SEMI-INTEGRAL EXTENSIONS AND PROPER MORPHISMS

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Dedicated to the memory of Gus Efroymson.

Semi-integral extensions were introduced by Brumfiel in [1] from an algebraic point of view, and applied to study algebraic varieties over real closed fields. In this note we develop the connection between semi-integral extensions and proper morphisms.

Let R be a real closed field. If X is an irreducible algebraic variety over R, we denote by X_c the maximum dimension locus of X, by R[X] the coordinate ring of X and by R(X) its field of rational functions; we denote by β_d the ordering in R[X] induced by the weakest ordering in R(X). A dominant morphism $\varphi: X \to Y$ of irreducible algebraic varieties over R is called semi-integral if $(R[X], \beta_d)$ is a semi-integral extension of $(R[Y], \beta_d \cap R[Y])$ via φ . Then we prove the following proposition.

PROPOSITION 1. The following are equivalent.

(a) φ is semi-integral

(b) For every bounded semialgebraic set $S \subset Y$, $[\varphi^{-1}(S)] \cap X_c$ is bounded.

(c) (i) For every closed semialgebraic set $S \subset X_c$, $\varphi(S)$ is closed, and, (ii) For every $y \in Y$, $[\varphi^{-1}(y)] \cap X_c$ is bounded.

From this proposition we can deduce a result of Brumfiel, with no need of Hironaka's desingularization theorem [2].

COROLLARY 2. Let $f, g \in R[X]$, Then f/g is semi-integral over R[X] if and only if f/g is bounded on bounded semialgebraic sets of $X_c - V(g)$.

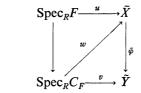
Condition (c) in proposition 1 can be taken as a definition of real properness. Then, in terms of real spectra [3], we obtain a valuative Criterion for properness analogous to the classical one. Note that $\tilde{X} = \text{Spec}_R R[X]$, $\tilde{Y} = \text{Spec}_R R[Y]$, and $\tilde{\varphi}: \tilde{X} \to \tilde{Y}$ the morphism induced by φ . We have

PROPOSITION 2. The following are equivalent.

- (a) (i) For every closed constructible set S ⊂ X̄, φ̃(S) is closed and
 (ii) For every rational point y ∈ Y, φ̃⁻¹(y) is real complete.
- (b) (Valuative Criterion) For every real closed field F, every real valuation

ring C_F of F and all morphisms u, v like those in (*), there is a unique morphism w which makes the diagram commutative.

(c) $\tilde{\varphi}$ is universally closed.



References

1. G. W. Brumfiel, *Partially ordered rings and semialgebraic geometry*, London Math. Soc. Lect. Not. **37** (1979).

2. ——, Real valuations and ideals. Springer Lectures Notes, 959 (1983).

3. M. Coste and Coste-Roy, M. F. *La topologie du spectre reel*. Contemporary Math. **8** (1982) 27–59.

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