symmetric $n \times n$-matrix) and hence is indefinite.

6) I conjecture that $u_r(F)$ as defined in Def. 6 turns out to be finite for all r and many classical fields F though up to now I cannot prove this for a single real field F. See however §3.

PROPOSITION 5. *Let E/F be a finite field extension of degree l. Let $q\colon V \to W$ be a quadratic map over E with $\dim_E V = n$, $\dim_E W = r$. By reducing constants from E to F we get a quadratic map $q_F\colon V_F \to W_F$ over F with $\dim_F V_F = nl$, $\dim_F W_F = rl$. ($V_F = V$, $W_F = W$ as sets, $q_F(v) = q(v)$ for $v \in V$) We have:*

a) *q isotropic (over E) $\Leftrightarrow$ q_F isotropic (over F)*

b) *$mq \sim 0$ (over E) $\Leftrightarrow$ $mq_F \sim 0$ (over F)*

COROLLARY. $u_r(E) \leqq (1/l)u_{rl}(F)$

REMARKS. 1) It is desirable to prove similar going-up results for the cases $E = F(t)$ (purely transcendental extension) and $E =$ field with a complete discrete valuation v and residue field F. For $r > 1$ even the nonreal case seems to be difficult. There is however the following known result [**4**]:

2) If F is nonreal and $u_r(F) \leqq 2^i r$(i.e., F is a C_i^q – field) then $u_r(E) \leqq 2^{i+1}r$ for $E = F(t)$ and for $E = F((t))$.

3) Question: Does 2) remain true for real fields F?

3. The invariants $u_r'(F)$.

DEFINITION 7. $u_r'(F) = \operatorname{Max}\{n \mid$ there exists a quadratic map $q\colon V \to W$ with the following properties: $\dim V = n$, $\dim W = r$, q anisotropic, $2q \sim 0\}$.

REMARK. For $r = 1$ this invariant appears in the paper of Elman and Lam [2] where it is denoted by N_1. The important fact is an inequality $u_1' \leqq u_1 < 2u_1'$ whenever $0 < u_1' < \infty$.

PROPOSITION 6. *Let* $s(F) < \infty$. *Then* $u_r(F) \leqq 2ru_r'(F)$.

REMARK. This result is never better than Leep's result [3] $u_r(F) \leqq (r(r+1)/2)u_1(F)$. The only interest of Prop. 6 is the possibility that the assumption $s(F) < \infty$ may be unnecessary.

We come to the main result of this paper.

THEOREM. *For* $F = \mathbf{R}$ $u_r'(\mathbf{R})$ *is an even number and satisfies the inequality* $2[2r/3] \leqq u_r'(\mathbf{R}) < 2r$ *for every* $r \geqq 1$.

IDEA OF PROOF. Let $q\colon V = \mathbf{R}^n \to \mathbf{R}^r$ be anisotropic with $n = u_r'(\mathbf{R})$ and $2q \sim 0$. Show successively:

1) There exists $T \in GL(V)$ with $q(Tv) = -q(v)$ for $v \in V$.
2) $|T|^2 = (-1)^n$, hence $n = 2m$ is even.
3) The only eigenvalues of T in $\mathbf{C}$ are $\pm$i.
4) $T^2 = -E$.
5) V allows a complex structure defined by $iv = Tv$.
6) As a complex vector space of dimension m, V carries a quadratic map $\varphi\colon V \to \mathbf{C}^r$ such that $q = \operatorname{Im}\varphi$ is the "imaginary part" of φ.
7) $n \geqq 2r$ implies q isotropic: contradiction.
8) Let $r = 3$, $m = 2$, $\varphi_1 = 2z_1z_2$, $\varphi_2 = z_1^2 - z_2^2$, $\varphi_3 = i(z_1^2 + z_2^2)$ and $q_j = \operatorname{Im}\varphi_j$ for $j = 1, 2, 3$. Then the system $q = \{q_1, q_2, q_3\}$ is anisotropic over $\mathbf{R}$ with $2q \sim 0$, $n = 4$. This shows $u_3'(\mathbf{R}) \geqq 4$.
9) Together with the trivial estimate $u_{r+s}' \geqq u_r' + u_s'$ the lower bound $u_r' \geqq 2[2r/3]$ follows.

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