ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 1, Winter 1986

HEIGHT ONE SEPARABILITY AND GALOIS THEORY IN SEMI-KRULL DOMAINS*

LAWRENCE NAYLOR

ABSTRACT. An R-algebra S is said to be height 1 separable over R if S_p is R_p-separable for each height 1 prime ideal p in R. In this paper we look at a class of domains called semi-Krull domains, and show that a semi-Krull domain has a unique height 1 separable closure. We then show that there is a Galois correspondence between certain subextensions of the base ring in its height 1 separable closure and certain subgroups of the automorphism group.

Introduction. This paper extends some of the results of Janusz dealing with separable algebras over connected commutative rings [3]. Janusz defines a separable closure for such a ring, and proves that separable closures do exist and are unique. He proves some basic properties of separable closures, and then describes how one could develop an infinite Galois theory for a connected commutative ring R inside its separable closure S by taking the full automorphism group of S over R, $\mathrm{Aut}_R(S)$, as the Galois group and exhibiting a one-to-one correspondence between closed subgroups of $\mathrm{Aut}_R(S)$ and certain R-subalgebras of S.

In this paper we continue with the idea of height 1 separability introduced in [6], i.e., an R-algebra S is said to be height 1 separable over R (where R is commutative with 1) if S_p is R_p-separable for each prime ideal p in the set $X'(R)$ of prime ideals of height $\leqq 1$. Some general properties of height 1 separable algebras are also found there. We make a slight modification in the meaning of $X'(R)$. Throughout this paper, $X'(R)$ will denote the set of all prime ideals in R of height 1. When R is a domain which is not a field and S_p is R_p-separable for a height 1 prime ideal p in R, then it follows (by a further localization) that $S_{(0)}$ is $R_{(0)}$-separable. Krull domains provide a good setting in which to study height 1 separa-

Received by the editors on December 8, 1982 and in revised form on September 3, 1984.

AMS (MOS) *subject classification* (1980). Primary 13B05.

Key words and phrases: Separability, separable closure, Galois correspondence, height one prime ideal, Krull domain.

* This paper is based on a portion of the author's Ph. D. thesis, written under the direction of Dr. Andy Magid.

bility, and a question arises: Is there a corresponding Galois theory? That is, is there a unique extension of a Krull domain which could be called a "height 1 separable closure" such that there is a correspondence between subextensions of the base ring inside the height 1 closure and subgroups of the automorphism group? It turns out that in this setting it is not really necessary to require that the domain be precisely the intersection of its height 1 localizations, and so we are led to consider a class of domains called semi-Krull domains (§1). We then show (§2) that a semi-Krull domain R does have a unique extension S which acts as a "height 1 separable closure", and we are able to show that there is a Galois correspondence between closed subgroups of finite index of $\mathrm{Aut}_R(S)$ and integrally closed subextensions T of R in S such that, for each p in $X'(R)$, T_p is a separable R_p-algebra and a finitely generated projective R_p-module (§3).

Many of the facts concerning separability that are used or referred to here can be found in [**1**]. Throughout this paper all rings are commutative with 1, and any unadorned tensors are taken over R.

1. Semi-Krull domains. For a given ring R, let $X'(R)$ denote the set of all prime ideals in R of height 1. We make the following definition.

DEFINITION 1.1. A domain R is a semi-Krull domain if:
(i) R is integrally closed;
(ii) R_p is a discrete valuation ring (DVR) for each p in $X'(R)$; and
(iii) each non-zero element of R is contained in at most a finite number of primes in $X'(R)$.

While a Krull domain is clearly semi-Krull, the converse is not true, as the next example shows.

EXAMPLE 1.2. (A semi-Krull domain R which is not a Krull domain). Let k be a field, let R' be the subring of $k[x, y]$ of all polynomials having no term in a power of x alone, and let M be the prime ideal in R' consisting of all polynomials with constant term zero. Then M is minimal over (y) and height $(M) \geqq 2$ (see [**6**, example 2.5]). Let $R = R'_M$. Then the element y is not contained in any height 1 prime ideal in R. It is shown in [**6**] that the domain R is integrally closed. We show next that if p is in $X'(R)$, then R_p is a DVR. This will follow if R'_p is a DVR whenever p is in $X'(R')$ with p contained in M. Note that x $(= xy/y)$ is in R'_p. Thus $k\,[x, y] \subseteq R'_p$. If $P_0 = pR'_p \cap k[x, y]$, it is straightforward to check that $k[x, y]_{p_0} = R'_p$. Then R'_p is a local PID (distinct from its quotient field) and is thus a DVR.

We show now that a non-zero element in R cannot lie in infinitely many height 1 prime ideals of R; again it suffices to show that a non-zero element of R' cannot lie in infinitely many height 1 primes of R' contained in M. Suppose $f \neq 0$ is in R' and $f \in p$, for p in $X'(R')$ with $p \subseteq M$. As above, let $P_0 = pR'_p \cap k[x, y]$. We have $k[x, y]_{p_0} = R'$, and so P_0 is a height 1 prime of $k[x, y]$ with $f \in P_0$. A straightforward argument shows that if P and Q are in $X'(R')$ with $P \neq Q$, then the corresponding P_0 and Q_0 (as defined above) are distinct. Since $k[x, y]$ is Krull, we see that f cannot lie in infinitely many height 1 primes of R . We have shown that R is semi-Krull. Finally, we note that R is not Krull since the element x is in R_p for each p in $(X'R)$ but x is not in R.

We present some terminology now. An R-algebra S is strongly separable over R if S is separable as an R-algebra and finitely generated and projective as an R-module. An R-algebra S is height 1 strongly separable over R if S_p is strongly separable over R_p for each p in $X'(R)$. An R-algebra S is a finite extension of R if S is finitely generated as an R-module. An R-algebra S is a locally height 1 separable extension of R if S is a direct limit of height 1 separable subextensions; i.e., if every finite subset of S is contained in some subalgebra of S which is height 1 separable over R. Likewise, an R-algebra S is a locally height 1 strongly separable extension of R if S is a direct limit of height 1 strongly separable subextensions, and S is a locally finite height 1 strongly separable extension if it is a direct limit of finite height 1 strongly separable subextensions.

The following is a general result that we will need on several occasions.

PROPOSITION 1.3. *If T and S are domains with the same quotient field such that T is contained in S and S is a finitely generated projective T-module, then $S = T$.*

PROOF. Let L be the common quotient field of T and S. T is a T-direct summand of S (e.g., [**1**; cor. 4.2, p. 56]), and so as a T-module, $S = T \oplus M$, where M is a projective T-module. Tensoring S with L, we get $L = L \otimes_T(T \oplus M) = L \oplus (L \otimes_T M)$. Thus, $L \otimes_T M = 0$, and so $M = 0$, completing the proof.

PROPOSITION 1.4. *Let R be semi-Krull and let S, a domain, be a finite extension of R such that the quotient field of S is a finite separable field extension of the quotient field of R. Then S_p is a separable R_p-algebra for all but possibly a finite number of primes p in $X'(R)$.*

PROOF. Let L, K be the quotient fields of S, R, respectively. Since L is a finite separable extension of K, $L = K(u)$, for some primitive element u in S. Let $f(x)$ be the minimal polynomial of u. Then $f(x)$ is in $R[x]$. Let r be the resultant of f and f', $r = \text{Res }(f, f')$. If b is the leading coefficient of f', then, since r and b are in R, there are at most a finite number of primes in $X'(R)$ containing r or b.

Suppose p is in $X'(R)$ and neither r nor b is in p. Let $T_p = R_p[x]/(f)$. We show that T_p is a strongly separable R_p-algebra. Clearly T_p is a finitely generated projective R_p-module. Let k be the field R_p/pR_p. Then $T_p/(pR_p)T_p = (R_p[x]/(f))/(pR_p)T_p = k[x]/(f)$. Since Res (f, f') is not in p (and b is not in p), Res $(f, f') \neq 0$ in k. Hence, f is a separable polynomial. It follows that T_p is a separable R_p-algebra.

Next we show that $S_p = T_p$. Note that the quotient field of T_p is L. Since S_p is R_p-torsion free and a finite R_p-module, where R_p is a PID, S_p is R_p-free. Now, since T_p is separable over R_p and S_p is projective over R_p, S_p is projective over T_p. Hence, by (1.3), $S_p = T_p$ and the proof is complete.

PROPOSITION 1.5. *If R is a* PID *and S is a domain which is strongly separable over R, then S is integrally closed.*

PROOF. Let x be an element in the integral closure of S in its quotient field. Then $S[x]$ is a torsion-free finite R-module, and thus projective over R [**2**; 2, p. 287]. Furthermore, $S[x]$ is projective as an S-module. Thus, by (1.3), we see that $S = S[x]$, and S is integrally closed.

We close this section with some results on semi-Krull domains. Before stating the next lemma, we recall the connection between a "separable algebra extension" and "separable field extension" for an extension of fields. If a field S is an extension of a field R then S is separable as an R − algebra if and only if S is a separable field extension of R of finite dimension over R(e.g., see [**1**, ch. 2]).

LEMMA 1.6. *Suppose S and T are domains with quotient fields K and L, respectively. If T is a height* 1 *separable extension of S, then L is a finite separable* (*algebraic*) *field extension of K.*

PROOF. Let p be in $X'(S)$. Then, since T_p is S_p-separable, we see that $(S - 0)^{-1} T_p = (S - 0)^{-1}T$ is a separable K-algebra. The field L is just a further localization of $(S - 0)^{-1}T$ and so by transitivity L is a separable K-algebra. Since K is a field, then L must be a finite separable (and hence, algebraic) field extension of K.

COROLLARY 1.7. *If S and T are domains such that T is a locally height* 1 *separable extension of S, then the quotient field of T is a separable algebraic field extension of the quotient field of S.*

PROOF. Let L and K be the quotient fields of T and S, respectively. Any finitely generated sub-K-algebra of L is contained in the quotient field of some height 1 separable extension of S. This quotient field is a separable algebraic extension of K by (1.6). Thus L itself is a separable algebraic field extension of K.

PROPOSITION 1.8. *Let T be an integrally closed domain extension of the* *semi-Krull domain R. If T is either a finite extension of R or an integral height* 1 *strongly separable extension of R, then T is semi-Krull.*

PROOF. First suppose that T is finitely generated as an R-module. Let p be in $X'(T)$. Then $Q = p \cap R$ is in $X'(R)$. The localization R_Q is Noetherian, and so T_Q and also T_p are Noetherian. Thus, T_p is a DVR.

Next, let t be a nonzero element of T, and suppose t belongs to infinitely many primes in $X'(T)$. Since t is integral over R, there are elements a_i in R with $t^n + a_1 t^{n-1} + \cdots + a_n = 0$. Thus,

$$a_n = -t^n - a_1 t^{n-1} - \cdots - a_{n-1} t. \tag{1}$$

The element a_n, then, belongs to infinitely many primes in $X'(T)$. Each prime in $X'(T)$ contracts to a prime in $X'(R)$, and, further, there are only a finite number of primes in $X'(T)$ lying over any prime in $X'(R)$, since T is a finite R-module. Thus, a_n belongs to infinitely many primes in $X'(R)$, and so $a_n = 0$. From (1) we then obtain $a_{n-1} = - t^{n-1} - a_1 t^{n-2} - \cdots - a_{n-2} t$. The above argument shows that $a_{n-1} = 0$. Repeating this argument, we eventually see that all $a_i = 0$, a contradiction since $t \neq 0$. Hence, t lies in at most finitely many primes in $X'(T)$. Since T is integrally closed by hypothesis, T is semi-Krull.

Now, assume that T is an integral extension of R and height 1 strongly separable over R. Let p be in $X'(T)$. As above, T_p is a DVR. The rest of the proof follows as above, except that the reason that there are only a finite number of primes in T lying over any prime in R follows from the fact that T is a height 1 separable extension of R. For then, the quotient field of T is a finite separable extension of the quotient field of R by (1.6).

2. Height 1 separable closures. In this section we define a height 1 separable closure for a semi-Krull domain, and show that such a closure exists and is unique (up to isomorphism).

DEFINITION 2.1. Let R be a semi-Krull domain. A height 1 separable closure of R is a domain extension S which satisfies the following:

(i) S is a locally finite height 1 strongly separable extension of R; and

(ii) Whenever T is a domain which is a finite height 1 strongly separable extension of S, then $T = S$; i.e., S is height 1 separably closed.

We will show that such a height 1 separable closure of a semi-Krull domain exists and is unique up to isomorphism. For reference, we note the following characterization of strong separability found in **[1]**.

S is a strongly separable extension of R if and only if there is an element t in $\mathrm{Hom}_R(S, R)$ and elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in S such that $\sum x_j y_j = 1$ and $\sum x_j t(y_j x) = x$ for all x in S.

THEOREM 2.2. *If R is a semi-Krull domain, then a height* 1 *separable closure of R exists.*

PROOF. Suppose S is a domain which is a locally finite height 1 strongly separable extension of R. If p is any prime in $X'(R)$, then S_p is a locally strongly separable extension of R_p, and so the cardinality of S_p is bounded by some cardinal number depending only on R_p [1; 3.3, p. 103]. Since $S \subseteq S_p$, we see that the cardinality of S is also bounded by a cardinal number depending only on R.

Now let T be a domain containing S which is a finite height 1 strongly separable extension of S. We will show that T is a locally finite height 1 strongly separable extension of R.

Let $x_1, \ldots, x_n$ generate T as an S-module. Let $x_ix_j = \sum a_{ijk}x_k$, for a_{ijk} in S. Pick a finite subset F of T. Each element f of F is of the form $f = \sum s_ix_i$, s_i in S. If $S = \text{dir lim } S_i$, where each S_i is a finite height 1 strongly separable R-subalgebra of S, then there is an index i, say $i = 0$, such that S_0 contains all the elements a_{ijk} and s_i described above. Let $T_0 = \sum S_0x_i$. Then T_0 is an S_0-algebra, finitely generated as an S_0-module.

Without loss of generality we may assume that the quotient field of T_0 is a finite separable field extension of the quotient field of R. (See the remark following this proof.) Thus, by (1.4), T_0 is height 1 separable over R at all but possibly a finite number of prime ideals in $X'(R)$.

Suppose that p is in $X'(R)$ and $(T_0)_p$ is not R_p-separable. We will show that we can replace S_0 by some S_i, $i \geqq 0$, so that the "new" $(T_0)_p$ is R_p-separable, and that raising S_0 like this will not affect the separability of T_0 at primes in $X'(R)$ where it was already separable. We show this last point first.

If $S_0 \subseteq S_i$, let $T_i = \sum S_ix_i$. Suppose that $(T_0)_Q$ is R_Q-separable for Q in $X'(R)$. By the construction of T_i, we see that the map $S_i \otimes_R T_0 \to T_i$ is onto, and, hence, the map $(S_i)_Q \otimes_{R_Q} (T_0)_Q \to (T_i)_Q$ is onto. Since $(S_i)_Q$ and $(T_0)_Q$ are each R_Q-separable, it follows that $(S_i)_Q \otimes (T_0)_Q$ is R_Q-separable, and therefore $(T_i)_Q$ is separable over R_Q. Clearly, $(T_i)_Q$ is a torsion-free R_Q-module, and, since $(T_i)_Q$ is finitely generated as a R_Q-module, we have $(T_i)_Q$ is projective over R_Q. Thus we see that $(T_i)_Q$ is strongly separable over R_Q. We have shown that if we replace S_0 by a larger S_i and therefore T_0 by T_i, T_i is strongly separable at any prime in $X'(R)$, where T_0 is strongly separable. (Notice that if p is in $X'(R)$ and $(T_0)_p$ is R_p-separable, then $(T_0)_p$ is automatically strongly separable over R_p just as in the argument above for $(T_i)_Q$.)

We now return to the first part of the claim. Suppose that p is in $X'(R)$ and T_{0_p} is not separable over R_p. It is easily shown that T_p is height 1 separable over S_p, and, since it is finitely generated over S_p, T_p is S_p-separable. Further, we show that T_p is S_p-projective. If M is a maximal ideal in S_p, then $M \cap R_p = pR_p$. Since $\text{ht}(M) \leqq \text{ht}\,(M \cap R_p) = 1$, it follows that M is actually in $X'(S_p)$. If Q is in $X'(S_p)$, then $Q' = Q \cap S$ is in $X'(S)$, and so $T_{Q'}$ is strongly separable over $S_{Q'}$. Therefore, $(T_{Q'})_p = T_{Q'} \otimes R_p$

is a strongly separable extension of $(S_{Q'})_p$. Since $(T_p)_Q = (T_{Q'})_p$ and $(S_p)_Q = (S_{Q'})_p$, we see that each localization of T_p at a maximal ideal of S_p yields a finitely generated flat module. Thus, T_p is a flat S_p-module, and so T_p is projective over S_p. Hence, T_p is strongly separable over S_p.

To simplify the notation we will drop the subscript p for the moment. We have the following situation. T is a strongly separable extension of S, S is a locally strongly separable extension of R, $S_0 \subseteq S$ is a strongly separable extension of R, and $T_0 \subseteq T$ is an S_0-algebra, finitely generated as a module. There are elements z_i, y_i $i = 1, \ldots, n$, in T and t in $\mathrm{Hom}_S(T, S)$ such that $\sum z_i y_i = 1$ and $\sum z_j\, t(y_j x) = x$, for each x in T. Let $z_i = \sum a_{ij} x_j$ and $y_i = \sum b_{ij} x_j$, for a_{ij} and b_{ij} in S. If necessary, replace S_0 by a larger S_i so that S_0 contains all a_{ij}, b_{ij}, and $t(x_i)$. Then $t(\sum s_i x_i) = \sum s_i t(x_i)$ is in S_0, and so $t|_{T_0}$ is in $\mathrm{Hom}_{S_0}(T_0, S_0)$. Also, T_0 contains the elements z_i, y_i. Hence, T_0 is strongly separable over S_0, and, thus, strongly separable over R. In our original notation, we have shown that T_{0_p} is strongly separable over R_p.

We may repeat this process finitely many times, if necessary, in order to obtain $T_0 \subseteq T$ such that T_0 is a finite height 1 strongly separable extension of R. Since we can find such a T_0 containing any finite subset of T, we have shown that T is a locally finite height 1 strongly separable extension of R.

A cardinality argument as in [**3**, 1.4] or [**1**, ch. III] shows that a height 1 separable closure of R must exist.

REMARK. In the proof of Theorem 2.2 it was noted that the quotient field of $T_0 = \sum S_0 x_i$ could be assumed to be a finite separable field extension of the quotient field of R. This can be assured by picking a large enough S_0, as we show now. Let L, K, L_0, K_0 be the quotient fields of T, S, T_0, S_0, respectively. We know that L is a finite separable field extension of K by (1.6). Hence, L is strongly separable as a K-algebra. Let $L = K(u)$, for a primitive element u in S. Each element x_i in $T \subseteq L$ (recall that $x_1, \ldots, x_n$ generate T over S) can be expressed as a polynomial in u, say $f_i(u)$, with coefficients in K. Now, if necessary, raise S_0 so that S_0 contains the element u and K_0 contains all the coefficients of the $f_i(u)$. Then $L_0 = K_0(u)$.

Since L is strongly separable over K, there are elements z_i, y_i, $i = 1, \ldots, m$, in L and t in $\mathrm{Hom}_K(L, K)$ with $\sum z_i y_i = 1$ and $\sum z_j t(y_j x) = x$, for each x in L. If necessary, raise S_0 again so that all the z_i, y_i are in L_0 and $t(u)$ is in K_0. Then, $t|_{L_0}$ is in $\mathrm{Hom}_{K_0}(L_0, K_0)$, and it follows that L_0 is a strongly separable extension of K_0. K_0 is a finite separable extension of the quotient field of R by (1.6), and so L_0 is also a finite separable field extension of the quotient field of R.

LEMMA 2.3. *If R is semi-Krull and $S =$* dir lim *S_i, where each S_i is a* *domain extension of R, finitely generated as an R-module, then whenever Q is in $X'(S)$, $Q \cap R$ is in $X'(R)$. (In this case, S is said to satisfy* NBU *over R,* i.e., *No Blowing Up.)*

PROOF. Let Q be $X'(S)$ and let $p = Q \cap R$. Suppose there is a prime p' in Spec(R) with $(o) \subsetneq P' \subsetneq p$. Note that S is a domain, and S is integral over R. Then, since R is integrally closed, there is a prime Q' in Spec(S) with $Q' \subseteq Q$ and $Q' \cap R = P'$, a contradiction. Thus p is in $X'(R)$.

THEOREM 2.4. *The height* 1 *separable closure of a semi-Krull domain R is unique, up to isomorphism.*

PROOF. If S is a height 1 separable closure of R, then the quotient field of S is a separable algebraic extension of the quotient field of R by (1.7). Hence, S is contained in a separable closure of the quotient field of R. We will show that, inside this separable closure, S is unique. Since any two separable closures are isomorphic, the result will follow.

Suppose that both S and T are height 1 separable closures of R contained in the same separable closure of the quotient field of R. Let $T =$ dir lim T_i, where each T_i is a finite height 1 strongly separable extension of R. We claim that ST_i is a finite height 1 strongly separable extension of S, for each i. Since R is integrally closed and S is integral over R, if Q is in $X'(S)$, then $Q \cap R$ is in $X'(R)$. Hence, by [6, 1.4], $S \otimes T_i$ is height 1 separable over S. Since $S \otimes T_i$ maps onto ST_i, ST_i is also height 1 separable over S by [6, 1.7]. Hence, we need only show that $(ST_i)_Q$ is projective over S_Q, for each Q in $X'(S)$. Let $p = Q \cap R$; then p is in $X'(R)$. To simplify the notation, we replace R_p, S_p, and T_{i_p} with R, S, and T, respectively. We have the following situation. R is a DVR, S is a locally strongly separable extension of R, and T is a strongly separable extension of R. Let $S =$ dir lim S_i, where each S_i is a strongly separable extension of R.

Since each $S_i \otimes T$ is a strongly separable R-algebra, $S_i \otimes T$ is a finite product of domains [3; 4.2, p. 473]. Since $S \otimes T$ is finitely generated and projective over S, $S \otimes T$ contains only a finite number of idempotents, $e_1, \ldots, e_n$, and $S \otimes T = (S \otimes T)e_1 \times \cdots \times (S \otimes T)e_n$. It is straightforward to check that each $(S \otimes T)e_i$ is a domain, and so $S \otimes T$ is also a finite product of domains.

Write $S \otimes T = D_1 \times \cdots \times D_n$, a product of domains. Since each D_i is strongly separable over S (because $S \otimes T$ is), we have $S \subseteq D_i$. Consider the surjection $D_1 \times \cdots \times D_n \to ST$, which is the multiplication map $S \otimes T \to ST$. Since $S \subseteq S \otimes T$ and $S \subseteq ST$, the kernel of this map intersects S in (0). This kernel is a prime ideal of the form $B = D_1 \times \cdots \times p_i \times \cdots \times D_n$, for some prime p_i in one of the D_i's. Then $B \cap S = (0)$ implies that $p_i \cap S = (0)$, and so $p_i = (0)$, since D_i is an integral extension of S. Thus, D_i is isomorphic to ST as an S-module, and ST is a projective $S \otimes$

T-module and therefore a projective S-module. In our original notation we have shown that $S_p T_{i_p} = (ST_i)_p$ is a projective S_p-module, for $p = Q \cap R$, Q in $X'(S)$. Localizing at p in Spec(R) and then localizing again at Q in Spec(S) is just the same as localizing at Q to begin with, since $R \subseteq S$ and $p \subseteq Q$. Hence, (ST_i) is a projective S_Q-module. Clearly, $(ST_i)_Q$ is finitely generated over S_Q. We have shown that ST_i is a finite height 1 strongly separable extension of S. Since S is height 1 separably closed, $S = ST_i$, for each i. Thus, $S = \text{dir lim } ST_i = ST$. The symmetry of the argument shows that $T = ST$, and so $S = T$, as desired. This completes the proof.

PROPOSITION 2.5. *If R is semi-Krull with height 1 separable closure S, then S is integrally closed.*

PROOF. We show first that S_p is integrally closed if p is in $X'(S)$. For such a p, let $Q = p \cap R$. Then Q is in $X'(R)$, by (2.3). Since S_Q is locally strongly separable over R_Q, we see that S_Q is a direct limit of strongly separable extensions of R_Q, each of which is integrally closed by (1.5). Hence, S_Q is integrally closed. It follows that S_p is integrally closed also. Now suppose that x is an element of the quotient field of S and x is integral over S. Then x is in the quotient field of S_p, for any p in $X'(S)$, and x is integral over S_p. Thus, x is in S_p. Consider now the finitely generated S-algebra $S[x]$. Note that $S[x]$ is actually finitely generated as an S-module. If p is in $X'(S)$, we have shown above that $S[x]_p = S_p$. Hence, $S[x]$ is a finite height 1 strongly separable extension of S. Since S is height 1 separably closed, $S[x] = S$, Thus, S is integrally closed.

PROPOSITION 2.6. *Let R be semi-Krull with height 1 separable closure S. If T is an integrally closed domain contained in S which is a finite height 1 strongly separable extension of R, then S is a height 1 separable closure of T also.*

PROOF. By (1.8), T is semi-Krull. If $S = \text{dir lim } S_i$, where each S_i is a finite height 1 strongly separable extension of R, there is an index i, say $i = 0$, such that $T \subseteq S_i$. Thus, $S = \text{dir lim } S_i$, where the limit is taken over $i \geqq 0$. We show that each S_i in this direct limit is also height 1 strongly separable over T. If p is in $X'(T)$, let $Q = p \cap R$. Then Q is in $X'(R)$. Since S_{i_Q} is a strongly separable extension of R_Q and T_Q is a strongly separable R_Q-extension contained in S_{i_Q}, S_{i_Q} is a strongly separable extension of T_Q. Then, since $R \subseteq T$ and $Q \subseteq p$, S_{i_p} and T_p are just further localizations of S_{i_Q} and T_Q. Hence, S_{i_Q} is a strongly separable extension of T_p. This finishes the proof.

We now present a few simple results on morphisms into height 1 separable closures.

PROPOSITION 2.7. *With R, T, and S as in* (2.6), *any R-algebra morphism $f\colon T \to S$ is induced by an automorphism of S.*

PROOF. Let S' denote S considered as a T-algebra by the action of f; i.e., $t \cdot s = f(t)s$, for t in T and s in S. S is a height 1 separable closure of T, and hence, so is S'. Thus, there is an isomorphism $g\colon S \to S'$. For all s in S and t in T we have $g(ts) = t \cdot g(s) = f(t)g(s)$. If $s = 1$, we get $g(t) = f(t)$, for all t in T.

PROPOSITION 2.8. *Let R and S be as above, and let T be a locally finite height* 1 *strongly separable domain extension of R. Then there is an R-algebra injection $T \to S$.*

PROOF. Clear, since T is contained in a height 1 separable closure of R which is isomorphic to S.

PROPOSITION 2.9. *Let R and S be as above, and let T be a height* 1 *strongly separable domain extension of R. Then there are only finitely many morphisms in* $\mathrm{Alg}_R(T, S)$.

PROOF. Let p be in $X'(R)$. Since T_p is a finitely generated separable R_p-algebra and S_p is connected, there are only finitely many R_p-homomorphisms from T_p to S_p. If $f\colon T \to S$ and $g\colon T \to S$ are R-algebra maps with $f \neq g$, then the induced R_p-algebra maps $f_p\colon T_p \to S_p$ and $g_p\colon T_p \to S_p$ are not equal. Hence there cannot be infinitely many R-algebra maps from T to S.

PROPOSITION 2.10. *Let R and S be as above, and suppose that S is contained in W, a separable closure of K, the quotient field of R. If f is a K-automorphism of W, then $f(S) = S$.*

PROOF. Clear, since $f(S)$ is also a height 1 separable closure of R contained in W.

We will show in (2.12) that if f is any R-endomorphism of S, then f is actually an automorphism. First, a lemma.

LEMMA 2.11. *Let T' be a height* 1 *strongly separable domain extension of R and let T be a height* 1 *separable domain extension of R. If f is an R-algebra homomorphism from T to T', then f is an injection.*

PROOF. If $f(x) = 0$, then $f_p(x) = 0$, for each p in $X'(R)$, where f_p is the induced map from T_p to T'_p. But the kernel of f_p is (0) by [**1**; 2.6, p. 96]. Thus, $tx = 0$, for some t in $R - p$, and so $x = 0$.

PROPOSITION 2.12. *Let R be semi-Krull with height* 1 *separable closure S. If f is any R-algebra homomorphism from S to S, then f is an automorphism of S.*

PROOF. Suppose $f(x) = 0$, for some x in S. Let T be a finite height 1 strongly separable extension of R contained in S such that x is in T. If $t_1, \ldots, t_n$ generate T as an R-module, let T' be a finite height 1 strongly separable extension of R in S containing $f(t_i)$, $i = 1, \ldots, n$. By (2.11), the restriction of f to T, mapping T to T', is an injection. Hence, $x = 0$, and f is an injection.

We show now that f is onto. Let y be in S and let T be a finite height 1 strongly separable extension of R in S containing y. There are only finitely many embeddings of T into S (2.9), say $g_1, \ldots, g_n$. Let g_1 be the inclusion map, $g_1(x) = x$ for x in T. Since $f|_T\colon T \to S$ is an injection, $\{f \circ g_1, \ldots, f \circ g_n\} = \{g_1, \ldots, g_n\}$. Thus, $g_1 = f \circ g_i$, for some i. Then $y = g_1(y) = f(g_i(y))$, showing f is onto.

3. Galois theory. Our objective in the rest of this paper is to examine the Galois theory connected with the height 1 separable closure of a semi-Krull domain. This means we will be interested in the groups $\mathrm{Aut}_T(S)$ and $\mathrm{Aut}_{T_p}(S_p)$, for certain subextensions T of R in S and p in $X'(R)$. We begin to look at these groups in the next proposition.

PROPOSITION 3.1. *Let R be semi-Krull with height 1 separable closure S. Let T be any extension of R contained in S, and let L, F be the quotient fields of S, T, respectively. Then, for any prime p in $X'(R)$, the groups* $\mathrm{Aut}_F(L)$, $\mathrm{Aut}_T(S)$, *and* $\mathrm{Aut}_{T_p}(S_p)$ *are isomorphic.*

PROOF. We show first that $\mathrm{Aut}_T(S)$ and $\mathrm{Aut}_{T_p}(S_p)$ are isomorphic. Let f be in $\mathrm{Aut}_T(S)$. Then $f_p\colon S_p \to S_p$, given by $f_p(s/r) = f(s)/r$, for s in S, r in $R - p$, is in $\mathrm{Aut}_{T_p}(S_p)$.

Suppose that $\bar{f}$ is in $\mathrm{Aut}_{T_p}(S_p)$. We will show that $f = \bar{f}|_S$ is in $\mathrm{Aut}_T(S)$. The map $\bar{f}$ extends to a map $g\colon L \to L$ given by $g(s/t) = \bar{f}(s)/\bar{f}(t)$, for s, t in S and $t \neq 0$. It is straightforward to check that g is in $\mathrm{Aut}_F(L)$. Now, by (1.7), L is a separable algebraic field extension of K, the quotient field of R. Hence, L is contained in a separable closure W of K. Since W is unique (inside a fixed algebraic closure of K), g extends to an automorphism $\bar{g}$ of W. As remarked earlier, $\bar{g}|_S$ is in $\mathrm{Aut}_T(S)$. By the construction of g, we see that $\bar{g}|_S = \bar{f}|_S = f$. It is straightforward to check that these maps describe the desired isomorphism.

The proof that $\mathrm{Aut}_F(L)$ and $\mathrm{Aut}_T(S)$ are isomorphic is similar. As we noted above, if f is in $\mathrm{Aut}_T(S)$, then $\bar{f}\colon L \to L$, given by $\bar{f}(a/b) = f(a)/f(b)$, for $a, b \neq 0$ in S, is in $\mathrm{Aut}_F(L)$. If we start with some $\bar{f}$ in $\mathrm{Aut}_F(L)$, $\bar{f}$ extends to an automorphism g of the separable closure of the quotient field of R, and $f = g|_S$ is in $\mathrm{Aut}_T(S)$.

The next proposition shows that integrally closed extensions of the base ring in the height 1 separable closure play an important part in the Galois correspondence we will establish.

PROPOSITION 3.2. *Let R be semi-Krull with height* 1 *separable closure S, and let H be a subgroup of* $\mathrm{Aut}_R(S)$. *Then S^H is integrally closed, where S^H is the set of elements of S invariant under H.*

PROOF. If x is in the quotient field of S^H and integral over S^H, then x is in S since S is integrally closed. It follows that x is in S^H.

We now state the Galois correspondence theorem for semi-Krull domains.

THEOREM 3.3. *Let R be a semi-Krull domain with height* 1 *separable closure S, and let $G = \mathrm{Aut}_R(S)$. Then there exists a one-to-one correspondence between integrally closed height* 1 *strongly separable extensions T of R contained in S and closed subgroups H of finite index in G, given by $T \to \mathrm{Aut}_T(S)$, $H \to S^H$.*

We will prove this theorem after Proposition 3.5.

PROPOSITION 3.4. *Let R be semi-Krull with height* 1 *separable closure S. Let T be an integrally closed height* 1 *strongly separable extension of R contained in S. If $H = \mathrm{Aut}_T(S)$ then $T = S^H$.*

PROOF. Let x be an element of $S - T$. Let F be the quotient field of T and K be the quotient field of R. Then F is a finite separable extension of K (1.6). Suppose that W is a separable closure of K containing both S and F. Since S is a direct limit of finitely generated R-modules, S is integral over R, and thus integral over T. Since T is integrally closed, x is in $W - F$. Thus, by standard Galois theory, there is a g in $\mathrm{Aut}_F(W)$ with $g(x) \neq x$. Hence, $f = g|_S$ is in $\mathrm{Aut}_T(S)$ with $f(x) \neq x$, and so $S^H = T$.

We note in particular that (3.4) implies that $S^G = R$, where $G = \mathrm{Aut}_R(S)$. If H is any subgroup of G, then H is almost finite, as defined in [4], since there are only finitely many restrictions of H to any finitely generated subalgebra of S. Hence, by [4, 1.13], $(S^H)_p = (S_p)^H$, where we have identified H_p with H. If $H = G$, then $R_p = S_p^H$, and we see that S_p is an infinite Galois extension of R_p. Further, we show that $L^G = K$, where L and K are the quotient fields of S and R, respectively, and G acts on L by $f(s/t) = f(s)/f(t)$, for s, t in S and f in G. Let W be a separable closure of K containing L. If x is any element of $L - K$, then x is in $W - K$. Hence, there is an f in $\mathrm{Aut}_K(W)$ such that $f(x) \neq x$. Let $g = f|_S$ and suppose $x = a/b$, for a, b in S. We see that g is in $\mathrm{Aut}_R(S)$ and g extends to $\bar{g}$ in $\mathrm{Aut}_K(L)$ as described above. Thus, $\bar{g}(a/b) = g(a)/g(b) = f(a)/f(b) = f(a/b) \neq a/b$. Therefore, $K = L^G$, and L is an infinite Galois extension of K.

We consider now the groups $\mathrm{Aut}_K(L)$ and $\mathrm{Aut}_R(S)$ (where, as above,

R is a semi-Krull domain with height 1 separable closure S, and K and L are the respective quotient fields). The group $\mathrm{Aut}_K(L)$ has the usual Krull topology, in which a subgroup H is open if and only if $H = \mathrm{Aut}_E(L)$, where E is a finite extension of K contained in L, and a subgroup is closed if and only if it is an intersection of open subgroups. By (3.1), $\mathrm{Aut}_K(L)$ is isomorphic to $\mathrm{Aut}_R(S)$, and we show that we can characterize the open subgroups of $\mathrm{Aut}_R(S)$ as follows. A subgroup H of $\mathrm{Aut}_R(S)$ is open if and only if $H = \mathrm{Aut}_T(S)$, where T is a height 1 strongly separable extension of R contained in S. For such an extension T, the quotient field of T, say F, is a finite separable field extension of K, so $\mathrm{Aut}_F(L)$ is an open subgroup of $\mathrm{Aut}_K(L)$. By (3.1), $\mathrm{Aut}_T(S)$ is isomorphic to $\mathrm{Aut}_F(L)$. On the other hand, if $H = \mathrm{Aut}_E(L)$ is an open subgroup of $\mathrm{Aut}_K(L)$, for some finite extension E of K, we claim that $H = \mathrm{Aut}_T(S)$, for some height 1 strongly separable extension T of R in S. Letting $T = S^H$ will produce the desired extension. Then the quotient field of T is E (actually, any element of E is of the form t/r, for some t in T and r in R). We need to show that T is height 1 strongly separable over R. Let p be in $X'(R)$. Since T_p is the integral closure of R_p in E, and R_p is Noetherian and integrally closed, T_p is a finite R_p-module. Then, since R_p is a PID and T_p is torsion-free, we see that T_p is a projective R_p-module. Since S_p is locally strongly separable over R_p there exists a strongly separable extension of R_p which contains T_p; call this extension S_0. S_0 is separable over T_p, and T_p is Noetherian (since R_p is) and integrally closed; thus, S_0 is a projective T_p-module, and therefore T_p is separable over R_p. Hence, T is a height 1 strongly separable extension of R, and we have $H = \mathrm{Aut}_E(L) = \mathrm{Aut}_T(S)$ by (3.1).

We note also that, since S_p is an infinite Galois extension of R_p for p in $X'(R)$, $\mathrm{Aut}_{R_p}(S_p)$ has the usual Krull topology (in which a subgroup is open if and only if it is of the form $\mathrm{Aut}_{T'}(S_p)$, for some strongly separable extension T' of R_p), and this topology coincides with that of $\mathrm{Aut}_K(L)$. We have seen that if F is a finite extension of K, then $T' = (S^H)_p$, where $H = \mathrm{Aut}_F(L)$, is a strongly separable extension of R_p and $\mathrm{Aut}_{T'}(S_p) = \mathrm{Aut}_F(L)$. If T' is a strongly separable extension of R_p then F, the quotient field of T', is a finite extension of K and $\mathrm{Aut}_{T'}(S_p) = \mathrm{Aut}_F(L)$.

PROPOSITION 3.5. *Let R be semi-Krull with height 1 separable closure S, and let H be a closed subgroup of finite index in $G = \mathrm{Aut}_R(S)$. Then $T = S^H$ is an integrally closed height 1 strongly separable extension of R and $\mathrm{Aut}_T(S) = H$.*

PROOF. S^H is integrally closed by (3.2). If p is in $X'(R)$, then S_p is locally strongly separable over R_p. Since H is a subgroup of $\mathrm{Aut}_R(S) = \mathrm{Aut}_{R_p}(S_p)$ of finite index, $T_p = (S^H)_p = S_p^H$ is a finitely generated separable extension of R_p, by standard Galois theory (See [5]). T_p is R_p-projective, since it is contained in some strongly separable extension of R_p. Thus T is a height 1 strongly separable extension of R.

To finish the proof, we need to show that $\mathrm{Aut}_T(S) = H$. Since $H_p(= H)$ is a closed subgroup of G_p $(= G)$, $R_p = S_p^{G_p}$, and $T_p = S_p^{H_p}$, we have $\mathrm{Aut}_{T_p}(S_p) = H_p$, i.e., $\mathrm{Aut}_T(S) = H$.

PROOF OF THEOREM 3.3. If T is an integrally closed height 1 strongly separable extension of R and $H = \mathrm{Aut}_T(S)$, then $T = S^H$, by (3.4). We show now that H is closed and has finite index in $G = \mathrm{Aut}_R(S)$. If p is in $X'(R)$, we have $G = \mathrm{Aut}_{R_p}(S_p)$ and $H = \mathrm{Aut}_{T_p}(S_p)$. Also, S_p is an infinite Galois extension of R_p and T_p is a strongly separable extension of R_p. Thus, H has finite index in G and is a closed subgroup of G. The rest of the proof of the theorem follows by (3.5).

PROPOSITION 3.6. *Let R be semi-Krull with height* 1 *separable closure S, and let T be a height* 1 *strongly separable extension of R contained in S. Then the integral closure of T is also height* 1 *strongly separable over R.*

PROOF. Let $H = \mathrm{Aut}_T(S)$. Then, if L and F are the quotient fields of S and T, respectively, we have $\mathrm{Aut}_T(S) = \mathrm{Aut}_F(L)$, and F is a finite (separable) extension of the quotient field of R. S^H is integrally closed by (3.2), and, as in the discussion preceding (3.5), S^H is a height 1 strongly separable extension of R, with quotient field F. Since T is contained in S^H we see that S^H must be the integral closure of T.

If R is Noetherian and semi-Krull, then an integrally closed height 1 strongly separable extension T is the integral closure of R in a finite separable extension of the quotient field of R, and so T is finitely generated as an R-module. The next example shows that T need not be finitely generated if R is not Noetherian. The example also shows that a finite height 1 strongly separable extension need not be integrally closed.

EXAMPLE 3.7. (*A* height 1 strongly separable extension T which is not integrally closed, and whose integral closure is not a finite R-module.) We return once more to the semi-Krull ring R in Example 1.2. Let $S = R[t]/(t^2 - y) = R[\sqrt{y}]$. Then, since y is not a unit in R but y is a unit in R_p, for each p in $X'(R)$, S is a finite height 1 strongly separable extension of R. We will show that (1) the integral closure of S, say T, is not finitely generated as an R-module, and (2) T is height 1 strongly separable over R.

Proof of(1). Let T' be the integral closure of $R'[\sqrt{y}]$, where R' is as in (1.2). An element of T' is of the form $a + b\sqrt{y}$, where a and b are in the quotient field of R'. Now, for each $i \geqq 0$, the element $t_i = x^i\sqrt{y}$ is in T', because $x^i = x^i y/y$ is in the quotient field of R' and $x^i\sqrt{y}$ is integral over R'. Let N' be the R'-module generated by the elements $1, t_i$ for $i \geqq 0$.

We claim that $N' = T'$. By the above, we see that $N' \subseteq T'$. Now suppose that $a + b\sqrt{y}$ is in T', for elements a, b in the quotient field of R'. Since $a+b\sqrt{y}$ is integral over $R'[\sqrt{y}]$, it is integral over R'. If K is the quotient field of R', then $a + b\sqrt{y}$ belongs to the finite extension of K with basis $\{1, \sqrt{y}\}$ and the characteristic polynomial of $a + b\sqrt{y}$ is $(t - (a + b\sqrt{y}))(t - (a - b\sqrt{y})) = t^2 - 2at + (a^2 - b^2y)$. The coefficients of this polynomial are in R' since $a + b\sqrt{y}$ is integral over R'. Thus, a is in R' (since $1/2$ is in R') and so b^2y is in R'. Let $b = f/g$, where f and g are in $R' \subseteq k[x, y]$, and, as elements of $k[x, y]$, f and g are relatively prime. Then f^2y/g^2 is in R', which implies that g^2 must divide y. Thus, g is actually a constant, and we see that the element b lies in $k[x, y]$. Write $b = \sum c_ix^i + h(x, y)$, where c_i is in k, and the polynomial $h(x, y)$ has no terms in a power of x alone. Then $a + b\sqrt{y} = a + h(x,y)\sqrt{y} + \sum c_it^i$, an element of N'. Therefore, $T' \subseteq N'$, and we have equality. We have shown that, as an R'-algebra, $T' = R'[t_0, t_1, \ldots]$. Now, since T' is integrally closed, so is T'_M and we have $T'_M = (R'[t_0, t_1, \ldots])_M = R'_M[t_0, t_1, \ldots] = R[t_0, t_1, \ldots]$. Thus, $R[t_0, t_1, \ldots]$ is integrally closed. Since $S \subseteq R[t_0, t_1, \ldots] \subseteq T$, the integral closure of S, we have $T = R[t_0, t_1, \ldots]$. Further, we have actually shown above that T is generated as an R-module by the elements $1, t_1, t_2, \ldots$.

We show that T cannot be finitely generated as R-module by showing that t_n cannot be written as an R-linear combination of $1, t_1, \ldots, t_{n-1}$, for $n \geqq 1$. Suppose that $x^n\sqrt{y} = r_0 + r_1x\sqrt{y} + \cdots + r_{n-1}x^{n-1}\sqrt{y}$, where each r_i is in R. Then, $r_0 = 0$, and $x^n = r_1x + \cdots + r_{n-1}x^{n-1}$. Let $r_i = f_i/g_i$, where f_i, g_i are in R' (and so have no terms in a power of x alone), and g is in $R' - M$ (and so g_i has a constant term). Clearing denominators, we obtain $(g_1g_2 \cdots g_{n-1})x^n = (g_2g_3 \cdots g_{n-1})f_1x + \cdots + (g_1g_2 \cdots g_{n-2})f_{n-1}x^{n-1}$. Setting $y=0$ in the above equation, we see that the left hand side is a nonzero polynomial in x of degree n, while the right hand side is a polynomial in x of degree at most $n - 1$, a contradiction. Thus T cannot be finitely generated as an R-module.

Proof of (2). This follows easily now. We know that T is generated as an R-module by the elements $1, \sqrt{y}, x\sqrt{y}, x^2\sqrt{y}, \ldots$. If p is any prime in $X'(R)$, $x^i = x^iy/y$ is in R_p, since y is in $R - p$. Hence, T_p is generated over R_p by the elements $1, \sqrt{y}$, and we have $T_p = S_p$, which is strongly separable over R_p.

While an integrally closed height 1 strongly separable extension T of R may not be finitely generated, it is true that T is the integral closure of a finitely generated height 1 strongly separable extension, as in the above example. We prove this next.

PROPOSITION 3.8. *If T is an integrally closed height* 1 *strongly separable extension of R, with T integral over R, then there is an R-subalgebra $F \subseteq$*

T, finitely generated and height 1 *strongly separable over R, such that the integral closure of F is T. Furthermore, $F_p = T_p$ for each p in $X'(R)$.*

PROOF. Let L and K be the quotient fields of T and R, respectively. Then L is a finite separable extension of K. Let a_i/b_i, $i = 1, \ldots, n$, generate L over K, where a_i and b_i, $b_i \neq 0$, are in T. Let B be the R-subalgebra of T generated by the elements a_i, b_i. Since B is integral over R, we see that B is actually finitely generated as a module over R. It is straightforward to check that the quotient field of B is L. By (1.4), B is height 1 separable at all but possibly a finite number of primes ir $X'(R)$.

We show next that if B_p is R_p-separable for p in $X'(R)$, then $B_p = T_p$. Since $R_p \subseteq B_p \subseteq T_p$ and T_p is strongly separable over R_p we see that T_p is strongly separable over B_p. Since T_p is finitely generated and projective over B_p, $B_p = T_p$ by (1.3).

Suppose now that p is in $X'(R)$ and B_p is not separable over R_p. By the strong separability of T_p over R_p there are elements z_i, y_i, $i = 1, \ldots, n$, in T_p and t in $\mathrm{Hom}_{R_p}(T_p, R_p)$ with $\sum z_i y_i = 1$ and $\sum z_j t(y_j x) = x$, for each x in T_p. Let T' be an R-subalgebra of T, finitely generated as an R-module, with $B \subseteq T' \subseteq T$, and such that T'_p contains the elements z_i, y_i. Then $t|_{T'_p}$ is in $\mathrm{Hom}_{R_p}(T'_p, R_p)$, and, since the z_j generate T_p as an R_p-module, $T'_p = T_p$. Hence, T'_p is strongly separable over R_p. Since there are only finitely many primes p in $X'(R)$ at which B_p is not separable over R_p, by (1.4), a finite number of iterations of this construction yields a finitely generated height 1 strongly separable R-algebra F with $B \subseteq F \subseteq T$ and such that $F_p = T_p$ for each p in $X'(R)$; hence the quotient field of F is L, the quotient field of T. Since T is integrally closed and integral over R, T must be the integral closure of F. This finishes the proof.

We finish with the following Galois correspondence for integrally closed extensions which are direct limits of height 1 strongly separable extensions.

PROPOSITION 3.9. *Let R be semi-Krull with height* 1 *separable closure S, and let $G = \mathrm{Aut}_R(S)$. Then there exists a one-to-one correspondence between integrally closed locally height* 1 *strongly separable extensions T of R contained in S and closed subgroups H of G, given by $T \to \mathrm{Aut}_T(S)$, $H \to S^H$.*

PROOF. Suppose T is integrally closed and $T = \text{dir lim } T_i$, where each T_i is height 1 strongly separable over R and $T_i \subseteq T$. Then we have $H = \mathrm{Aut}_T(S) = \mathrm{Aut}_{\cup T_i}(S) = \bigcap \mathrm{Aut}_{T_i}(S)$, and so H is closed in G. Since the quotient field of T is a separable algebraic extension of the quotient field of R, $T = S^H$ just as in the proof of (3.4).

Conversely, suppose H is closed in G. Then $H = \bigcap \mathrm{Aut}_{T_i}(S)$ for some set of height 1 strongly separable extensions T_i of R. Let $\{T_{i_1}, \ldots, T_{i_n}\}$ be any finite subset of the T_i's, and let

$$J = \mathrm{Aut}_{T_{i1} \cdot T_{i2} \cdot \ldots \cdot T_{in}}(S) = \mathrm{Aut}_{T_{i1}}(S) \cap \cdots \cap \mathrm{Aut}_{T_{in}}(S).$$

Then J is a closed subgroup of finite index in G, so $S^J = T_{i_1} \cdot T_{i_2} \cdot \cdots \cdot T_{i_n}$ is an integrally closed height 1 separable R-algebra by (3.5). Let T' be the union of all the finite products $T_{i_1} \cdot T_{i_2} \cdot \cdots \cdot T_{i_n}$. Then T' is an integrally closed, locally height 1 strongly separable extension of R. Thus, $S^H = S^{\mathrm{Aut}_{T'}(S)} = T'$ by the first part of the proof, and so S^H is integrally closed and height 1 locally strongly separable over R. To complete the proof, we need to show that $\mathrm{Aut}_{S^H}(S) \subseteq H$, since it is always true that $H \subseteq \mathrm{Aut}_{S^H}(S)$. Let $H_i = \mathrm{Aut}_{T_i}(S)$. Noting that $S^H = S^{\cap H_i} \supseteq \bigcup S^{H_i}$ and $S^{H_i} = T_i$, by (3.3), we have

$$\mathrm{Aut}_{S^H}(S) \subseteq \mathrm{Aut}_{\cup S^{H_i}}(S) \subseteq \bigcap \mathrm{Aut}_{S^{H_i}}(S) = \bigcap \mathrm{Aut}_{T_i}(S) = H.$$

This completes the proof.

REFERENCES

1. F. DeMeyer, and E. Ingraham, *Separable Algebras over Commutative Rings*. Springer-Verlag **181**, New York, 1971.

2. S. Endo, *On flat modules over commutative rings*, J. Math. Soc. Japan, **14** (1962), 284–291.

3. G. Janusz, *Separable algebras over commutative rings*, Trans. Amer. Math. Soc., **122** (1966), 461–479.

4. A. Magid, *Locally Galois algebras*, Pac. J. Math, **33** (1970), 707–724.

5. T. Nagahara, *A note on Galois theory of commutative rings*, Proc. Amer. Math. Soc., **18** (1967), 334–340.

6. L. Naylor, *Height one separable algebras over commutative rings*, Rocky Mt. J. of Math **11** (1981), 593–603.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DRAKE UNIVERSITY, DES MOINES, IA 50311