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ORLICZ SPACES WHICH ARE RIESZ ISOMORPHIC TO ℓ^∞

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ABSTRACT. The main purpose of this paper is to describe, in terms of the function φ and the measure μ, Orlicz spaces $L^\varphi(S,\sum,\mu)$ which are Riesz isomorphic to ℓ^∞. The "thickness", in the sense of Baire category, of the subset of measures for which $L^\varphi(S,\sum,\mu)$ is Riesz isomorphic to ℓ^∞ is also investigated.

I. Basic notation and auxiliary results. Throughout the note, in what concerns Riesz spaces (= vector lattices) we use the terminology of [**2**]. When two Riesz spaces L and K are Riesz isomorphic, then this fact will be noted by $L \simeq K$. The symbols $\mathbf{R}^S$ and N are reserved for the space of functions from a set S into $\mathbf{R}$ with the standard pointwise order and for the set of positive integers, respectively. Moreover, e_s denotes the characteristic function of the set $\{s\}, L_+$ is the cone of positive elements of a Riesz space L and $\ell_0^\infty(S)$ is the ideal in $\ell^\infty(S)$ consisting of functions with at most countable support. When S is countable then, of course, $\ell_0^\infty(S) = \ell^\infty(S)$.

We start with two simple lemmas.

LEMMA 1. *Let $L_i (i = 1, 2)$ be Riesz subspaces of $\mathbf{R}^S$ containing all e_s's. If $T : L_1 \to L_2$ is a Riesz isomorphism onto, then there exists a function $g \in \mathbf{R}_+^S$ and a bijection $\alpha : S \to S$ such that*

$$T(x)(s) = g(s)x(\alpha(s))$$

for all $x \in L_1$.

The above statement follows immediately from two facts: $T(e_s)$ is an atom in L_2 (so it has the form $a_s e_{s'}$) and T is a normal Riesz homomorphism.

The next Lemma will be frequently used.

LEMMA 2. *Let L be a Riesz subspace of $\mathbf{R}^S$ containing all e_s's, and let A be a subset of S. If L is Riesz isomorphic to $\ell_0^\infty(S)$, then*

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(a) *there exists a function* $h \in \mathbf{R}_+^S$ *such that the operator* $T : \ell_0^\infty(S) \to L$ *given by the formula*

$$T(x)(s) = h(s)x(s)$$

is a Riesz isomorphism onto.

(b) $P_A L$ *is Riesz isomorphic to* $\ell_0^\infty(A)$, *where* P_A *denotes the projection onto the band generated by the set* $\{e_s : s \in A\}$.

PROOF. Let $H : \ell_0^\infty(S) \to L$ be a Riesz isomorphism onto. Then there exists a bijection $\beta : S \to S$ such that $H(e_s) = a_s e_{\beta(s)}$, where $a_s > 0$. Put $h(s) = a_{\alpha(s)}$, where $\alpha = \beta^{-1}$. The Dedekind completeness of L implies that T defined in (a) is a Riesz isomorphism onto.

Part (b) follows by the equality $T(P_A(\ell_0^\infty(S))) = P_A L$, where T is the operator given in (a).

Let $(S, \sum, \mu)$ be a measure space. A function $G : [0,\infty) \times S \to [0,\infty)$ is called a Musielak-Orlicz function, if

1° $G(\cdot, s) : [0,\infty) \to [0,\infty)$ is left continuous, continuous at zero, non-decreasing and $G(r,s) = 0$ if and only if $r = 0$,

2° $G(r,\cdot) : S \to [0,\infty)$ is $\sum$-measurable.

Every Musielak-Orlicz function G together with $(S, \sum, \mu)$ generates a space of measurable functions called Musielak-Orlicz space:

$$\begin{aligned} L^G(S,\textstyle\sum,\mu) = \{x \in L^0(S,\textstyle\sum,\mu) : m_G(tx) \\ = \int_S G(t|x(S)|, s)d\mu < \infty \text{ for some } t > 0\}. \end{aligned}$$

Here $L^0(S,\sum,\mu)$ is the space of all μ-equivalence classes of $\sum$-measurable real-valued functions on S.

Any Musielak-Orlicz space $L^G(S,\sum,\mu)$, with respect to the standard $\mu - a.e.$ order and with the monotone F-norm $||x||_G = \inf\{r > 0 : m_G(x/r) \le r\}$is a super Dedekind complete F-lattice whose topology has the Fatou and σ-Levi properties. It is easy to observe that the sets of the form $a \cdot B(r)$ constitute a base of neighbourhoods of zero for the topology given by $||\cdot||_G$ where $a, r > 0$ and $B(r) = \{x \in L^G(S,\sum,\mu) : m_G(x) < r\}$.

The largest ideal in $L^G(S,\sum,\mu)$ on which $\|\cdot\|_G$ has the Lebesgue property is usually denoted by $L^G_a(S,\sum,\mu)$. It is known that

$$L^G_a(S,\sum,\mu)=\{x\in L^G(S,\sum,\mu):m_G(tx)\\ <\infty \text{ for all } t>0\}.$$

Moreover, L^G_a is super order dense in $L^G(S,\sum,\mu)$ (for details see [**6**]).

The class of Musielak-Orlicz functions contains Orlicz functions, i.e., functions $\varphi:[0,\infty)\to[0,\infty)$ with properties listed in 1^0. A Musielak-Orlicz space generated by Orlicz functions is called an Orlicz space. Orlicz functions will be denoted by small greek letters φ,ψ. Two Orlicz functions φ,ψ are equivalent (see [**5**]) if

$$a\varphi(br)\le\psi(r)\le c\varphi(dr)$$

for some positive constants a,b,c,d and all $r\in\mathbf{R}_+$. Equivalent Orlicz functions generate the same Orlicz spaces and therefore the identity is a Riesz and topological isomorphism between them. Thus if we are interested in isomorphic invariants of an Orlicz space, we may replace a given Orlicz function φ by an equivalent one $\overline{\varphi}$ possessing "better" properties than φ. For example, for every Orlicz function φ there exists a continuous and strictly increasing $\overline{\varphi}$ equivalent to φ. Indeed, putting

$$(*)\qquad \overline{\varphi}(r)=\frac{1}{r}\int_0^r\varphi(t)dt,$$

we obtain

$$\frac{1}{2}\varphi(\frac{1}{2}r)\le\overline{\varphi}(r)\le\varphi(r).$$

In this paper Orlicz spaces which are Riesz isomorphic to $\ell_0^\infty(S)$ will be investigated and therefore only purely atomic measure spaces $(S,\sum,\mu)$ will be considered. We can assume that $\sum$ is the σ-algebra generated by one-point sets $\{s\}$ and $\mu(\{s\})=a_s\in(0,\infty)$. It is possible to restrict our considerations to semi-finite measures because if $S_\infty=\{s:\mu(\{s\})=\infty\}$, then $S_\infty\cap\operatorname{supp}x=\emptyset$ for every $x\in L^\varphi(S,\sum,\mu)$, and so the spaces $L^\varphi(S,\sum,\mu)$ and $L^\varphi(S\backslash S_\infty,\wedge,\mu|\wedge)$ are Riesz isomorphic, where $\wedge$ is the σ-algebra of subsets of $S\backslash S_\infty$ generated by sets $\operatorname{supp}x$ $(x\in L^\varphi(S,\sum,\mu))$.

We can also assume, in the case when μ is σ-finite, that $S=N,\sum$ is the σ-algebra of all subsets of $N,\mu(\{n\})=a_n$.

We will write $\ell^G(a_s)$ ($\ell^G_a(a_s)$) instead of $L^G(S,\sum,\mu)$ ($L^G_a(S,\sum,\mu)$), and ℓ^G (ℓ^G_a) when S is countable and $a_s = 1$ for all s.

In the second part of this paper we will need the following simple Lemma (due to Drewnowski [**4**]).

LEMMA 3. *Let $G : [0,\infty)\times S \to [0,\infty)$ be a Musielak-Orlicz function. The following conditions are equivalent:*

(a) $\ell^G(a_s) = \ell^\infty_0(S)$ *(algebraically),*

(b) *for every countable subset $S_0 \subset S$ there exist $0 < v < u$ such that $\inf_{s\in S_0} G(u,s)a_s > 0$ and $\sum_{s\in S_0} G(v,s)a_s < \infty$.*

The next Lemma exhibits a class of Orlicz spaces over a σ-finite purely atomic measure space which are not Riesz isomorphic to ℓ^∞.

LEMMA 4. *If $\sum_1^\infty a_n = \infty$ and $\sup a_n < \infty$, then the Orlicz space $\ell^\varphi(a_n)$ has no strong unit.*

PROOF. Suppose $x = (x_n)$ is a strong unit in $\ell^\varphi(a_n)$. Then $x_n > 0$ for all n and zero must be an accumulation point of (x_n). Thus, by the continuity of φ at zero, we can choose a subsequence (x_{n_k}) such that $\varphi(2^k x_{n_k}) < 2^{-k}$. Put

$$y_m = \begin{cases} kx_{n_k} & \text{for } m = n_k \\ 0 & \text{for the others } m's. \end{cases}$$

We obtain $y = (y_m) \in \ell^\varphi_a(a_n)$, and so $y \le Mx$ for some number M. In other words, $kx_{n_k} \le Mx_{n_k}$ for all k which is impossible.

Lemma 2(b) and Lemma 4 imply the following fact:

If S is uncountable, then $\ell^\varphi(a_s)$ is never Riesz isomorphic to $\ell^\infty_0(S)$.

Indeed, suppose that $\ell^\varphi(a_s)$is Riesz isomorphic to $\ell^\infty_0(S)$ and let $S(a,b) = \{s : a_s \in (a,b)\}$. Since S is uncountable, $S(m^{-1},m)$ contains imfinitely many elements for some $m \in N$. Taking an arbitrary countable subset A of $S(m^{-1},m)$ and using Lemma 2(b) we have $P_A(\ell^\varphi(a_s))$ is Riesz isomorphic to ℓ^∞. Hence the spaces ℓ^∞ and ℓ^φ would be Riesz isomorphic which is impossible because, by lemma 4, ℓ^φ has no strong unit.

Therefore, in further considerations we will assume that μ is a σ-finite purely atomic measure.

The analogous arguments as above give

LEMMA 5. *If $\ell^\varphi(a_n)$ is Riesz isomorphic to ℓ^∞, then the set of accumulation points of the sequence (a_n) is included in $\{0,\infty\}$.*

II. Main results. We recall that an Orlicz function φ is said to satisfy the Δ_2^∞-condition (Δ_2^0-condition), shortly $\varphi \in \Delta_2^\infty (\varphi \in \Delta_2^0)$, if $\varphi(2r) \leq k\varphi(r)$ for some $k > 1$ and all r's from some neighbourhood of infinity (of zero).

THEOREM 1. *The following conditions are equivalent:*

(a) $\ell^\varphi(a_n) \simeq \ell^\infty$ *for some sequence* (a_n)*;*

(b) $\ell^\varphi_a(a_n) \sim c_0$ *(i.e., these spaces are isomorphic as topological vector spaces) for some sequence* (a_n);

(c) $\ell^\varphi(a_n) \sim \ell^\infty$ *for some sequence* (a_n)*; and*

(d) $\varphi \notin \Delta_2^\infty$ *or* $\varphi \notin \Delta_2^0$.

PROOF. (a)$\Rightarrow$(b) is obvious. (b)$\Rightarrow$(c). Since $\ell^\varphi_a(a_n)$ and c_0 are isomorphic, they are Riesz isomorphic (see [**1**; Theorem 6]. Let $T : c_0 \to \ell^\varphi_a(a_n)$ be Riesz isomorphism onto. The σ-Levi property of $\ell^\varphi(a_n)$ implies the existence of the element $e = \sup_n T(e_n)$. Take an arbitrary element $x \in \ell^\varphi(a_n)$. Thus $|x| = \sup_n T(z_n)$ for some increasing sequence $(z_n) \subset c_{0+}$ according to supper order density of $\ell^\varphi_a(a_n)$ in $\ell^\varphi(a_n)$. We have $z_n(k)e_k \leq ||z_n||_{\ell^\infty} e_k$ for all k; so $T(z_n) = T(\sup_k z_n(k)e_k) = \sup_k z_n(k)T(e_k) \leq ||z_n||_{\ell^\infty} e$. The inequality $T(z_n) \leq |x|$ gives that (z_n) is topologically bounded, thus $x = \sup_n T(z_n) \leq \sup_n || z_n ||_{\ell^\infty} \cdot e$. In other words, the element e is a strong unit in $\ell^\varphi(a_n)$, and so $\ell^\varphi(a_n)$ must be Riesz isomorphic (not only isomorphic) to ℓ^∞ (see [**7**; Theorem 12]). (c)$\Rightarrow$(d). Suppose $\ell^\varphi(a_n) \sim \ell^\infty$ but $\varphi \in \Delta_2^\infty$ and $\varphi \in \Delta_2^0$, Therefore $\varphi(2r) \leq K \cdot \varphi(r)$ for some $K > 1$ and all $r \geq 0$. The last inequality implies $\ell^\varphi(a_n) = \ell^\varphi_a(a_n)$ and we have obtained a contradiction because $\ell^\varphi_a(a_n)$ is separable. (d)$\Rightarrow$(a). Suppose first $\varphi \notin \Delta_2^\infty$. Then there exists a sequence (b_n) increasing to infinity such that $\varphi(2b_n) > 2^n\varphi(b_n)$ for all n. Putting $a_n = (\varphi(2b_n))^{-1}$ and $G(r,n) = \varphi(rb_n)a_n$ we obtain $\sum_1^\infty G(1,n) < \infty$ and $\inf_n G(2,n) > 0$. According to Lemma 3, $\ell^G = \ell^\infty$, and so $(x_n) \in \ell^\infty$ if and only if $(b_n x_n) \in \ell^\varphi(a_n)$. Hence the operator $T : \ell^\infty \to \ell^\varphi(a_n)$ defined by the formula $T((x_n)) = (b_n x_n)$ is a Riesz isomorphism onto.

In the case $\varphi \notin \Delta_2^0$ the proof is analogous.

REMARKS. 1. The theorem implies in particular that $\ell^\varphi(a_n)$ may be locally convex even if φ is not equivalent to a convex function.

2. Applying Lemma 2(b) and Lemma 5, we obtain: if $\ell^\varphi(a_n) \simeq \ell^\infty$, then

(a) $a_n \to 0$ when $\varphi \notin \Delta_2^\infty$ and $\varphi \in \Delta_2^0$,

(b) $a_n \to \infty$ when $\varphi \in \Delta_2^\infty$ and $\varphi \notin \Delta_2^0$,

(c) $a_n \to 0$ or $a_n \to \infty$ or zero and infinity are the accumulation points of (a_n) when $\varphi \notin \Delta_2^\infty$ and $\varphi \notin \Delta_2^0$.

THEOREM 2. *Let $(a_n) \subset R, a_n > 0$. Assume additionally that the Orlicz function φ is strictly increasing. Then the following conditions are equivalent:*

(a) $\ell^\varphi(a_n) \simeq \ell^\infty$;

(b) $\sum_1^\infty \varphi(a\varphi^{-1}(a/a_n))a_n < \infty$ *for some* $a \in (0,1)$; *and*

(c) $(\varphi^{-1}(a/a_n))$ *is a strong unit in* $\ell^\varphi(a_n)$ *for some* $a > 0$.

PROOF. (a)$\Rightarrow$(b). According to Lemma 2(a) there exists a sequence $(c_n) \in \mathbf{R}_+^N$ such that the operator $T : \ell^\infty \to \ell^\varphi(a_n)$ defined by $T((x_n)) = (c_n x_n)$ is a Riesz isomorphism onto. Thus $\ell^G = \ell^\infty$, where $G(r,n) = \varphi(rc_n)a_n$. Using Lemma 3 we obtain $\inf_n G(u,n) = d > 0$ and $\sum_1^\infty G(v,n) < \infty$ for some $0 < v < u$. In other words $d = \inf_n \varphi(uc_n)a_n > 0$ and $\sum_1^\infty \varphi(vc_n)a_n < \infty$. The inequality $c_n \geq u^{-1}\varphi^{-1}(d/a_n)$ implies $\sum_1^\infty \varphi((v/u)\varphi^{-1}(d/a_n))a_n < \infty$. The proof will be finished if we put $a = \min(v/u, d)$. (b)$\Rightarrow$(c) is obvious. (c)$\Rightarrow$(a). Since $\ell^\varphi(a_n)$ possesses a strong unit, it is Riesz isomorphic to ℓ^∞ (see [**7**; Theorem 12]).

If E is a subset of $\mathbf{R}^n$, then E_{++} denotes the subset of E consisting of sequences with strictly positive terms.

EXAMPLE. Applying Theorem 2 to the function $\varphi(r) = e^r - 1$ we obtain $\ell^\varphi(a_n) \simeq \ell^\infty$ if and only if $(a_n) \in \cup_{0<p<1}\ell^p_{++}$.

Theorem 2 implies also the following properties of the set $W_\varphi = \{(a_n) \in R_{++}^N : \ell^\varphi(a_n) \simeq \ell^\infty\}$ (φ strictly increasing): $W_\varphi + W_\varphi \subset W_\varphi; tW_\varphi \subset W_\varphi$ for all $t > 0$; and $(x_n), (y_n) \in W_\varphi$ implies $(x_n \vee y_n) \in W_\varphi$.

It was already noticed that if $\ell^\varphi(a_n) \simeq \ell^\infty, \varphi \notin \Delta_2^\infty$ and $\varphi \in \Delta_2^0$ (respectively $\varphi \in \Delta_2^\infty$ and $\varphi \notin \Delta_2^0$), then $(a_n) \in c_0$ (respectively $(a_n^{-1}) \in c_0$). Let $c^\infty = \{(a_n) : a_n \to \infty\}$ be equipped with the topology of uniform convergence. Then we may ask about the "thickness" of the set of sequences $(a_n) \in c_{0++}((a_n) \in c^{\infty}_{++})$ for which $\ell^{\varphi}(a_n) \simeq \ell^{\infty}$ in the sense of Baire category.

The part (b) of Theorem 2 implies that if $\ell^{\varphi}(a_n) \simeq \ell^{\infty}$ and $a_n \to 0$, then $(a_n) \in \ell^1_{++}$. Denoting $W^1_{\varphi} = \{(a_n) \in \ell^1_{++} : \ell^{\varphi}(a_n) \simeq \ell^{\infty}\}$ we have

THEOREM 3. *For every Orlicz function φ the set W^1_{φ} is of the first category in ℓ^1_{+}.*

We need the following Lemma

LEMMA 6. *Let X be a metric space and let (f_n) be a sequence of real continuous functions on X. If $\{x : \sup_n f_n(x) < \infty\}$ is of the second category, then there is a non-empty open set U such that $\sup_n \sup_{x \in U} f_n(x) < \infty$.*

The proof of the above Lemma can be found in [**3**, p.111].

PROOF OF THEOREM 3. As $W^1_{\varphi} = W^1_{\overline{\varphi}}$ (for the definition of $\overline{\varphi}$ see (*)), we can assume that φ is strictly increasing and continuous. If φ is bounded, then $W^1_{\varphi} = \emptyset$ ($\varphi \in \Delta^{\infty}_2$, thus $\ell^{\varphi}(a_n) = \ell^{\varphi}_a(a_n)$ for every $(a_n) \in \ell^1_{++}$). Let φ be unbounded. According to Theorem 2, $W^1_{\varphi} = \cup_{a \in Q} W^1_a$, where $W^1_a = \{(a_n) \in \ell^1_{++} : \sum_1^{\infty} \varphi(a\varphi^{-1}(a/a_n))a_n < \infty\}$ and Q rationals in $(0,1)$. Fix $a \in Q$. The functions $s_n : \ell^1_{++} \to [0,\infty)$ defined by

$$s_n((a_k)) = \sum_{j=1}^{n} \varphi(a\varphi^{-1}(a/a_j))a_j$$

are continuous. Moreover, $W^1_a = \{(a_n) \in \ell^1_{++} : \sup_n s_n((a_k)) < \infty\}$. We claim W^1_a is of the first category in ℓ^1_{++} for all a. If not, then W^1_a is of the second category for some a, and so, by Lemma 6, there exists a ball B with the radius ε such that

$$\sup_n \sup\{s_n((a_k)) : (a_k) \in B\} < \infty.$$

Let $c = (c_j)$ be the center of B. Fix k such that $\sum_{k+1}^{\infty} c_j < \varepsilon/2$. Put

$$b_n(j) = \begin{cases} c_j & \text{for } j \in \{1, \cdots, k\} \cup \{k+1+n, \cdots\} \\ \varepsilon/2n & \text{for } j \in \{k+1, \cdots, k+n\}. \end{cases}$$

We have $||c - b_n||_{\ell^1} < \varepsilon$ for all n, thus $b_n \in B$. Moreover

$$\begin{aligned} s_{n+k}(b_n) &\geq \sum_{j=k+1}^{k+n} \varphi(a\varphi^{-1}(2an/\varepsilon))(\varepsilon/2n) \\ &= (\varepsilon/2)\varphi(a\varphi^{-1}(2an/\varepsilon)). \end{aligned}$$

Since $\sup_n \sup\{s_n((a_k)) : (a_k) \in B\} \geq \sup_n s_{n+k}(b_n) \geq \sup_n(\varepsilon/2)\varphi (a\varphi^{-1}(2an/\varepsilon)) = \infty$, we have obtained a contradiction. Therefore W_a^1 is of the first category in ℓ_{++}^1 for all a and thus W_φ^1 is likewise. It is clear that ℓ_{++}^1 is a dense G_δ subset of ℓ_+^1. Hence W_φ^1 is of the first category in ℓ_+^1.

Now we will consider the case of sequences (a_n) tending to infinity and $\ell^\varphi(a_n) \simeq \ell^\infty$. Comparing with $(a_n) \in \ell_{++}^1$, the situation changes essentially.

THEOREM 4. *The set of sequences $(a_n) \in c_{++}^\infty$ such that $\ell^\varphi(a_n) \simeq \ell^\infty$ is open in c_+^∞.*

PROOF. The uniform convergence in c_+^∞ is determined by the metric $d((x_n),(y_n)) = \sup_n \min(|x_n - y_n|, 1)$. We can assume, as before, that φ is strictly increasing. Let W_φ^∞ and W_a^∞ be defined similarly as the sets W_φ^1 and W_a^1 in the previous proof replacing ℓ_{++}^1 by c_{++}^∞. Take an arbitrary sequence $(a_n) \in W_\varphi^\infty$. Then, by Theorem 2, $(a_n) \in W_a^\infty$ for some $a \in (0,1)$. Let $p \in (0,1)$ be so that $a_n - p > 0$ for all n. Let $(b_n) \in B((a_n), p) = \{(c_n) : d((c_n),(a_n)) < p\}$. Then there exists $n_0 \in N$ such that, for all $n \geq n_0$,

$$(1-p)a_n < a_n - p < b_n < a_n + p < (1+p)a_n.$$

Using the above inequalities and putting $b = a(1-p) < a$, we have

$$\begin{aligned} \sum_{n_0}^{\infty} \varphi(b\varphi^{-1}(b/b_n))b_n &\leq \sum_{n_0}^{\infty} \varphi(b\varphi^{-1}(b/(1-p)a_n))(1+p)a_n \\ &\leq (1+p)\sum_{n_0}^{\infty} \varphi(a\varphi^{-1}(a/a_n))a_n < \infty. \end{aligned}$$

Thus, $B((a_n), p) \subset W_b^\infty \subset W_\varphi^\infty$ and W_φ^∞ is open.

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