ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

ON PYTHAGOREAN REAL IRREDUCIBLE ALGEBROID CURVES

JESÚS M. RUIZ

Dedicated to the Memory of Gus Efroymson

In this note we deal with the pythagoras number p of certain 1-dimensional rings, i.e., real irreducible algebroid curves over a real closed ground field k. The problem we are concerned with is to characterize those real irreducible algebroid curves which are pythagorean (i.e., $p = 1$). We obtain two theorems involving the value-semigroup. Then we apply them to solve the cases of: (a) Gorenstein curves, (b) planar curves, (c) monomial curves, and (d) curves of multiplicity $\leqq 5$. Finally, two conjectures are stated.

1. Statement of the theorems. Let k be a fixed real closed field. A real irreducible algebroid curve is any real 1-dimensional complete local integral domain A whose residual field is k.

Let pA denote the pythagoras number of A (i.e., the least $p \geqq 0$ such that any sum of squares is a sum of p squares). It can be shown that pA is finite. When $pA = 1$, A is called pythagorean.

Now we recall some definitions [1]. As is known, the derived normal ring $\bar{A}$ of A is a discrete valuation ring and we denote by ν its valuation. The semigroup $\Gamma = \nu(A - \{0\})$ is called value-semigroup of A. Then

i) The multiplicity of A = least positive integer m in Γ.

ii) The degree of the conductor of A in $\bar{A}$ = least positive integer $c \in \Gamma$ such that each $n \geqq c$ is in Γ.

Conversely, if Γ is a numerical semigroup (i.e., $\Gamma \subset N$ and $N - \Gamma$ is finite) the right sides above give definitions of m and c. Finally we denote by $\mathscr{M}_\Gamma$ the class of all curves whose value-semigroup is Γ and by $\mathscr{P}yth_\Gamma$ the class of all pythagorean curves in $\mathscr{M}_\Gamma$. Then we have

THEOREM I. *$\mathscr{P}yth_\Gamma \neq \phi$ if and only if for each $q \in \Gamma$ the set $\Gamma_q = \{p - q \mid p \geqq q, p \in \Gamma\}$ is a semigroup.*

Now set $d = \min\{p \in \Gamma \mid p \not\equiv 0(\overline{m})\} \cup \{c\}$ and $E = \{p \in \Gamma \mid p \geqq d\}$. Then

THEOREM II. $\mathcal{P}yth_\Gamma = \mathcal{M}_\Gamma$ if and only if for each $q \in \Gamma$, $p \in E$ with $q < p$, we have $(1/2)(q + c) \leqq p$.

2. Sketch of the proofs. We may assume $A \subset \bar{A} = k[[t]]$, and we let ν be the standard valuation in A. The condition that A is pythagorean can be rephrased as follows. If g, $h \in A$, then $\sqrt{g^2 + h^2} \in A$. Notice that $\sqrt{g^2 + h^2} \in k\,[[t]]$. After this, the method to prove I and II consists of: (a) finding suitable $A \in \mathcal{P}yth_\Gamma$, $g, h \in A$ and identifying $\nu(\sqrt{g^2 + h^2})$; (b) finding suitable "equations" for an element $f \in k[t]]$ to be in A. Let us show now how this works in some cases.

PROOF (of I). For the "only if" part, let $A \in \mathcal{P}yth_\Gamma$ be such that $t^q \in A$. Then if $p_1, p_2 \in \Gamma$, $q < p_1 \leqq p_2$ there are $g_1, g_2 \in k[[t]]$ with $t^q g_1$, $t^q g_2 \in A$ and $\nu(g_1) = p_1 - q$, $\nu(g_2) = p_2 - q$. We have

$$\sqrt{t^{2q} + (h_1 + h_2)^2} = -t^q + \sqrt{t^{2q} + h_1^2} + \sqrt{t^{2q} + h_2^2} + f,$$

where $\nu(f) = p_1 + p_2 - q$. As A is pythagorean, $f \in A$, $p_1 + p_2 - q \in \Gamma$, and $(p_1 - q) + (p_2 - q) \in \Gamma_q$.

For the "if" part, it is checked that the monomial curve $A = \{f \in k[[t]]\colon f^{(n)}(0) = 0,\ n \notin \Gamma\}$ is pythagorean as a consequence of the hypothesis on the Γ_q.

PROOF ("Only if" of II). The proof is developed in four steps. The first one is the inequality $c \leqq 2d$. To do that, write $d = \lambda m + r$, $0 < r < m$ (case $r = 0$ is trivial). If $c > 2d$ a curve $A \in \mathcal{M}_\Gamma$ is obtained such that t^m, $t^d + t^{c-(r+1)}$, $t^{d+jr} \in A$, $j \geqq 1$. Then

$$\sqrt{t^{2\lambda m} + (t^d + t^{c-(r+1)})^2} = \sum_{l \geqq 1} M_l\, t^{d+(2l-1)r} + t^{\lambda m} g, \qquad M_l \in k,$$

where $\nu(g) = r + c - (d + 1)$. Since $\mathcal{P}yth_\Gamma = \mathcal{M}_\Gamma$ we conclude $t^{\lambda m} g < A$ and $\lambda m + r + c - (d + 1) = c - 1 \in \Gamma$, which is absurd.

The remaining steps run along the same lines. Once the suitable square root has been found, the hard part is to obtain effectively the curve $A \in \mathcal{M}_\Gamma$

PROOF ("If" of II) Let $A \in \mathcal{M}_\Gamma$, $g, h \in A$ and $f = \sqrt{g^2 + h^2} \in k\,[[t]]$. To show that $f \in A$ we distinguish two cases:

i) $q = \nu(f) \geqq d$. Then we can assume $g = t^q$ and the hypothesis applies to deduce a formula $f = ag + bh + f^*$, $\nu(f^*) \geqq c$, and so $f \in A$.

ii) $q = \nu(f) < d$. Then we can assume $t^m \in A$ and find numbers $a_{jl} \in k$ such that $f \in A$ if and only if it is true that

$$f^{(l)}(0) = 0 \text{ for } l < d,\ l \not\equiv 0\ (m), \text{ and}$$

$$\frac{1}{l!} f^{(l)}(0) = \sum_{j=1}^{s} \frac{1}{p_j!} f^{(p_j)}(0)\, a_{jl} \text{ for } l > d,\ l \notin \Gamma,$$

where $p_1 < \cdots < p_s$ are the integers $< c$ and $\not\equiv 0(m)$ in Γ. This is of course related to the moduli of Γ (see **[2]**). Finally, as g and h verify these equations, it follows by induction on q that so does f.

3. Applications. Recall that A is called Gorenstein if the length of the A module $\mathfrak{M}^{-1}A$ is 1 (where $\mathfrak{M}$ is the maximal ideal of A) **[3]**, and it is called Arf if emb $-$ dim(B) $=$ mult(B) for every local ring B infinitely near to A, **[4]**. Then from I and II, and general properties of the value-semigroup, one deduces:

(3.1) Assume A Gorenstein. Then $pA = 1$ if and only if mult $A \leqq 2$.

(3.2) Assume A plane. Then (a) $pA = 1$ if mult $A \leqq 2$; (b) $pA = 2$ if mult $A \geqq 3$.

(3.3) Assume A monomial. Then $pA = 1$ if and only if A is Arf.

Finally let us say that I and II furnish a useful device for exploring pythagorean curves of low multiplicity. Actually, we have obtained the list of all pythagorean curves of multiplicity $\leqq 5$. For instance, the ones of multiplicity 3 are

$$A_n = k[[t^3, t^{3n+1}, t^{3n+2}]], \qquad B_n = k[[t^3, t^{3n+2}, t^{3n+4}]] \ (n \geqq 1).$$

(Complete details are given in **[5]** and **[6]**.)

4. Two conjectures. In the light of the previous results the following conjectures are suggested:

(4.1) Every pythagorean curve is Arf.

(4.2) Every local ring infinitely near to a pythagorean curve is pythagorean too. Both of them can be tested for multiplicity $\leqq 5$, of course.

REFERENCES

1. A. Campillo, *Algebroid curves in positive characteristic*. Lecture Notes in Mathematics **813**, Springer-Verlag, New York, 1980.

2. S. Ebey, *The classification of singular points of algebraic curves*, Trans. A.M.S. **118** (1965), 454–471.

3. E. Kunz, *The value-semigroup of a one-dimensional Gorenstein ring*. Proc. A.M.S. **25** (1970), 748–751.

4. J. Lipman, *Stable ideals and Arf rings*, Amer. J. Math. **93** (1971), 649–685.

5. J. M. Ruiz, Ph. D. Dissertation, Univ. Complutense de Madrid, 1982.

6. ———, On pythagorean real curve germs, preprint, 1983.

DEPARTAMENTO DE ALGEBRA, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE, MADRID, SPAIN