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HOVANSKY' THEOREM AND COMPLEXITY THEORY.

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Dedicated to the memory of Gus Efroymson

The additive complexity of $P \in R[X_1, \ldots, X_n]$ is related to the set of zeros of P in R^n.

1. Hovansky's theorems. (Cf. [2], [3]). The results of Hovansky are in the spirit of Bezout's theorem, but in the real case. Let us recall Descartes's lemma.

LEMMA 1.1. *If $P = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, the number of positive real roots of P is smaller than the number of changes of signs in the sequence $a_0, \ldots, a_n$.*

PROOF. This is very simple by induction on n, using Rolle's theorem.

COROLLARY 1.2. *The number of positive real roots of P is smaller than the number of non-zero monomials in P.*

The result of Hovansky is a generalisation of this-corollary.

THEOREM 1.3. *Let $F_1, \ldots, F_n \in R[X_1, \ldots, X_n, Y_1, \ldots, Y_k]$, $\deg. F_i = m_i$, where $Y_i = e^{\langle a^i, X\rangle} (1 \leqq i \leqq k)$ with $\langle a^i, X\rangle = \sum_{j=1}^n a^i_j X_j$, $a^i_j \in R$. Then the number of non-degenerate roots in R^n of the system $\{F_i(X, Y(X)) = 0, 1 \leqq i \leqq n\}$ (with $X = (X_1, \ldots, X_n)$) is $\leqq 2^{(1/2)k(k-1)}(1 + \sum m_i)^k \prod m_i$.*

The proof is by induction on k, beginning with the classical Bezout theorem, and using an old method of Liouville to "kill" the exponentials, and a variant of Rolle's theorem.

COROLLARY 1.4. *Let $P_1, \ldots, P_n \in R[X_1, \ldots, X_n]$, the total number of monomials in $(P_1, \ldots, P_n)$ being k; then the number of non-degenerate solutions in R^n_+ of the system $P_1 = \cdots = P_n = 0$, is $\leqq (1 + n)^k 2^{k(k-1)/2}$.*

PROOF. Put $X_i = e^{Y_i}$, and use Theorem 1.3.

REMARKS 1.5. a) Probably the bound in the corollary can be greatly improved.

b) Theorem 1.3 can be generalised to a large set of analytic functions.

2. Additive complexity of polynomials in one variable over R. The additive complexity of $P \in R[X]$, denoted $L_+^R(P)$, is by definition the minimum number of additions and subtractions required to evaluate P over R. $L_+^R(P) \leqq k$ if and only if there exists a system of $k+1$ equations:

$$(1) \qquad \begin{cases} S_0 = X \\ S_k = c_k \prod_{i=0}^{k-1} S_i^{m(i,k)} + d_k \prod_{i=0}^{k-1} S_i^{m'(i,k)}, \\ P_* = c_{k+1} \prod_{i=0}^{k} S_i^{m(i,k+1)} \end{cases}$$

with $m(i, j)$ and $m'(i,j)$ in Z, c_i and d_i in R, and $P(X)$ being evaluated from P_* by successive elimination of the $S_i (1 \leqq i \leqq k)$.

THEOREM 2.1. *Let $\rho(k)$ be the l.u.b. of the distinct real zeros of P such that $L_+^R(P) \leqq k$; then there exists a constant $C > 0$ such that $\rho(k) \leqq C^{k^2}$.*

PROOF. Make a little perturbation to P, and apply Hovansky's result to the system (1) (the last equation being $P = 0$), c.f. [**4**].

REMARKS 2.2. a) This result is an amelioration of a result of Borodin and Cook. [**1**].

b) The best lower bound known for $\rho(k)$ is 3^k (this bound is attained for Chebyshev polynomials).

3. Additive complexity of polynomials in several variables (over R). If $P \in R[X_1, \ldots, X_n]$, the definition of $L_+^R(P)$ is the same as in §2; let $C(P)$ be the number of connected components of $Z(P)$.

THEOREM 3.1. *There exists a function $\psi(k, n)$: $N \times N \to N$* such that *$C(P) \leqq \psi(k, n)$ for all $P \in R[X_1, \ldots, X_n]$ with $L_+^R(P) \leqq k$.*

PROOF. Induction on n, the case $n = 1$ having been solved in §2.

Let $C_b(P)$ be the number of bounded components of $Z(P)$, and $C_n(P)$ be the number of non bounded components.

LEMMA 3.2. (Cf. [**2**], [**4**]). *There exists an affine hyperplane $H \subset R^n$ intersecting at least $C_n(P)/2$ unbounded components of $Z(P)$.*

This lemma bounds $C_n(P)$, because $P|H$ is a polynomial in $n-1$ variables and one can apply induction hypothesis.

To majorize $C_b(P)$, one uses the fact that if C is a smooth compact component of $Z(P)$, then the function $X_n|C$ has at least two critical points, and the following lemma.

LEMMA 3.3. *If $L_+^R(P) \leqq k$, then $L_+^R(\partial P/\partial X_i) \leqq 3k(k+2)/2$.*

To prove Theorem 3.1, one must then apply Hovansky's theorem to the system of equations satisfied by the critical points of $X_n|Z(P)$ (c.f. [**4**]).

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